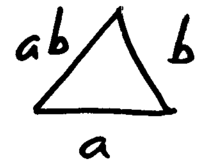
$$\mathrm{Hom}(\mathbb{C}_g, \mathbb{C}_h) = \delta_{g,h}\, \mathbb{C}_g \Longrightarrow$$

$$Z(M,T,\ell) = \begin{cases} 1 & \text{if } \ell: \\ 0 & \text{otherwise} \end{cases}$$



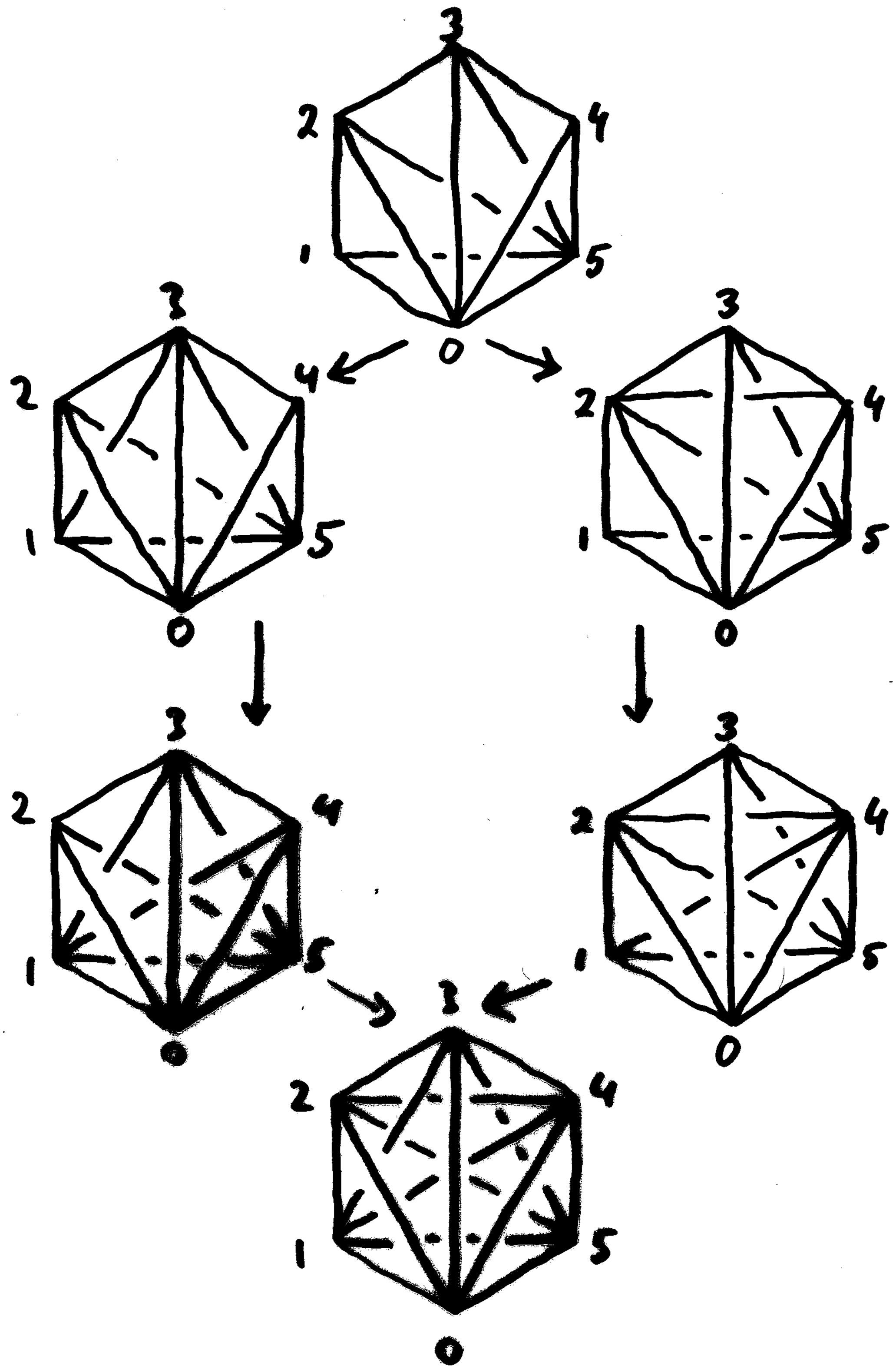
$$Z(M,T) = |G|^{-V} \,\#\{\text{non-zero labellings}\}$$

$$= |G|^{-V}\, |G|^{V-1} \,\#\mathrm{Hom}(\pi_1 M, G)$$

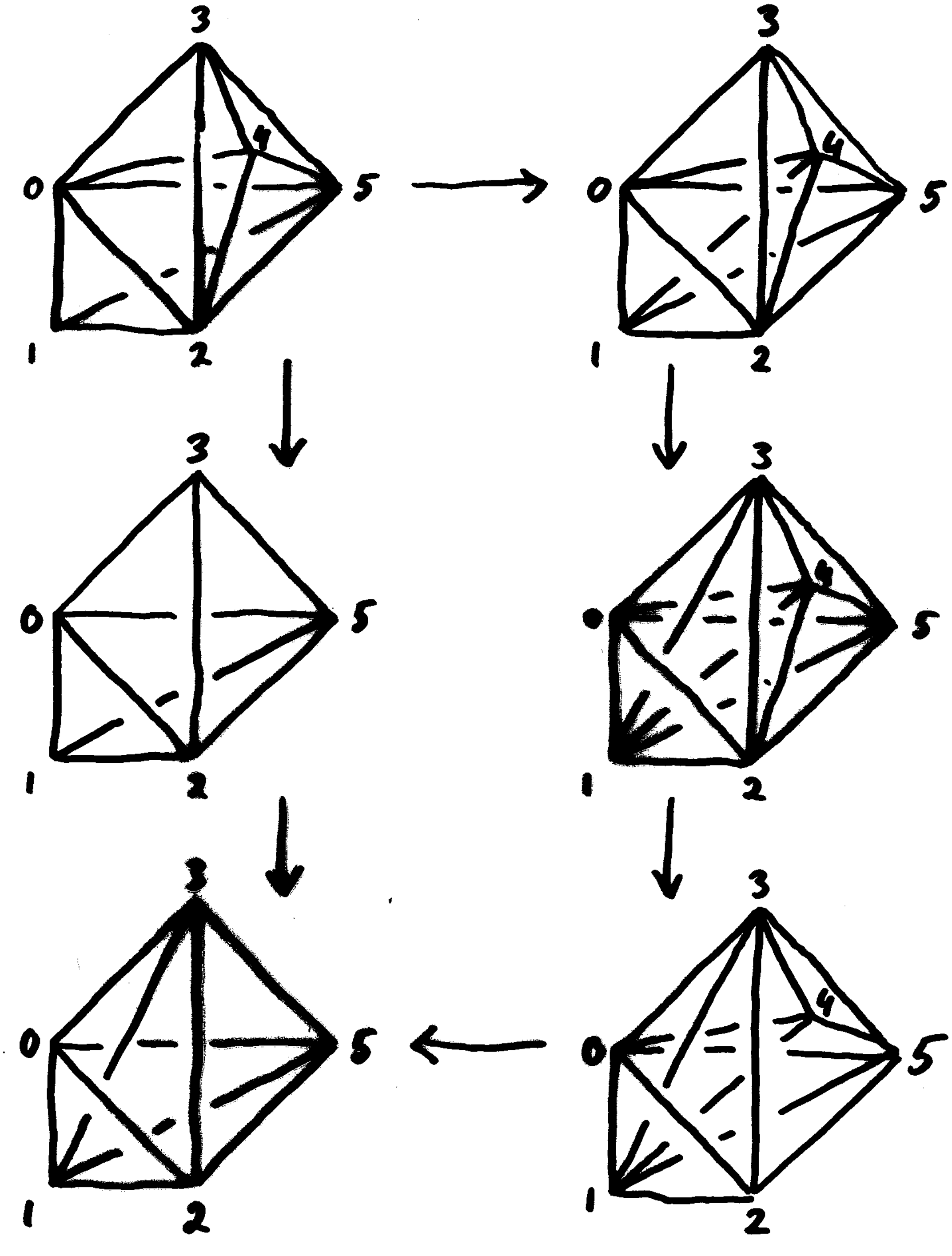
$$= |G|^{-1} \,\#\mathrm{Hom}(\pi_1 M, G)$$

(Dijkgraaf-Witten, Freed-Quinn)

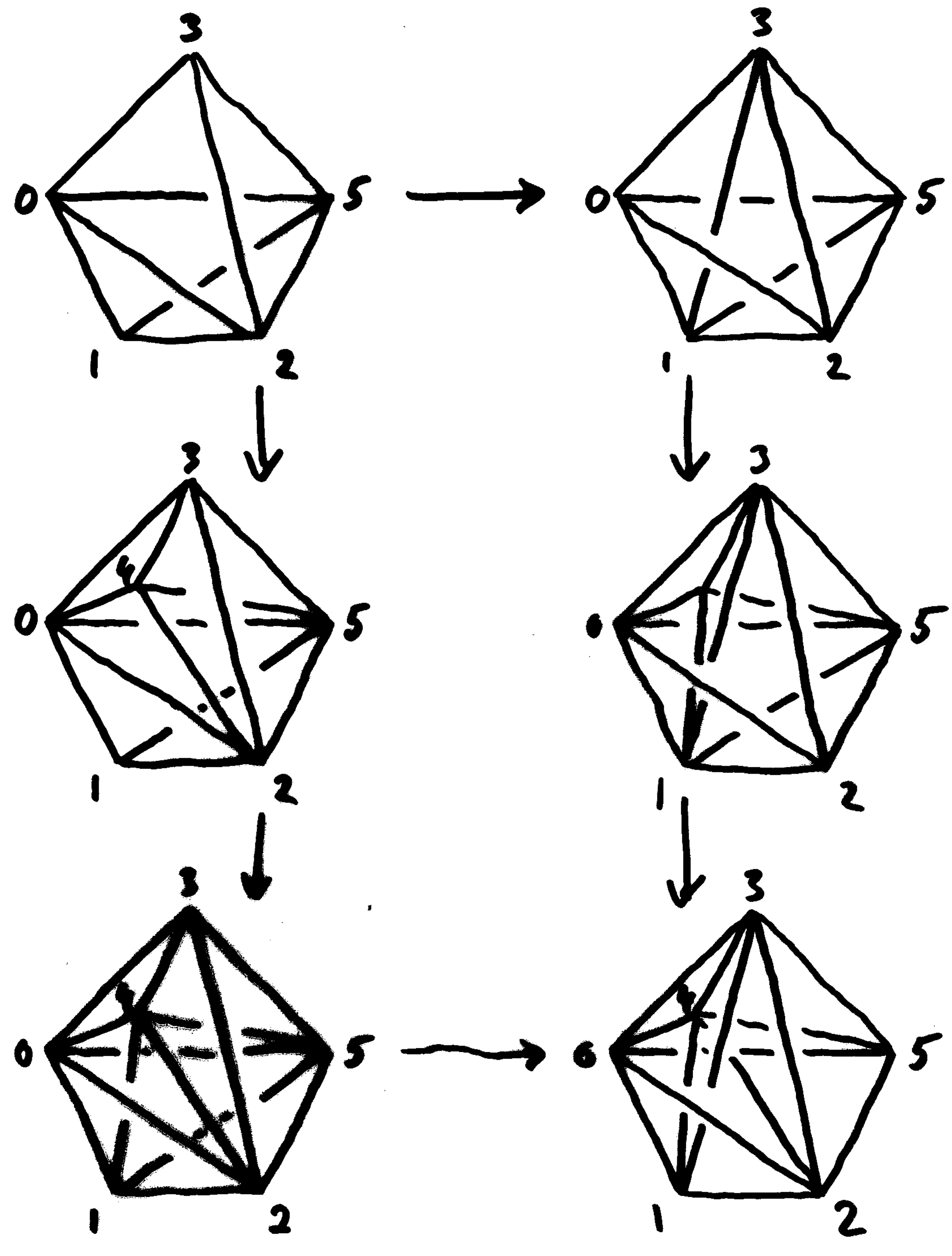
$3 \rightleftarrows 3$ 4D



$2 \rightleftarrows 4$ **4D**



$1 \rightleftharpoons 5$ 4D



4D state-sums

?

Crane-Frenkel 1994
f.d. involutory Hopf
Categories

Crane-Yetter 1993
Art. s.s. tortile categories

15-j symbols

Yetter 1993
homotopy 2-types

(Porter
homotopy n-types)

Crane-Frenkel 1994

$$\begin{tikzcd}
\text{s.s. 2-cat.} \arrow[r, dash] \arrow[d, dash] & \text{f.d. Hopf cat.} \arrow[r, dash] \arrow[d, dash] & \text{tri-alg} \arrow[dl, dash] \\
\text{s.s. cat.} \arrow[r, dash] \arrow[d, dash] & \text{f.d. Hopf alg.} \arrow[dl, dash] & \\
\text{s.s. alg.} & &
\end{tikzcd}$$

Monoidal 2-categories (semi-strict!)

$$A \otimes B \qquad \forall A, B \in Obj(C)$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$A \otimes I = I \otimes A = A$$

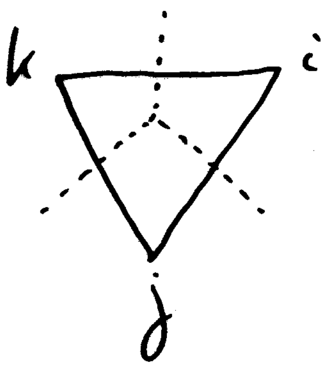
$$A \otimes f \qquad \forall A \in Obj(C), \forall f \in Hom(B, B')$$

$$f \otimes A$$

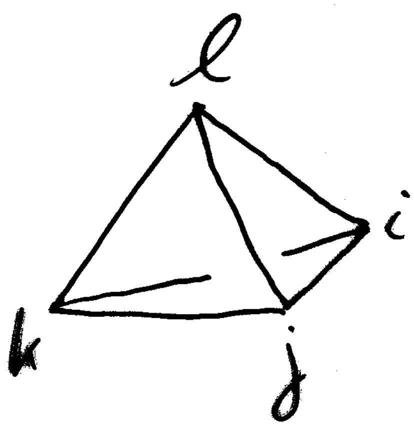
$$\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes f"] \arrow[d, "g \otimes B"'] \arrow[dr, phantom, "\Downarrow g \otimes f"] & A \otimes B' \arrow[d, "g \otimes B'"] \\
A' \otimes B \arrow[r, "A' \otimes f"] & A' \otimes B'
\end{tikzcd}$$

Crane, Yetter, Porter, Birmingham-Rakowski,
Freed-Quinn, Baez, M.M.

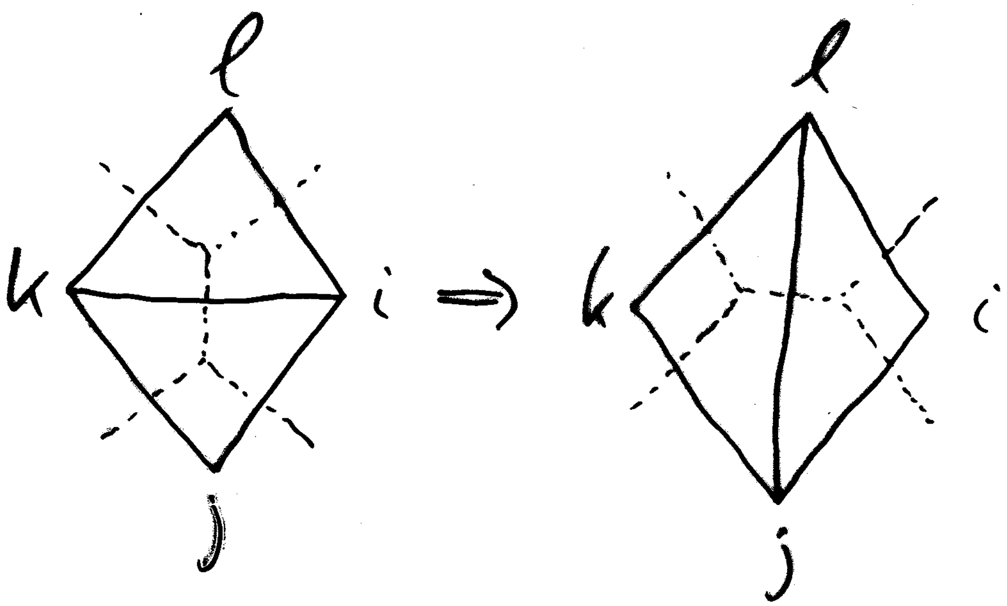
$\longleftrightarrow \quad e_{ij} \in \mathrm{Obj}(C)$

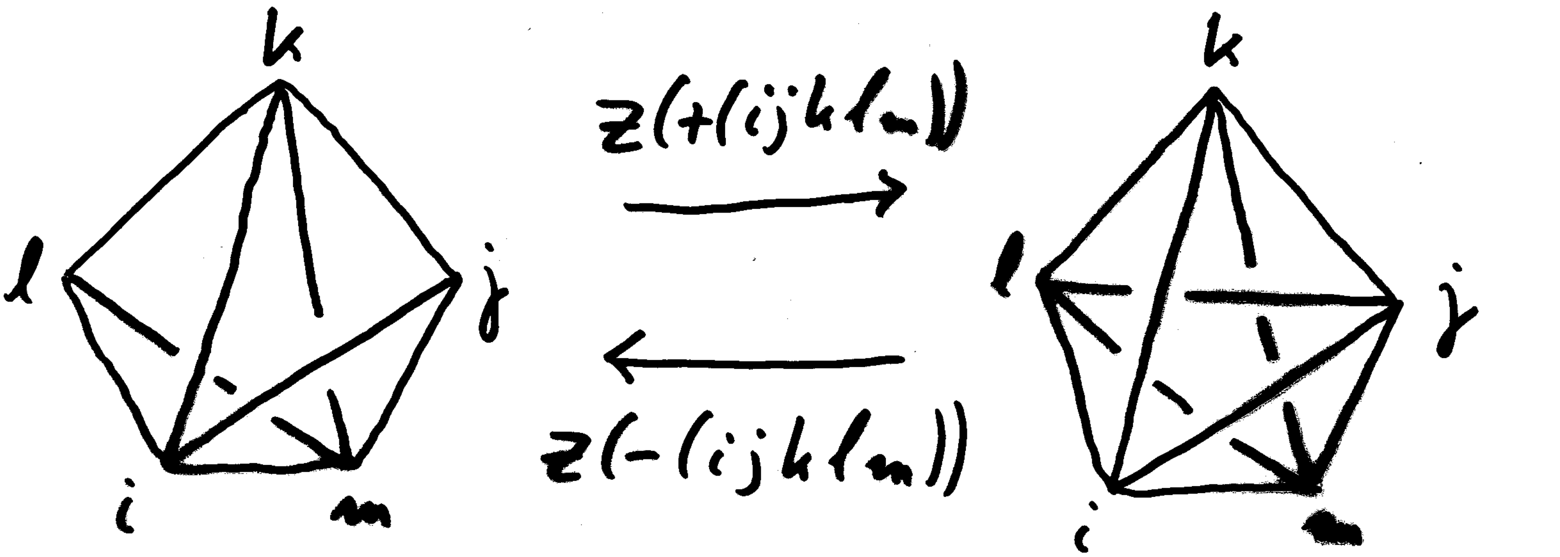


$\longleftrightarrow \quad f_{ijk} \in \mathrm{Hom}(e_{ik}, e_{jk} \otimes e_{ij})$
$= H(ijk)$



$\longleftrightarrow \quad 2\mathrm{Hom}\big(f_{ikl}(e_{kl} \otimes f_{ijk}),$
$f_{ijl}(f_{jkl} \otimes e_{ij})\big)$
$= 2H(ijkl)$





$$2H(iklm) \otimes 2H(ijkm) \underset{z^-}{\overset{z^+}{\rightleftarrows}} 2H(jklm) \otimes 2H(ijlm) \otimes 2H(ijkl)$$

4D State-sum

$$I(M,T) = k^{-v} \sum_{l} Z(M,T,l) \prod_{e} \dim_q([illegible])$$

Th (M.M. 1990)

$I(M,T)$ does not depend on chosen ordered triangulation T.

- $\mathrm{Hom}(A,B)$ is an [redacted] finite dim. Vect-module

- $\oplus$ and $\otimes$ of Vect-modules exists

- C finitely semi-simple if $\exists$ finite set of equivalence classes of objects $\mathcal{J}$ s.t. $\forall A, B \in \mathrm{Obj}(C)$:

$$\bigoplus_{X \in \mathcal{J}} \mathrm{Hom}(A,X) \otimes \mathrm{Hom}(X,B) \simeq \mathrm{Hom}(A,B)$$

- A is simple $\Longleftrightarrow$ $A \in \mathcal{J}$ $\Longleftrightarrow$ $\mathrm{End}(A) \simeq \mathrm{Vect}$

Duality

(Carter-Rieger-Saito, Fisher, Kharlamov-Turaev, Baez-Langford)

1.) $X \rightsquigarrow X^*$ $\quad \begin{cases} i_X : I \longrightarrow X \otimes X^* \\ e_X : X^* \otimes X \longrightarrow I \end{cases}$

$$(i_X \otimes X)(X \otimes e_X) \xRightarrow{T_X} 1_X \quad \text{(triangulator)}$$

2.) $\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \rightsquigarrow \begin{array}{c} Y \\ f^*\downarrow \\ X \end{array}$ $\quad \begin{cases} i_f : 1_X \longrightarrow f f^* \\ e_f : f^* f \longrightarrow 1_Y \end{cases}$

3.) $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow\alpha^*"] & Y \end{tikzcd}$

+ 3 pages of conditions

C spherical if $Tr_L(f) \cong Tr_R(f)$

Pairing

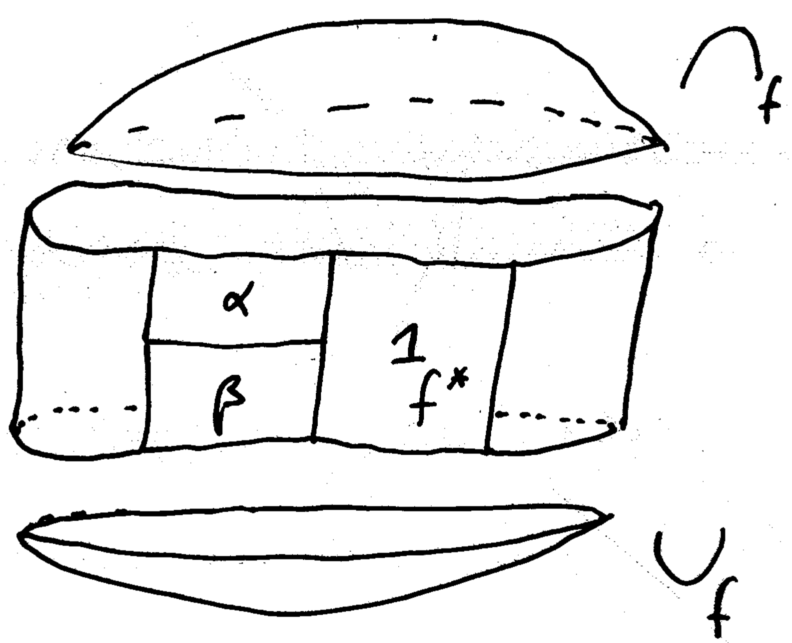
$$Tr_R(ff^*) = \left(i_A(f \otimes A^*)\right)\left(i_A(f \otimes A^*)\right)^*$$

$$\cap^R_f : 1_I \longrightarrow Tr_R(ff^*)$$

$$\cup^R_f : Tr_R(ff^*) \longrightarrow 1_I$$

$$\langle\, , \rangle : 2Hom(f,g) \otimes 2Hom(g,f) \longrightarrow \mathbb{C}$$

$$\langle \alpha, \beta \rangle = \cap^{R,L}_f \, Tr_{R,L}\left((\alpha\beta) \circ 1_{f^*}\right) \cup^{R,L}_f$$



Assume non-degenerate

$$\partial(01234) = (1234) - (0234) + (0134) - (0124) + (0123)$$

$$_2H(1234)^* \otimes {_2H}(0234) \otimes {_2H}(0134)^* \otimes {_2H}(0124) \otimes {_2H}(0123)^*$$

$$\longrightarrow \mathbb{C}$$

$$\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \longmapsto T_2(\text{``}\alpha\beta\gamma\delta\varepsilon\text{''})$$

non-degeneracy $\rightsquigarrow$

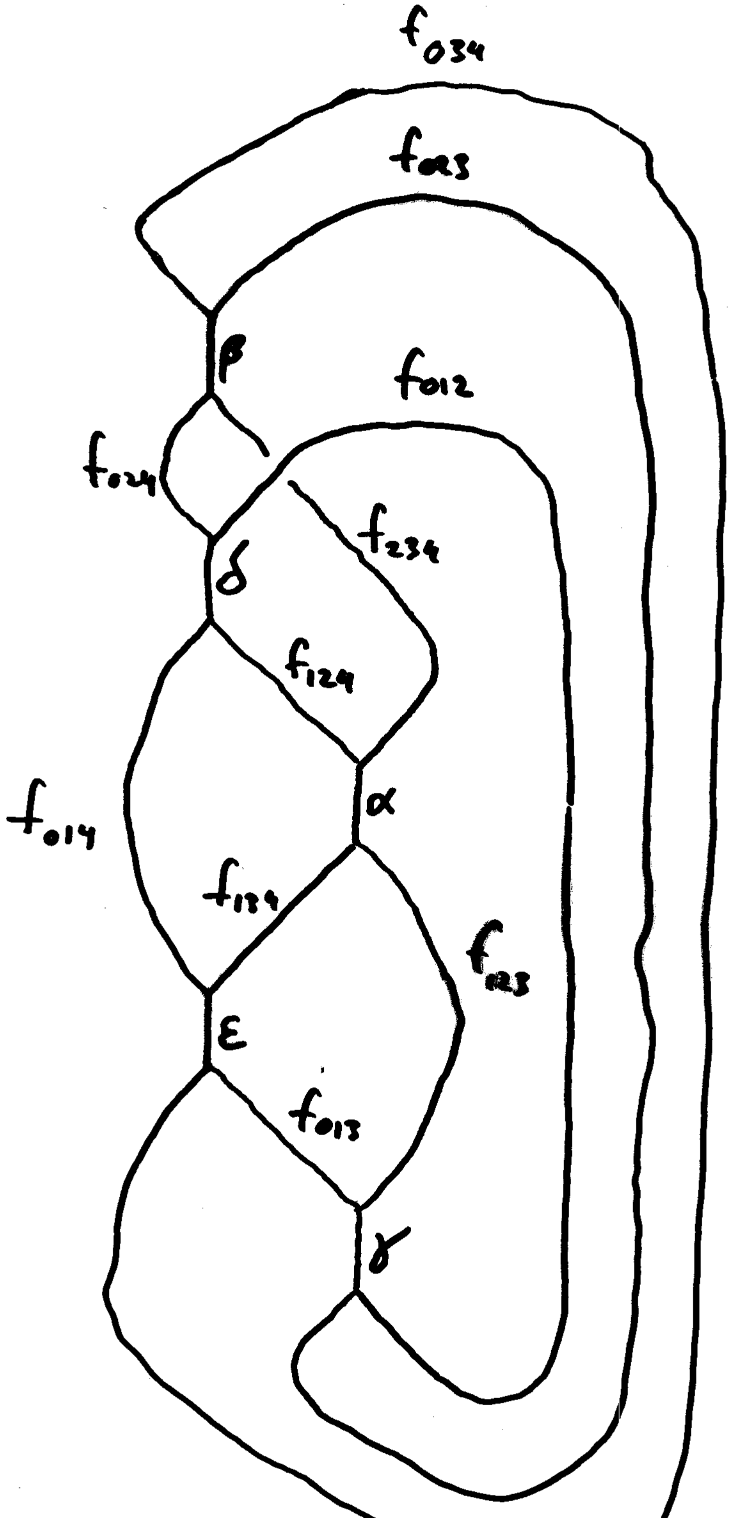
$$_2H(0234) \otimes {_2H}(0124) \xrightarrow{Z(+(01234))} {_2H}(1234) \otimes {_2H}(0134) \otimes {_2H}(0123)$$

$$\xleftarrow[Z(-(01234))]{}$$

$$V(M,l) = \bigotimes_{\Delta \text{ in } T} {_2H}(\Delta)$$

$$V(M,l) \xrightarrow{\otimes Z(\pm\Delta)} V(M,l)^{\pi} \xrightarrow{P} V(M,l)$$

<u>Def</u>: $Z(M,T,l) = T_2(\otimes Z(\pm\Delta)P)$


f_{034}
f_{023}
β
f_{012}
f_{024}
f_{234}
δ
f_{124}
α
f_{014}
f_{134}
f_{123}
ε
f_{013}
γ

$Z(+(01234))$

Example (Barrett, M.M.)

$\mathcal{G}$ cat. group $\longleftrightarrow$ $E \xrightarrow{\partial} G$, $\rho : G \to \mathrm{Aut}(E)$

$$\rho(\partial e_1) e_2 = e_1 e_2 e_1^{-1}$$

$$\partial(\rho(g) e) = g \partial e g^{-1}$$

- - - - - - - - - - -

$$\mathcal{G} \longmapsto \bigcup_X \mathcal{G}(1, X) \xrightarrow{\partial = t} \mathcal{G}_0$$

$$\rho(x) f = x \otimes f \otimes x^{-1}$$

- - - - - - - - - -

$\mathcal{G}_0 = G \quad \longleftarrow\!\!\dashv \quad E \xrightarrow{\partial} G$, $\rho : G \to \mathrm{Aut}(E)$

$$\mathrm{Hom}(x, y) \equiv \partial^{-1}(x^{-1} y)$$

$$f \circ g \equiv fg$$

$$f \otimes g \equiv (\rho(s(g))^{-1} f) g$$

$\mathcal{F}(\mathcal{G})$ = category of functors
$\mathcal{G} \to$ Vect + natural ~~transf~~

Lemma. $\mathcal{F}(\mathcal{G})$ = cat. of G-graded vector spaces w/ compatible right E-action: $|v \triangleleft e| = |v| \partial(e)$. Morphisms are grading preserving intertwiners

Example $V = \mathbb{C}[E]$, $|e| = \partial(e)$, $e_1 \triangleleft e_2 = e_1 e_2$

$V_{x,\phi} = \mathbb{C}_x \otimes_{\mathbb{C}[\ker \partial]} \mathbb{C}[E]$, $\ker\partial$ acts on $\mathbb{C}_x$ by character ϕ. $|[1_x \otimes e]| = x\,\partial(e)$

$$[1_x \otimes e_1] \triangleleft e_2 = [1_x \otimes e_1 e_2]$$

$x \in G$

Note: $V_{x,\phi}$ are irreducible.
$\mathcal{F}(\mathcal{G})$ is a Hopf category

$\mathcal{C} = F(\mathcal{G}) - \text{mod}$

$\mathcal{C}_0 \equiv G$

$\text{Hom}(x, y) \equiv$ cat. of $\partial^{-1}(x^{-1}y)$-graded vector spaces

Composition of 1-morphisms:

$$V_f \circ W_g \equiv (V \otimes W)_{fg}$$

Monoidal product of 1-morphisms

$$V_f \otimes W_g \equiv (V \otimes W)_{(\rho(w^{-1})f)g}$$

if $g \in \partial^{-1}(w^{-1}z)$.

$$\text{Tr}_L(V_f) = V_{\rho(a)f} \qquad f \in \partial^{-1}(a^{-1}a) = \partial^{-1}(1)$$

$$\text{Tr}_R(V_f) = V_f$$

$\mathcal{C}$ spherical $\Longleftrightarrow$ G action trivial on $\ker \partial$