IMA 2004 Summer programme on *n-Categories: foundations and applications.*

Joachim Kock: *Topological Quantum Field Theories (TQFT)*

Summary and 1 slide

Summary

I started by explaining the general idea: a TQFT as a correspondence between space and space-time on one side and state spaces and time evolution operator on the other side — the picture is that of a cobordism on one side and a linear map on the other side. Then I showed a slide with Atiyah's axioms, and derived the usual definition: a TQFT is a symmetric monoidal functor from a cobordism category to the category of vector spaces. For this to make sense you need to arrange the cobordisms into a category, and this requires passing to a quotient, identifying diffeomorphic cobordisms. The fact that you can decompose a cylinder into a snake implies that all the vector spaces that arise in a TQFT come equipped with a non-degenerate pairing, and in particular they must be of finite dimension. A TQFT can be seen as a correspondence between shape and algebraic operations. In dimension 2 this is particularly clear: the cobordism category ***2Cob*** was described explicitly in terms of generators and relations, which in the image vector spaces correspond exactly to the structures and properties defining a commutative Frobenius algebra. I explained the abstract version of this result, namely that ***2Cob*** is the free symmetric monoidal category on a commutative Frobenius object. Then I made brief mention of the problems encountered in higher-categorical approaches: either you get manifolds with corners, or you might have to use double or multiple categories instead of n-categories. To finish, taking the clue from axiom (iv) on the slide, I briefly explained how to get a non-algebraic definition of ***nCob*** in terms of Tamsamani categories; this involves the so-called *fat delta* to account for weak units, which are the cylinder cobordisms.

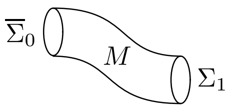
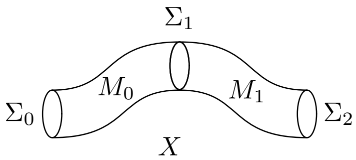
Reference

JOACHIM KOCK. *Frobenius algebras and 2D topological quantum field theories.* No. 59 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2003.

ATIYAH'S AXIOMS FOR TQFT

(a)	closed oriented $(n-1)$-mfld Σ	$\mapsto$	vector space $\mathscr{A}(\Sigma)$
(b)	n-mfld M with boundary $\partial M = \Sigma$	$\mapsto$	vector $\mathscr{A}(M) \in \mathscr{A}(\Sigma)$

such that

(i)	diffeo $\Sigma \xrightarrow{\simeq} \Sigma'$ extending to $M \xrightarrow{\simeq} M'$	$\mapsto$	$\mathscr{A}(\Sigma) \xrightarrow{\simeq} \mathscr{A}(\Sigma')$, $\mathscr{A}(M) \mapsto \mathscr{A}(M')$
(ii)	opposite orientation $\overline{\Sigma}$	$\mapsto$	dual vector space $\mathscr{A}(\Sigma)^*$
(iii)	disjoint union $\coprod$	$\mapsto$	tensor product $\otimes$
	empty manifold $\emptyset$	$\mapsto$	ground field $\Bbbk$
Observation:	cobordism	$\mapsto$	linear map
		$\mapsto$	$\mathscr{A}(M) \in \mathscr{A}(\Sigma_0)^* \otimes \mathscr{A}(\Sigma_1) = \mathrm{Hom}(\mathscr{A}(\Sigma_0), \mathscr{A}(\Sigma_1))$
(iv)	decomposition of cobordisms	$\mapsto$	composition of linear maps
		$\mapsto$	$\mathscr{A}(\Sigma_0) \xrightarrow{\mathscr{A}(X)} \mathscr{A}(\Sigma_2)$, $\mathscr{A}(\Sigma_0) \xrightarrow{\mathscr{A}(M_0)} \mathscr{A}(\Sigma_1) \xrightarrow{\mathscr{A}(M_1)} \mathscr{A}(\Sigma_2)$
(v)	cylinders	$\mapsto$	idenitity maps
	$\Sigma \times I$	$\mapsto$	$\mathrm{id} : \mathscr{A}(\Sigma) \to \mathscr{A}(\Sigma)$