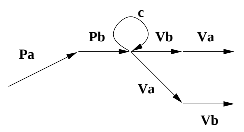
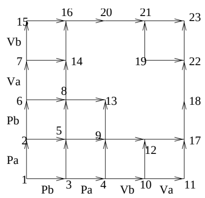
Some properties of interest

All processes are (discrete in general) dynamical systems = directed graphs of actions $\longrightarrow$ some kind of fibred product.

$$PbPaVbVa \mid PaPbVaVb:$$

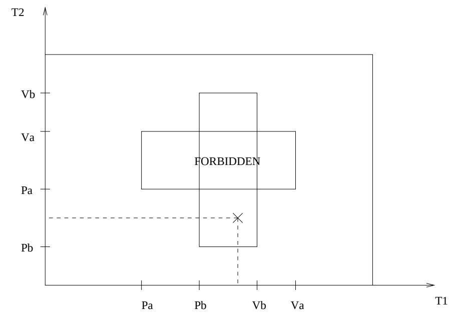




Geometry

First idea: "Progress graphs" E.W.Dijkstra'68 (later V.Pratt'91)

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



"Continuous image": x_i = local time; dashed rectangles=forbidden!

Directed topology and Concurrency: an introduction

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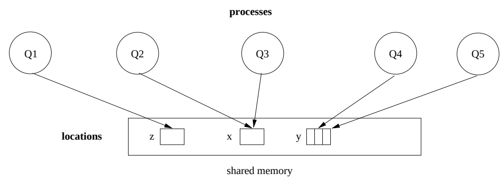
Concurrency and Geometry?

(here, shared memory style)



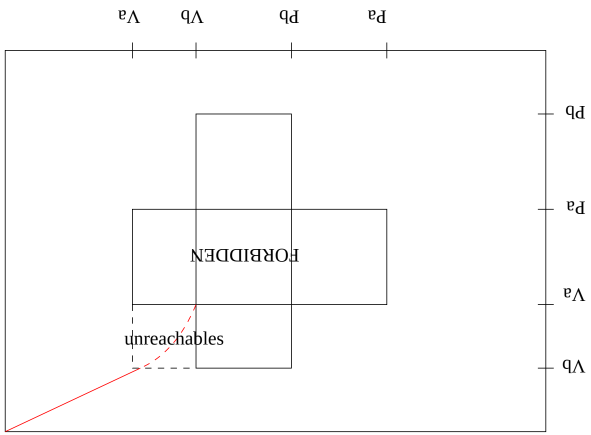
$\neq$ sequential; problem of coordination (need for locks P, V)

Is the coordination too constrained? $\longrightarrow$ deadlocks

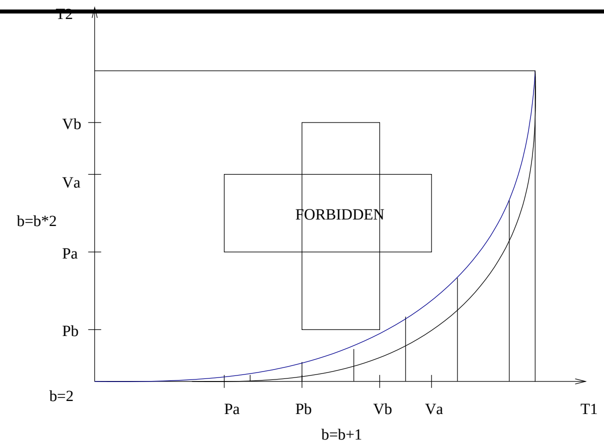
Do we have "bad" states? $\longrightarrow$ reachability

Are the sequences of accesses "right"? $\longrightarrow$ paths...

And dually...

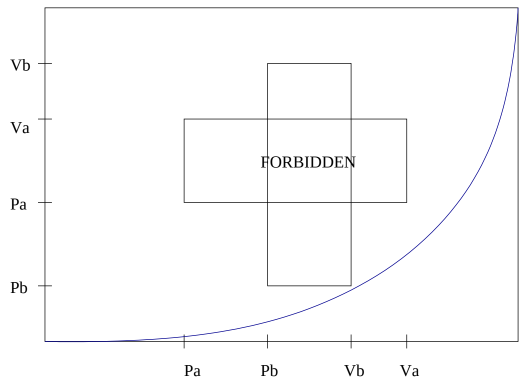


Deformation of execution paths



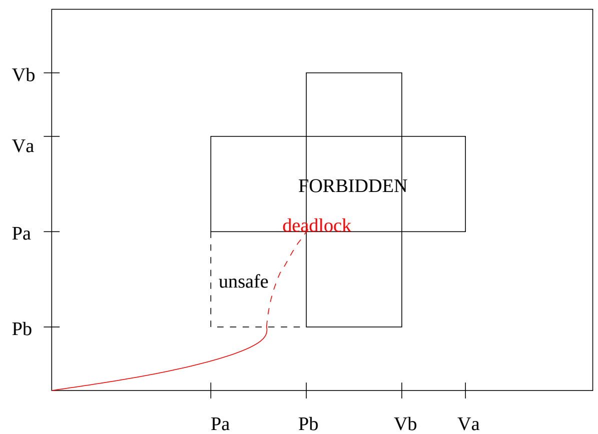
T1 gets a and b before T2 does: b=6

Execution paths

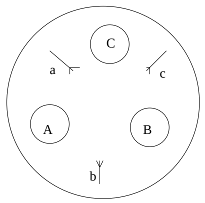


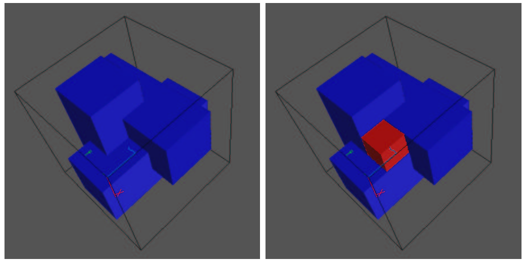
Traces = paths increasing in each coordinate = “di-paths”

First remark



In higher-dimension

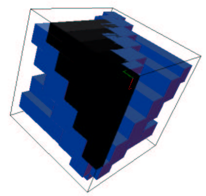




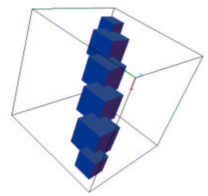
A=Pa.Pb.Va.Vb
B=Pb.Pc.Vb.Vc
C=Pc.Pa.Vc.Va

Also: effect of the level of sharing

```
A=Pa.Pb.Va.Pc.Vb.Pd.Vc.Pe.Vd.Pf.Ve.Vf
B=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va
C=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va
```

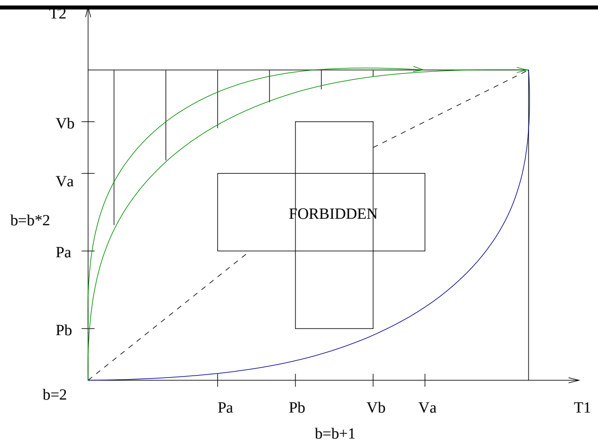


$a, \ldots$ binary sem.



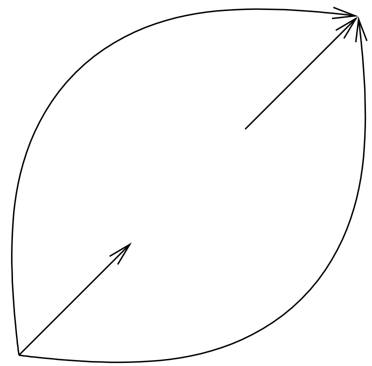
$a, \ldots$ counting sem.

Non-deformable paths



T2 gets a and b before T1 does: b=5!

Ideally...(*not quite true though*)



We will get back to this later...

Po-spaces

For "Progress Graphs", one only needs (*MFPS'98*):

A topological space with a (global) closed partial order

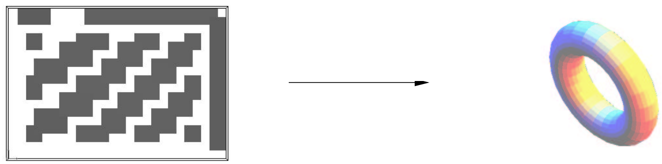
- Morphisms are increasing and continuous maps
- (Finite) Traces on $(X, <)$ are dimaps from $\vec{I} = ([0,1], \leq)$ to $(X, <)$ (dipaths).
- Dihomotopies between dipaths f and g with fixed extremities α and β are dimaps $H : \vec{I} \times I \rightarrow X$ such that for all $x \in \vec{I}, t \in I$,

$$H(x, 0) = f(x), \; H(x, 1) = g(x), \; H(0, t) = \alpha, \; H(1, t) = \beta$$

Another preferred definition - by M. Grandis, where $H : \vec{I} \times \vec{I} \rightarrow X$ (some equivalence when X is a finite precubical set - see Lisbeth's talk).

Loops?

```
A=PdPa(PbVaVdPcVbPaPdVcPbVaPcVbPaVcPb
                                VaPcVbPaVc)*VaPeVdVe
B=PePa(PbVaPcVbPaVcPbVaPcVbPaVc)*VaPdVeVd
```



Correspondences (almost...)

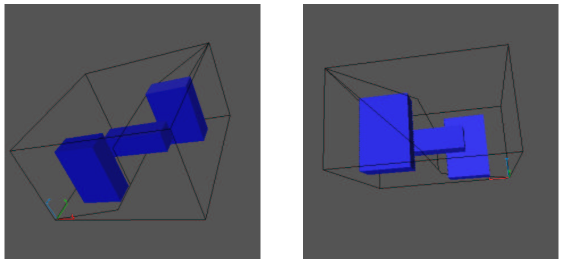
Model [discrete]	combinatorial complex
Model [continuous]	topological space
Relation discrete/continuous	geometric realisation
Parallel composition	product
Action refinement	subdivision
Compositionality	Seifert/van Kampen
Deadlocks/reachability	connected components
Scheduling properties	fundamental group
Observational equivalence	homotopy equivalence (weak/strong)
Computable properties	topological invariants (homology etc.)

Models

- Po-spaces, local po-spaces, (pre-)cubical sets (see MFPS'98, with L. Fajstrup and M. Raussen)
- globular CW-complexes: with P. Gaucher, "Topological Deformation of Higher-Dimensional Automata", *arXiv:math.AT/0107060, HHA 2003*
- Ω-categories, Category "Flow" (Philippe Gaucher)
- d-spaces (Marco Grandis)
- Higher-Dimensional Transition Systems (Vladimiro Sassone and Gian Luca Cattani, LICS'96)
- ECHIDNA (Richard Buckland and Michael Johnson, AMAST'96)
- etc.

Second subtlety...

```
#sem c 2
A=Pa.Pc.Va.Pb.Vc.Vb
B=Pa.Va.Pc.Vc.Pb.Vb
C=Pc.Vc
```



Directed homotopy is not classical homotopy plus fixed extremities:

A typical object of study

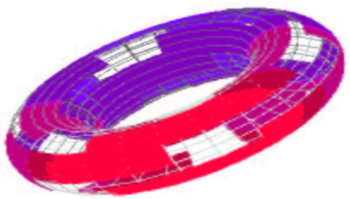
Use of fundamental category (and functor):

$\vec{\pi}_1(X)$:

- its objects are the points of X,
- its morphisms are dihomotopy classes of dipaths: a morphism from x to y is a dihomotopy class $[f]$ of a dipath f going from x to y.

Generalizes the fundamental group $\pi_1(X)$ of a topological space X (a single object=base point; morphisms=homotopy classes of loops).

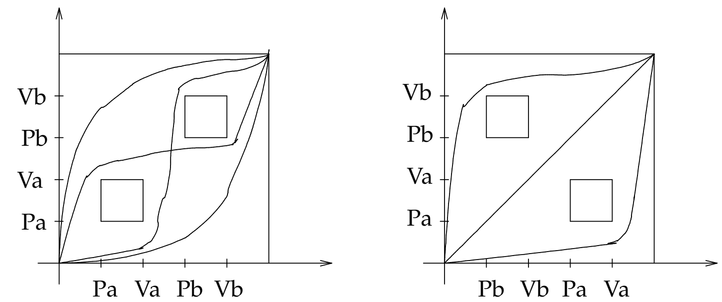
Gives...



$\longrightarrow$ local po-space (*MFPS'98*); manifold like definition, locally a po-space. Extensions of algorithms (see L. Fajstrup, M. Raussen, S. Sokolowski).

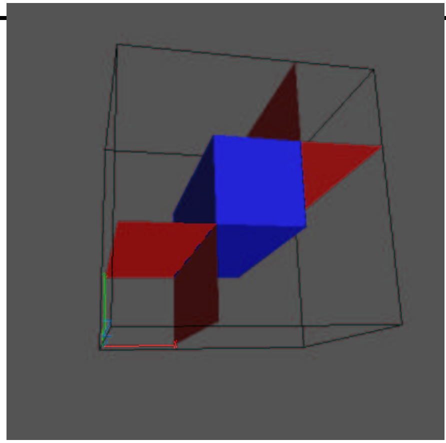
First subtlety

$PaVaPbVb \mid PaVaPbVb$: $PbVbPaVa \mid PaVaPbVb$:

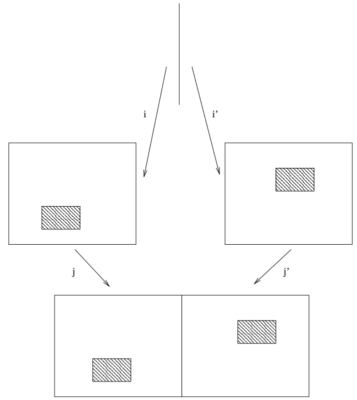


Directed homotopy is not "standard" homotopy

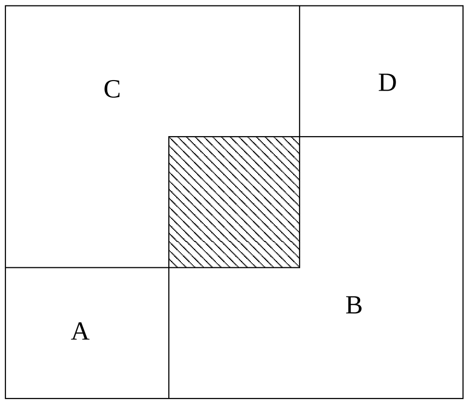
$\vec{\pi}_1$ of a product [parallel "independent" composition]



A Seifert/van Kampen theorem [compositionality]



Example of a $\vec{\pi}_1$



Homotopy classes

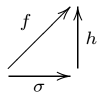
	A	B	C	D
A	1	1	1	2
B	0	1	0	1
C	0	0	1	1
D	0	0	0	1

$\longrightarrow$ compression or (finite) presentation of the fundamental category?

Future components

Let $\mathcal{C}$ be a directed category. Σ is a subcategory of future weak equivalences (f-WE) if:

- Σ contains all isos of $\mathcal{C}$.
- all σ in Σ are epis in $\mathcal{C}$.
- Σ is stable under pushout (which always exists, with any morphim in $\mathcal{C}$).
- If there is $u : \beta \to \gamma$ in $\mathcal{C}$, then for all $\sigma : \alpha \to \beta$ in Σ, and all $f : \alpha \to \gamma$ in $\mathcal{C}$, f factors through σ, that is, there exists $h : \beta \to \gamma$ such that the following diagram commutes

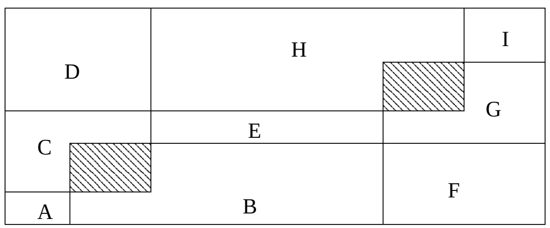


Duals and existence

- (other definitions possible - such as weaker version by Lisbeth)
- Dual notion p-WE, bi-notion WE.
- In directed categories, there always exists f-WE, p-WE and WE subcategories, namely the subcategory of isomorphims of $\mathcal{C}$.

A finite "invariant"?

"diconnected components" (category of fractions of the fundamental category):



(with L. Fajstrup, E. Haucourt and M. Raussen)

Directed categories

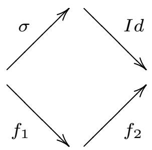
- Let $\mathcal{C}$ be a category, Σ a subcategory of $\mathcal{C}$. Σ is pure in $\mathcal{C}$ if for all f, g in $\mathcal{C}$, $f \circ g \in \Sigma$ implies $f \in \Sigma$ and $g \in \Sigma$.
- A directed category is a category whose subcategory of isomorphisms is pure.
- For X a po-space, local po-space etc. $\vec{\pi}_1(X)$ is a directed category.

Purity

Let Σ be a subcategory of WE (respectively, f-WE, p-WE) in $\mathcal{C}$, then Σ is (respectively left, right) pure.

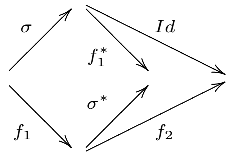
Sketch of proof.

Suppose σ in Σ is equal to $f_2 \circ f_1$ with f_1 and f_2 in $\mathcal{C}$:



Sketch of proof

As σ is in Σ, we have a pushout between σ and f_1, hence we have the commutative diagram



WE and left/right calculi

- p-WE form right calculi, and
- f-WE form left calculi,
- WE form left and right calculi.

Left and right purity

- Σ is left pure if for all σ in Σ, and for all f_1 and f_2 in $\mathcal{C}$ such that $\sigma = f_2 \circ f_1$, then f_1 is in Σ.
- Σ is right pure if (...), then f_2 is in Σ.
- Σ is pure if it is both left and right pure.

Maximal WE subcategories

Let $\mathcal{C}$ be a directed category. There exists a maximal subcategory of WE (f-WE, p-WE respectively) in $\mathcal{C}$.

Sketch of proof.

There exists one, as we saw earlier.

Now, it suffices to show that if τ is a morphism of $\mathcal{C}$ satisfying all axioms of a subcategory of WE (as the category generated by τ) and if Σ is a subcategory of WE, then all composites $\tau \circ \sigma$ and $\sigma \circ \tau$, together with Σ, is a subcategory of WE. By induction, we see that being WE is inductive. Using Zorn's lemma, we find the maximal WE.

Consequence...

Using this maximal WE, we have all the results from APCS'04, and in particular:

Let $\mathcal{C}$ be a category in which all endomorphisms are identities. Then there is a maximal WE subcategory Σ in $\mathcal{C}$. Furthermore, let $C_1, C_2 \subset Ob(\mathcal{C})$ denote two components such that the set of morphisms (in $\mathcal{C}/\Sigma$) is *finite*. Then, for every $x_1 \in C_1$ there exists $x_2 \in C_2$ such that the quotient map

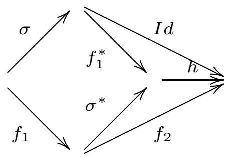
$$\mathcal{C}(x_1, x_2) \to \mathcal{C}/\Sigma(C_1, C_2),\ ;\ f \mapsto [f]$$

is *bijective*.

Similarly with just f-WE (we can choose the target point only) and the p-WE (we can choose the source point only).

Sketch of proof

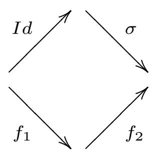
hence, by the universality of the pushout, we have a unique h in $\mathcal{C}$ such that the diagram below commutes



This implies that $Id = h \circ f_1^*$. Hence h is a left inverse in $\mathcal{C}$ hence is in Σ. This also implies that $f_2 = h \circ \sigma^*$, a composite of two elements of Σ, hence is in Σ.

Sketch of proof

Now we can rewrite $\sigma = f_2 \circ f_1$ as the commutative diagram



and use the co-universal property of the pullback of σ along f_2, to conclude in the same manner that f_1 is in Σ.

Theorem

Let Σ be the inessential morphisms in the future, in the category $\mathcal{C} = \vec{\pi}_1(X)$ for some local po-space X.

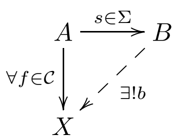
Suppose that Σ is cocomplete (it is automatically the case in mutual exclusion models, i.e. $\vec{I}^n$ minus isothetic hyperrectangles), then $\mathcal{C}_\Sigma$ is reflective in $\vec{\pi}_1(X)$.

Computations?

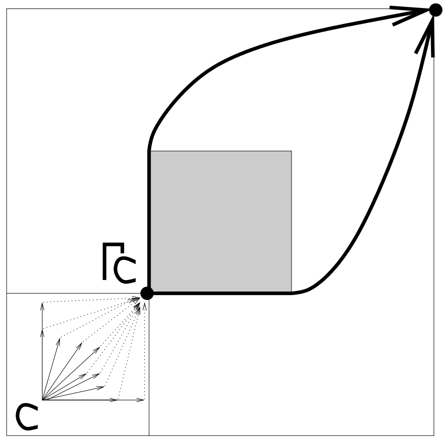
- We have a Seifert/van Kampen for local po-spaces (last ATMCS - or M. Grandis' proof)
- We also have a form of Seifert/van Kampen for components categories, "up to subdivision" (Emmanuel Haucourt), which is of value for practical computations.
- Also, some specific algorithms for mutual exclusion models (M. Raussen in dimension 2, and sub-optimal algorithm by E. Goubault in all dimensions).

Orthogonal subcategories

Let $\mathcal{C}$ be a category and Σ a class of morphisms of $\mathcal{C}$. By the orthogonal subcategory of $\mathcal{C}$ determined by Σ, we mean the full subcategory $\mathcal{C}_\Sigma$ of $\mathcal{C}$, whose objects are those $X \in \mathcal{C}$ such that $s \perp X$ for every $s \in \Sigma$, i.e., such that for every $s : A \to B \in \Sigma$, for every morphism $f : A \to X$, there exists a unique morphism $b : B \to X$ such that $b \circ s = f$.



The orthogonal subcategory of Σ_+ is reflective



A proposal for strong dihomotopy equivalence

~~For X a local po-space, we note $\vec{\pi}_1(X)$ its fundamental category~~, and Σ_Y the maximal subcategory of inessential morphisms of $\vec{\pi}_1(X)$.

Let $f : X \to Y$. We say that f is a strong invariant if $\vec{\pi}_1(f)$ maps the subcategory Σ_X of $\vec{\pi}_1(X)$ into the subcategory Σ_Y of $\vec{\pi}_1(Y)$.

Let X and Y be two local po-spaces. We say that X and Y are dihomotopy equivalent if and if there exist strong invariant dimaps

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

such that $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$.

Examples

- The directed interval is dihomotopy equivalent to a point, but...
- The wedge of two directed intervals (glued together at their start point) is not dihomotopy equivalent to a point nor a directed interval...

If we want to go further

- $\vec{\pi}_0$!
- (Strong) dihomotopy equivalence
- Higher-dimensional fundamental categories (difficult)
- Model-category theoretic explanation of dihomotopy (P. Gaucher, or K. Hess/K. Worytkiewicz/P. Bubenik)

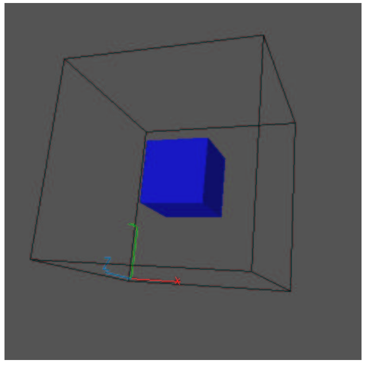
A proposal for $\vec{\pi}_0$

Let $U : Cat \to Poset$ be the forgetful functor from the category of small categories to the category of posetal categories (the ones with at most one arrow between two objects). Let X be a local po-space. Let $\vec{\pi}_0(X)$, the set of diconnected components, be defined as the set of arrows of $U(\vec{\pi}_1(X))$.

The set of diconnected components is to be thought of as the set of possible bibase points that we need to "rigidify" the deformations of dimaps.

Higher-dimensional dihomotopies

(in homology: in my thesis [1995]; also, in homotopy, see S. Sokolowski - more recently, M. Grandis)
Observation of $k > 1$ degrees of freedom



$\longrightarrow$ higher-dimensional loops (π_k): ¡¡around dipaths¿¿.

Possible π_n ($n \geq 2$)

$$\boxed{\pi_n(X)(x_0, x_1) = \overrightarrow{\pi}_1 \left((X, \leq)^{(S^{n-1},=)}(x_0, x_1)\right)}$$

(where x_0 et x_1 are identified with $(n-1) - loops$)

Therefore:

- $\overrightarrow{\pi}_2$ $(\vec{I}^3)(i, f) = \{*\}$

- $\overrightarrow{\pi}_1$ $(\vec{I}^3 \backslash \{(1/2, 1/2, 1/2)\})(i, f) = \{*\}$ but its $\overrightarrow{\pi}_2$ is $\mathbb{Z}$ (with "lateral composition"!) and its $\overrightarrow{\pi}_3$ is not $\{*\}$ but $\mathbb{Z}$ (because of an analogous of the *Hopf function*)

- all the homotopies of spheres are ¡¡lifted¿¿ in $\overrightarrow{\pi}_n$ $(\vec{I}^k \backslash \{*\})$

Algebraic model of dihomotopy n-types?