In the mid-1980s, Ross Street wrote a 52-page paper called "The algebra of oriented simplexes". For most of the paper he was concerned with <u>strict</u> ω-categories, but in the very last sentence he tentatively suggested a definition of <u>weak</u> ω-category. I will now write his definition on the board in complete detail. It's very short. Nick will then explain some of the intuition behind it. He'll also tell you about some of Ross's more recent work, in which he's come up with a new and improved version of his original definition.

Simplicial terminology

- Δ has objects $[0], [1], \ldots$ where $[m] = \{0, 1, \ldots, m\}$
- For each $m \geq k \geq 0$, have horn $\Lambda^k[m] : \Delta^{op} \longrightarrow \mathbf{Set}$, where

$$\Lambda^k_{m'}[m] = \{\psi \in \Delta([m'],[m]) \mid im(\psi) \text{ is not } [m] \text{ or } [m]\setminus\{k\}\}$$

(It's already been mentioned that in a Kan complex, which is something like a weak ω-gpd, all horns have fillers)

- Let $X : \Delta^{op} \to \mathbf{Set}$, $m \geq 0$, and $x \in X_m$. Then x is <u>degenerate</u> if there exist $p < m$, a surjection $\sigma : [m] \longrightarrow [p]$ and $y \in X_p$ such that $x = (X\sigma)y$.
- A <u>stratified simplicial set (SSS)</u> is a simp. set X equipped with a subset $t_m \subseteq X_m$ $\forall m \geq 1$ (the <u>thin</u> or <u>hollow</u> elements) (to be thought of as like opetopic universal cells — but watch it, this is extra structure, not property), s.t. degenerate $\Rightarrow$ thin.

Admissible horns

(The next few defins will look completely opaque. They embody the answer to the following question: ~~when~~ in a weak ω-<u>category</u> (not ω-groupoid), which horns should have fillers? The answer is so complicated because of the issue of orientation.)

- A set of natural numbers is <u>alternating</u> if its elements, written in ascending order, alternate in parity.
- Let $m \geq k \geq 0$ and $S \subseteq [m]$. Then S is <u>k-alternating</u> if (not Ross's original terminology)
  - $\{k-1, k, k+1\} \cap [m] \subseteq S$
  - $[m] \setminus (S \setminus \{k-1, k, k+1\})$ is alternating.
- Let (X,t) be a SSS and $m \geq k \geq 0$. A horn $h : \Lambda^k[m] \longrightarrow X$ in X is <u>admissible</u> if $\forall m' \geq 1$, $\forall \psi \in \Lambda^k_{m'}[m]$,

$$im(\psi) \subseteq [m] \text{ is } k\text{-alternating} \Rightarrow h(\psi) \text{ is thin.}$$

The definition

- A <u>weak ω-category</u> is a SSS (X, t) such that
  - * every admissible horn has a thin filler
  - * if $m \geq 2$, $0 \leq k \leq m$, and $x \in X_m$ with $x, d_0 x, \ldots, d_{k-1} x, d_{k+1} x, \ldots, d_m x$ all thin and forming an admissible horn, then $d_k x$ is thin.

(Compare ~~defn~~ opetopic def-n!)

- A <u>weak n-category</u> is a weak ω-category (X, t) such that
  - * $\forall m > n$, $t_m = X_m$
  - * every admissible horn of dimension $> n$ has a <u>unique</u> filler.