GOAL

M closed, oriented, n-mfd

T triangulation of M

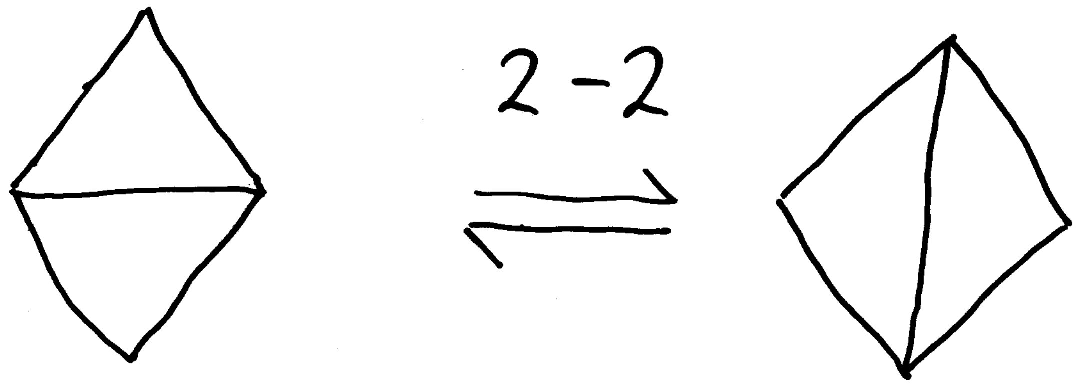
Want:

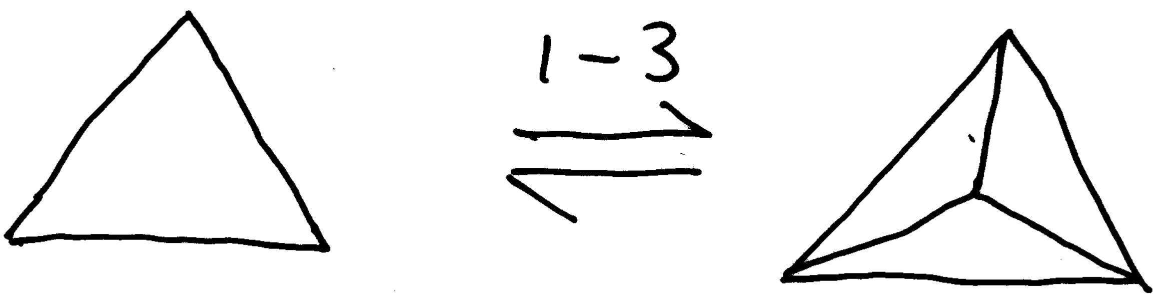
$$(M,T) \longrightarrow Z(M,T) \in \mathbb{C}$$

$$T_1 \sim T_2 \implies Z(M,T_1) = Z(M,T_2)$$

"$\sim$" Pachner moves

2d Pachner moves



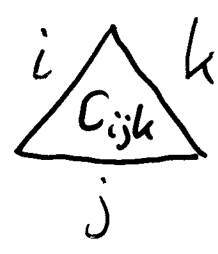


State-sums

labels $\{1, 2, 3, \ldots, n\}$

weights $C_{ijk} \in \mathbb{C}$

$$Z(M,T) = \sum_{lab.} \prod_{\Delta \in T}$$



A f.d. associative algebra

$L = \{ \phi_i \mid i = 1, \dots, n \}$ basis

$$\phi_i \phi_j = C_{ij}^{k} \phi_k$$

Associativity: $(\phi_i \phi_j) \phi_k = \phi_i (\phi_j \phi_k)$

$$\Longleftrightarrow \quad C_{im}^{\ell} C_{jk}^{m} = C_{ij}^{m} C_{mk}^{\ell} \quad \forall \ell$$

$$g_{ij} = \mathrm{tr}\,(R(\phi_i) R(\phi_j)) = C_{i\ell}^{k} C_{jk}^{\ell}$$

A semisimple if $(g_{ij})_{i,j=1}^{n}$ invertible

$$g^{im} g_{mj} = \delta_j^i \qquad C_i^{jk} = g^{jm} C_{mi}^{k}$$

Choose total ordering on vertices of T.

$$\partial(+(012)) = (12) - (02) + (01)$$
$$\partial(-(012)) = -(12) + (02) - (01)$$

$$L \ni e_{ij} = \text{label of } (ij)$$

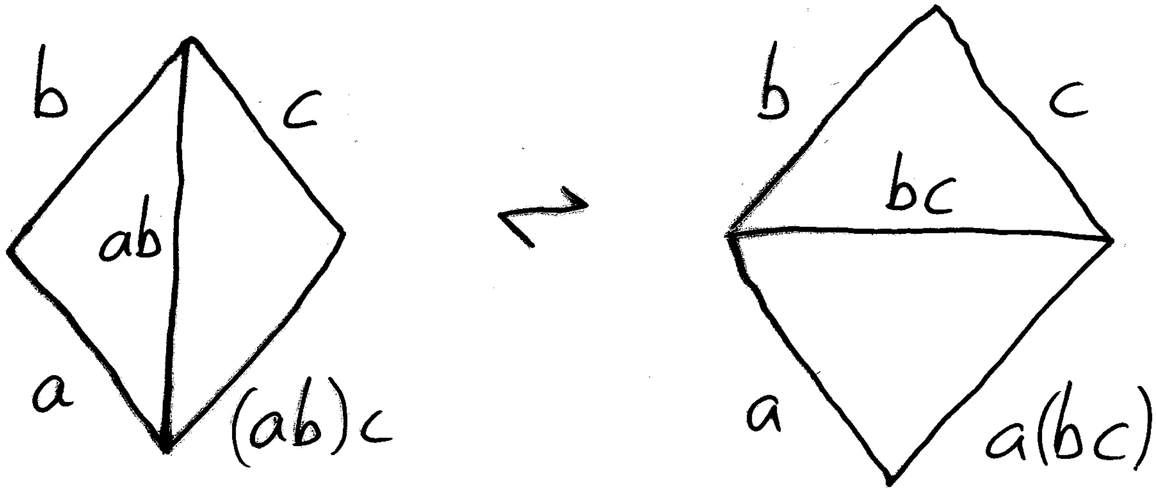
$$Z(+012) = C^{e_{01}\, e_{12}}_{e_{02}}$$

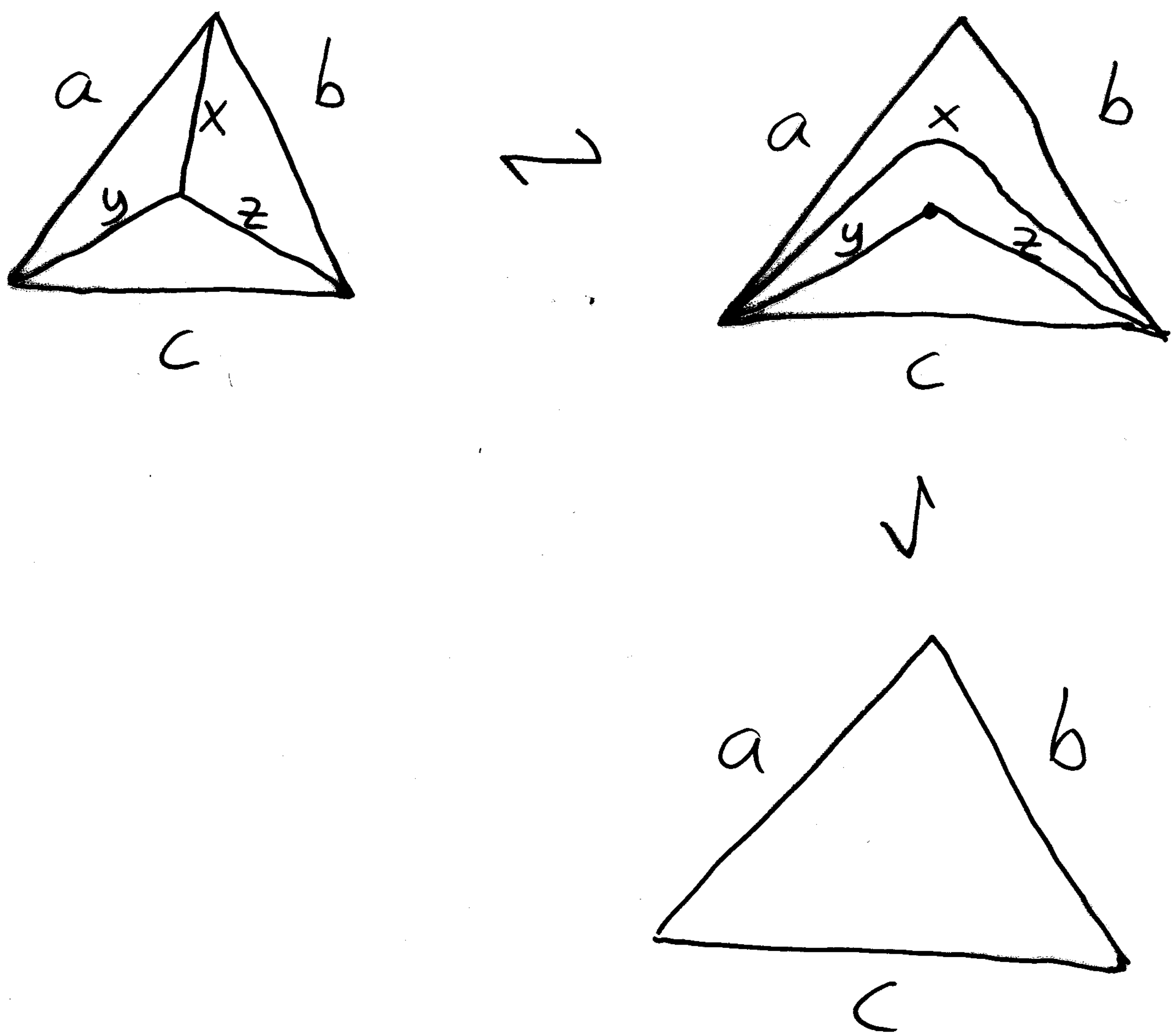
$$Z(-012) = C^{e_{02}}_{e_{01}\, e_{12}}$$

$$Z(M,T) = \sum_{l \in L} \prod_{\Delta} Z(\Delta)$$

(Fukuma - Hosono - Kawai '92)

Invariance





Example

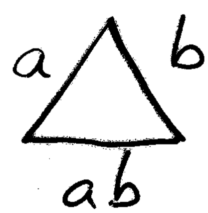
$$A = \mathbb{C}[G] = \bigoplus_{x \in G} \mathbb{C}\,\phi_x \qquad G \text{ finite group}$$

$$\phi_x \phi_y = \phi_{xy} \implies C_{xy}^{\;z} = \delta_{xy}^{\;z}$$

$$g_{xy} = |G|\,\delta_{xy}^{\;1} \qquad g^{xy} = |G|^{-1}\,\delta_1^{\;xy}$$

$$C_x^{\;yz} = |G|^{-1}\,\delta_x^{\;yz}$$

ℓ non-zero :



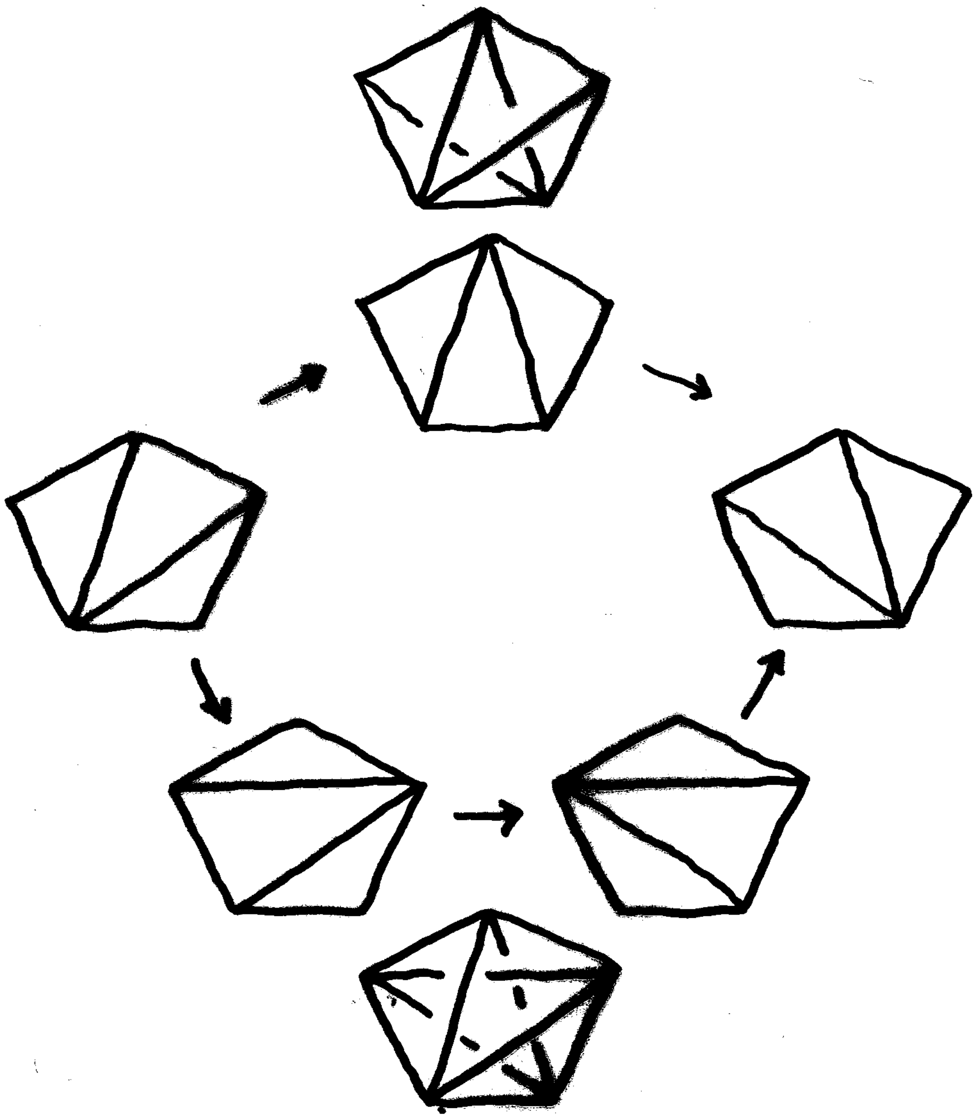
$$\Downarrow$$

$$\ell \in \mathrm{Hom}(\pi_1 \Sigma, G)$$

$$Z(\Sigma, T) = \frac{|G|^{v-1}\,\#\,\mathrm{Hom}(\pi_1 \Sigma, G)}{|G|^{\frac{1}{2}f}}$$

$$= |G|^{\chi(\Sigma)-1}\,\#\,\mathrm{Hom}(\pi_1 \Sigma, G)$$

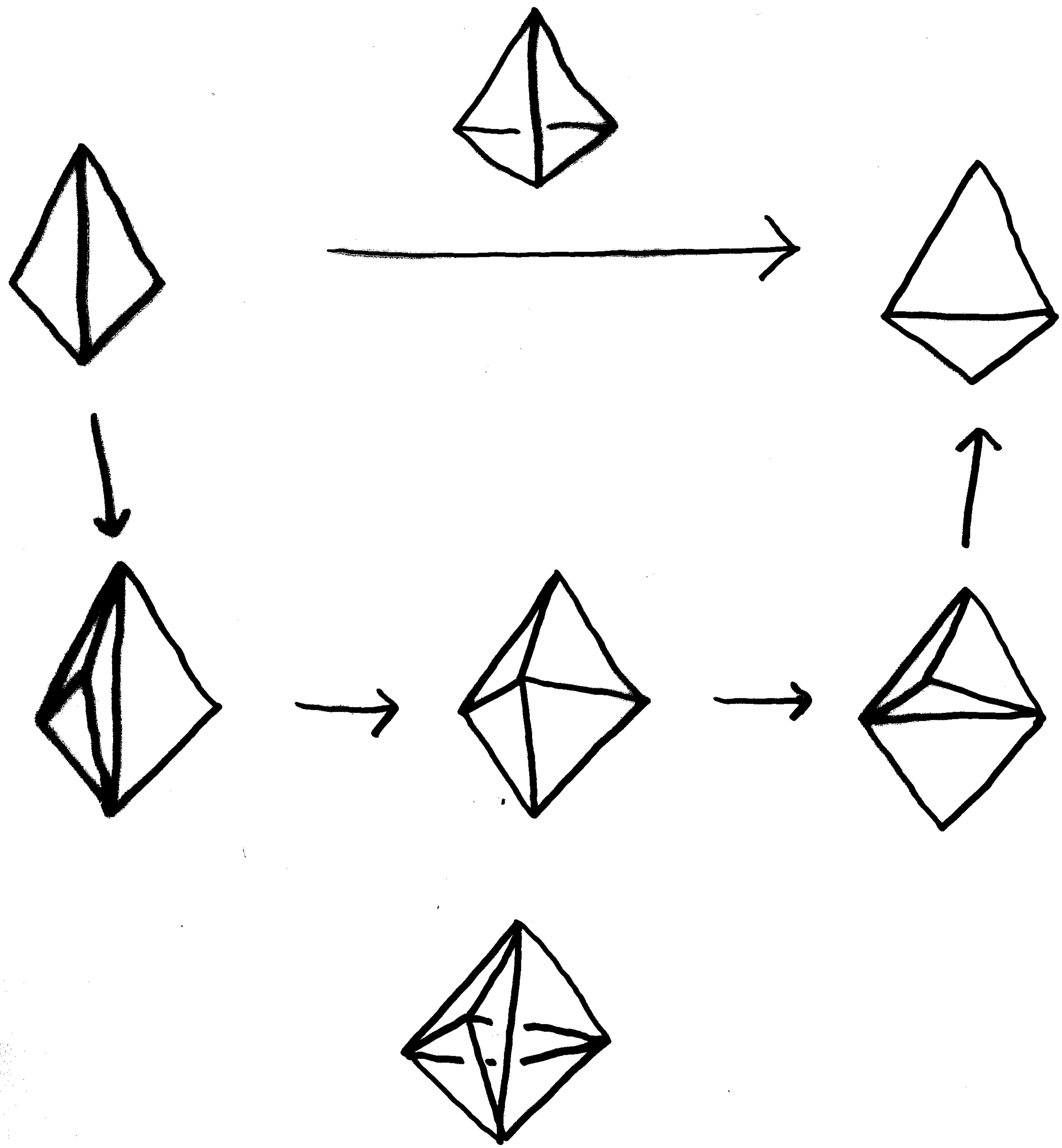
$2 \rightleftarrows 3$



2D —— 3D

Carter - Kauffman - Saito

$1 \rightleftharpoons 4$



2D — 3D

3D state-sums

Barrett-Westbury, 1996
s.s. non-deg. spherical cat. of dim $\neq 0$

Kuperberg 1991
Involutory f.d. Hopf
algebras

[illegible]-Fukuma-Shapere
1994

"

Turaev-Viro 1992
q-$6j$ symbols,
s.s. subquotient of
$U_q(sl_2)$-mod, $q^{4r}=1$

Yetter 1994
Artinian s.s.
tortile categories

monoidal - categories. (strict!)

$$A \otimes B \qquad \text{(tensor product)}$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$A \otimes I = I \otimes A = A$$

<u>semi-simplicity</u> $\qquad \mathcal{J} \subseteq \mathrm{Obj}(\mathcal{C})$.

$\forall A, B \in \mathrm{Obj}(\mathcal{C})$:

$$\bigoplus_{X \in \mathcal{J}} \mathrm{Hom}(A, X) \otimes \mathrm{Hom}(X, B) \cong \mathrm{Hom}(A, B)$$

Assume $|\mathcal{J}| < \infty$

<u>simple object</u>

A is simple $\iff A \in \mathcal{J} \iff \mathrm{End}(A) = k$

(k [illegible] field)

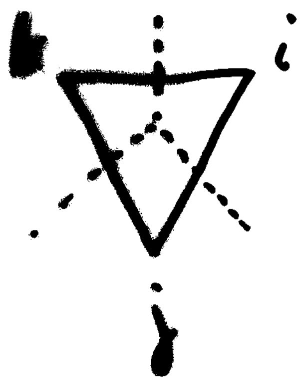
$$V(M,T,\ell) = \bigotimes_{\Delta} H\left({}^{i}\Delta^{j}_{k} \right)$$

$$\begin{tikzcd}
V(M,T,\ell) \arrow[r, "\otimes Z(\pm \Delta)"] \arrow[dr, "\tilde{Z}(M,T,\ell)"'] & V(M,T,\ell)^{\pi} \arrow[d, "\pi^{-1}"] \\
& V(M,T,\ell)
\end{tikzcd}$$

$$Z(M,T,\ell) = \mathrm{tr}\left(\tilde{Z}(M,T,\ell)\right)$$

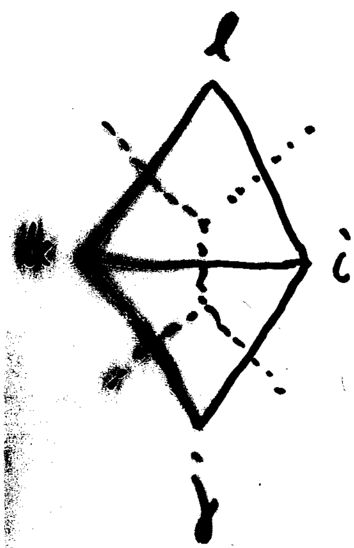
Barrett - Westbury 1996

$\longleftrightarrow \; e_{ij} \in Obj(C)$

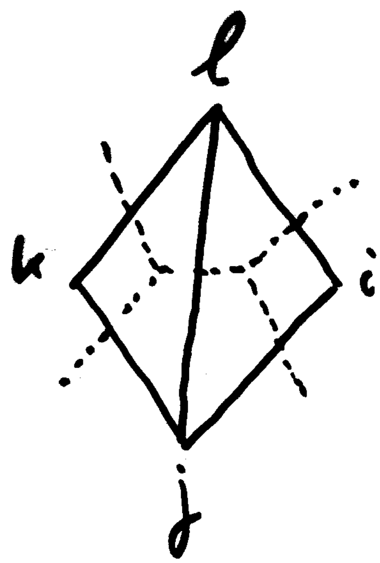


$\longleftrightarrow \; \mathrm{Hom}(e_{ik}, e_{jk} \otimes e_{ij}) =$

$H(ijk)$



$\underset{Z(-(ijkl))}{\overset{Z(+(ijkl))}{\rightleftarrows}}$



$$\mathrm{Hom}(e_{il}, e_{kl} \otimes e_{ik}) \otimes \mathrm{Hom}(e_{ik}, e_{jk} \otimes e_{ij}) \underset{Z^-}{\overset{Z^+}{\rightleftarrows}}$$

$$\mathrm{Hom}(e_{jl}, e_{kl} \otimes e_{jk}) \otimes \mathrm{Hom}(e_{il}, e_{jl} \otimes e_{ij})$$

3D state-sum

$$I(M,T) = K^{-v} \sum_{\ell} Z(M,T,\ell) \prod_{e} \dim_q(\ell(e))$$

v = # vertices

$\dim_q$ = quantum dimension

Thm (BW 1996):

$I(M,T)$ does not depend on the chosen ordered triangulation T.

Duality

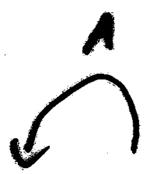
Each object A has a dual A^*

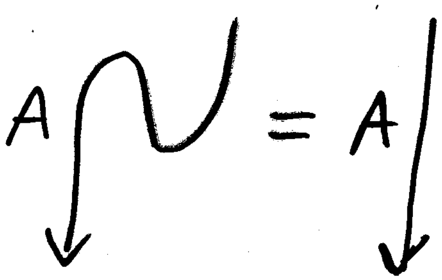
and $i_A : I \to A \otimes A^*$ $\quad e_A : A^* \otimes A \to I$.

$$(i_A \otimes A)(A \otimes e_A) = 1_A \qquad (1)$$

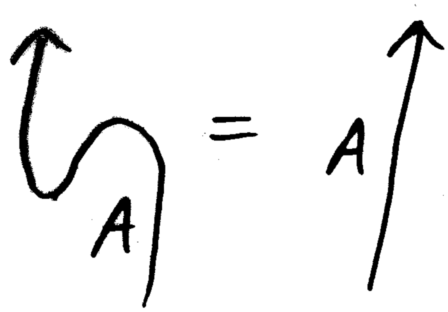
$$(A^* \otimes i_A)(e_A \otimes A^*) = 1_{A^*} \qquad (2)$$

Graphical calculus

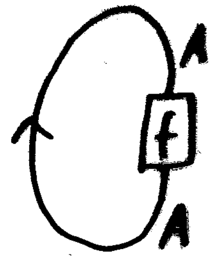
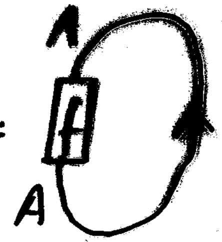
$A \downarrow = 1_A$ $\quad A \uparrow = 1_{A^*}$ $\quad$  $= i_A$ $\quad$  $= e_A$



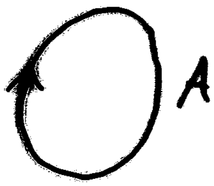
(1)



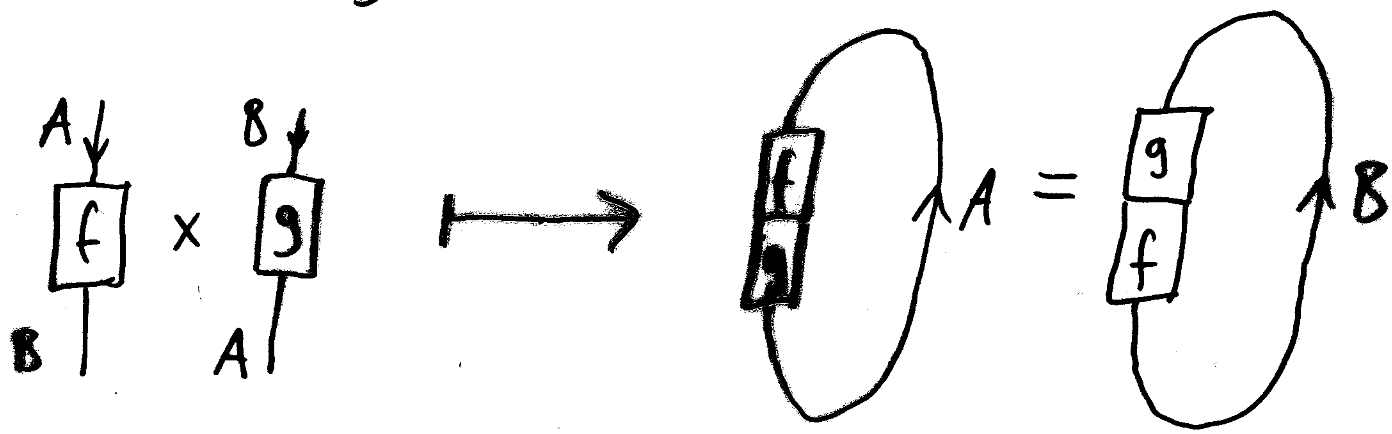
(2)

$\mathrm{Tr}_L(f) =$  $\mathrm{Tr}_R(f) =$ 

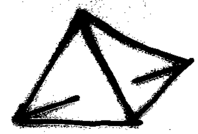
C spherical if $\mathrm{Tr}_L = \mathrm{Tr}_R$

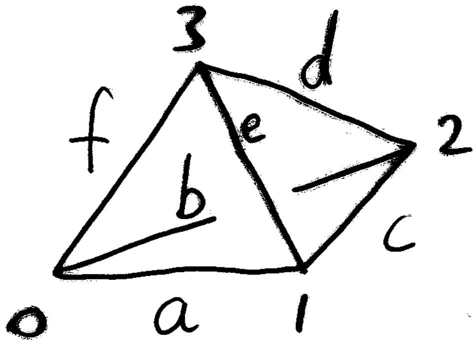
$\dim_q(A) =$  $=$ 

$$K_C = \sum_{A \in J} \dim_q^2(A) \neq 0 \quad \text{(assumption)}$$



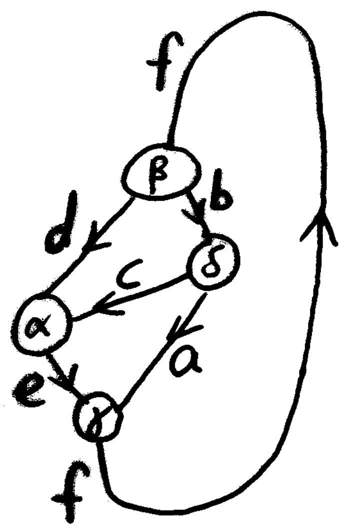
Assume this pairing to be non-degenerate

$Z(\pm$  $)$



$$\mathrm{Hom}(d\otimes c, e)\otimes \mathrm{Hom}(f, d\otimes b)\otimes \mathrm{Hom}(e\otimes a, f)\otimes$$
$$\mathrm{Hom}(b, c\otimes a) \xrightarrow{Z^+} \mathbb{C}$$

$\alpha\otimes\beta\otimes\gamma\otimes\delta \mapsto$



Non-deg. $\Rightarrow \mathrm{Hom}(d\otimes c, e)^* \cong \mathrm{Hom}(e, d\otimes c)$

$Z($+  $): H(f, d\otimes b)\otimes H(b, c\otimes a) \longrightarrow$

$$H(e, d\otimes c)\otimes H(f, e\otimes a)$$

Example

$$F(G) = \{ f : G \longrightarrow \mathbb{C} \}$$

$F(G)$ is a Hopf algebra

$$\delta_x : G \longrightarrow \mathbb{C} \qquad \delta_x(y) = \delta_{x,y}$$

multiplication $\quad \delta_x \delta_y = \delta_{x,y} \delta_x$

unit $\quad 1 = \sum_x \delta_x$

comultiplication $\quad \Delta(\delta_x) = \sum_{yz=x} \delta_y \otimes \delta_z$

counit $\quad \varepsilon(\delta_x) = \delta_{x,1}$

antipode $\quad S(\delta_x) = \delta_{x^{-1}}$

Note: $F(G)$ commutative but NOT cocommutative

$F(G)\text{-mod} \cong G\text{-graded vector}$ [redacted]

$$\mathbb{C}_g \in F(G)\text{-mod}: \quad \delta_x \triangleright 1 = \delta_{x,g}$$

$$\mathrm{Hom}(\mathbb{C}_g, \mathbb{C}_h) \cong \delta_{g,h}\,\mathbb{C} \implies$$

$F(G)\text{-mod}$ semisimple, $\mathbb{C}_g$ simple objects

$$\delta_x \triangleright (\mathbb{C}_y \otimes \mathbb{C}_z) = \sum_{uv=x} (\delta_u \triangleright \mathbb{C}_y) \otimes (\delta_v \triangleright \mathbb{C}_z)$$

$$\implies \mathbb{C}_y \otimes \mathbb{C}_z = \mathbb{C}_{yz}, \quad I = \mathbb{C}_1$$

$$\mathbb{C}_g^* = \mathbb{C}_{g^{-1}}, \quad i_g \cong 1_{\mathbb{C}_1} \cong e_g$$

$$\phi \in \mathrm{Hom}(\mathbb{C}_g, \mathbb{C}_g) \cong \mathbb{C} : \mathrm{Tr}_L(\phi) = \mathrm{Tr}_R(\phi) = \phi$$

$$\dim_q(\mathbb{C}_g) = 1$$

$$\mathbb{K}_{F(G)\text{-mod}} = |G| \neq 0$$