Model structures on Cat, 2-Cat, Bicat

(IMA 17/6/04)

<u>model structure on a category</u>: consists of three classes of maps called weak equivalences, cofibrations, and fibrations; obtain the "homotopy category" by inverting the weak equivalences

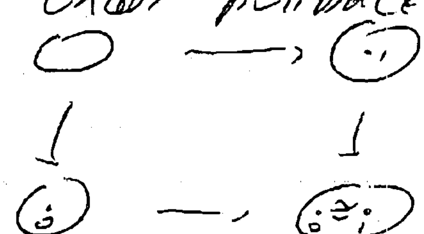
<u>Cat</u> (has many model structures; today talk about one)

- weak equivalences are the equivalences
- for a general pullback square

$$\begin{array}{ccc} A & \xrightarrow{v} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[w]{} & D \end{array}$$

of categories, with w an equivalence, no reason why v should be an equivalence ("equivalences are not stable under pullback")

e.g.



- but equivalences are stable under pullback along functors p with the "isomorphism lifting property".

$$\begin{array}{ccc} \exists b' \xrightarrow[\cong]{f} b & & B \\ & & \downarrow{\scriptstyle p} \\ d \xrightarrow[\delta]{\cong} pb & & D \end{array}$$

Such functors are the fibrations

- similarly equivalences not stable under pushout but are stable under pushout along functors with the "isomorphism extension property". Joyal & Street showed that a functor has the latter property iff it is injective on objects. These functors are the cofibrations

- remarks: (i) all objects fibrant & cofibrant
  (ii) homotopy category is the category of categories & isomorphism classes of functors
  (iii) trivial fibrations are the surjective (= retract = easy) equivalences
  (iv) model structure is proper (w.e. stable under pullback along fibration, pushout along cofibration)

<u>2-Cat / Bicat</u> (treat both cases together)

- in each case use only the strict morphisms (they're better behaved and the weak morphisms will show up later anyway)

- w.e. are biequivalences: strict map $\mathcal{A} \xrightarrow{F} \mathcal{B}$ hom categories
  - each functor $\mathcal{A}(A,B) \to \mathcal{B}(FA,FB)$ an equivalence
  - each $B \in \mathcal{B}$ have $A \in \mathcal{A}$ and equivalence $FA \simeq B$

- fibrations
  - each $\mathcal{A}(A,B) \to \mathcal{B}(FA,FB)$ a fibration in Cat
  - "equivalence lifting property": given $A \in \mathcal{A}$ and equivalence $B \simeq FA$ in $\mathcal{B}$ can lift to equivalence $A' \simeq A$ in $\mathcal{A}$ over it (here can take equivalence to mean either (a) arrow $B \to FA$ which is an equivalence, or (b) all data for an adjoint equivalence

- cofibrations defined by left lifting property

<u>remarks</u>: (i) trivial fibrations are the easy/surjective equivalences (i.e. surjective on objects and a surjective equivalence (in <u>Cat</u>) on the homs)
(ii) all objects fibrant, not all cofibrant

can't be sure the lifting ~~will be strict~~

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "p"', "\text{t.f.}"] \\ \mathcal{C} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

in fact 2-category $\mathcal{C}$ is cofibrant iff its underlying category is free on a graph

<u>weak morphisms</u> (for simplicity consider only <u>2-Cat</u>)

- for $\mathcal{A}$ 2-category there's a 2-category $\mathcal{A}'$ with same objects as $\mathcal{A}$, morphisms paths in $\mathcal{A}$ (i.e. underlying category of $\mathcal{A}'$ is free on underlying graph of $\mathcal{A}$) and 2-cells of $\mathcal{A}'$ defined so that the canonical $\mathcal{A}' \xrightarrow{q} \mathcal{A}$ is locally fully faithful (fully faithful on 2-cells)
- this $\mathcal{A}' \xrightarrow{q} \mathcal{A}$ is a very special cofibrant replacement for $\mathcal{A}$:

$\mathcal{A}' \longrightarrow \mathcal{B}$ 2-functors (strict)

bijection → $\mathcal{A} \longrightarrow \mathcal{B}$ pseudofunctors (weak)

(~~actually an isomorphism~~ of 2-cats since involves not just morphisms, but 2-cells & 3-cells)

So weak morphisms $\mathcal{A} \to \mathcal{B}$ <u>are</u> (can be identified with) strict morphisms $\mathcal{A}' \to \mathcal{B}$

- $\mathcal{A}$ is cofibrant iff $\mathcal{A}' \to \mathcal{A}$ has a section

"cofibrant = flexible" ← important notion in 2-category theory

Theorem: these definitions make 2-Cat and Bicat into model categories, so that the inclusion

$$\text{2-Cat} \longrightarrow \text{Bicat}$$

is the right adjoint part of a Quillen equivalence

key fact: this inclusion has a left adjoint
for a general bicategory $\mathcal{B}$ the reflection into 2-Cat is not a biequivalence but it is one if $\mathcal{B}$ is cofibrant.

remark: model structure on 2-Cat is proper (this is quite hard) haven't checked Bicat

Gray tensor product

- 2-Cat is cartesian closed, but the cartesian monoidal structure is not compatible with the model structure (not a "monoidal model category")

- 2-Cat is a monoidal model category with respect to the Gray tensor product $\otimes$. (Symmetric monoidal closed structure with internal hom $[\mathcal{A}, \mathcal{B}]$ consisting of 2-functors, pseudo-natural transformations, modifications)

- key aspect of this "compatibility": tensor product of cofibrant objects should be cofibrant.
$\mathbb{2} = (\cdot \to \cdot)$ is a cofibrant 2-category

$$\mathbb{2} \times \mathbb{2} = \begin{tikzcd} \cdot \arrow[r] & \cdot \\ \cdot \arrow[u] \arrow[r] \arrow[ur, phantom, "="] & \cdot \arrow[u] \end{tikzcd} \qquad \mathbb{2} \otimes \mathbb{2} = \begin{tikzcd} \cdot \arrow[r] & \cdot \\ \cdot \arrow[u] \arrow[r] \arrow[ur, phantom, "\cong"] & \cdot \arrow[u] \end{tikzcd}$$

not cofibrant $\qquad$ cofibrant