HIGHER ENRICHMENT: N-FOLD OPERADS AND ENRICHED N-CATEGORIES, DELOOPING AND WEAKENING

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ABSTRACT. The most familiar example of higher, or vertically iterated enrichment is that in the definition of strict n-category. We begin with strict n-categories based on a general symmetric monoidal category $\mathcal{V}$. Motivation is offered through a comparison of the classical and extended versions of topological quantum field theory. A sequence of categorical types that filter the category of monoidal categories and monoidal functors has been given by Balteanu, Fiedorowicz, Schwanzl and Vogt. These subcategories of **MonCat** are called n-fold monoidal categories. A k-fold monoidal category is n-fold monoidal for all $n \leq k$, and a symmetric monoidal category is n-fold monoidal for all n. Operads and enriched categories were originally defined as enriched over a symmetric monoidal category $\mathcal{V}$. The symmetry in $\mathcal{V}$ is required in order to describe the associativity axiom the operad composition must obey, to describe the associativity that must hold in the action of an operad on one of its algebras and to define the product of operads. For enriched categories the symmetry is required in order to describe the opposite of a $\mathcal{V}$-category and the product of $\mathcal{V}$-categories. We demonstrate how all this can be accomplished instead by use of the interchange transformations of an n-fold monoidal category. There are suggestive results about higher enrichment over n-fold categories. Briefly, it turns out that for $\mathcal{V}$ k-fold monoidal, $\mathcal{V}$-n-Cat is a $(k-n)$-fold monoidal $(n+1)$-category. Since iterated monoidal categories have iterated loop space nerves, we suspect a delooping process. This is shown to be the case in the example of group torsors. Next we discuss how weakening higher enrichment may shed light on comparisons of n-category definitions. We pictorially describe the process of weakening and the appropriate morphisms using polytopes, and then define an operad whose action provides a concise equivalent description.

Thanks to XY-pic for the diagrams.

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Contents

1. Introduction: iterated enrichment and TQFT

In a topological quantum field theory we use structure in a category to model spaces. Classically TQFT's reflected the topology of a space, but more recently extended TQFT's also reflect the embedding, as in the case of n-tangles. In the latter vision, k-cells in an n-category with one object, one morphism, etc. are used to model k-dimensional manifolds embedded in a higher dimensional space. The brading and duality structure reflect the possibilities of embedding, allowing for instance, the creation of knot invariants. Another (possibly simultaneous) use of higher cells in an n-category would be to model a polygonal decomposition of a topological space. In this picture the k-cells model k-dimensional faces and thus though allowed to be arranged more freely they only reflect topology as opposed to embedding. In the case of manifolds, this viewpoint leads us to enrichment since up to a choice of orientation there is at most one top-dimensional face between any pair of codimension 1 faces. Consider a 2-disk with a given decomposition. Objects are endpoints of the directed edges. Any two objects determine a set of arrows. Any two arrows determine at most one directed disk, or 2-arrow. Thus we can use the picture to generate an enriched 2-category! Dual cells also play an important role here in moves on decompositions.

2. k–fold monoidal categories

This sort of category was developed and defined in [Balteanu et.al, 2003]. The authors describe its structure as arising from its description as a monoid in the category of $(k-1)$–fold monoidal categories. Here is that definition altered only slightly to make visible the coherent associators as in [Forcey, 2004]. In that paper I describe its structure as arising from its description as a tensor object or pseudomonoid in the category of $(k-1)$–fold monoidal categories.

2.1. Definition. *An n-fold monoidal category is a category $\mathcal{V}$ with the following structure.*

1. *There are n distinct multiplications*

$$\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

for each of which the associativity pentagon commutes

$$\begin{array}{ccc}
((U \otimes_i V) \otimes_i W) \otimes_i X & \xrightarrow{\alpha^i_{UVW} \otimes_i 1_X} & (U \otimes_i (V \otimes_i W)) \otimes_i X \\
\downarrow{\scriptstyle \alpha^i_{(U \otimes_i V)WX}} & & \downarrow{\scriptstyle \alpha^i_{U(V \otimes_i W)X}} \\
(U \otimes_i V) \otimes_i (W \otimes_i X) & & U \otimes_i ((V \otimes_i W) \otimes_i X) \\
& \searrow{\scriptstyle \alpha^i_{UV(W \otimes_i X)}} \quad \swarrow{\scriptstyle 1_U \otimes_i \alpha^i_{VWX}} & \\
& U \otimes_i (V \otimes_i (W \otimes_i X)) &
\end{array}$$

$\mathcal{V}$ has an object I which is a strict unit for all the multiplications.

2. *For each pair (i, j) such that $1 \le i < j \le n$ there is a natural transformation*

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) *Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A \otimes_j B}$*

(b) *External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A \otimes_i B}$*

(c) *Internal associativity condition: The following diagram commutes*

$$\begin{array}{ccc}
((U \otimes_j V) \otimes_i (W \otimes_j X)) \otimes_i (Y \otimes_j Z) & \xrightarrow{\eta^{ij}_{UVWX} \otimes_i 1_{Y \otimes_j Z}} & \big((U \otimes_i W) \otimes_j (V \otimes_i X)\big) \otimes_i (Y \otimes_j Z) \\
\downarrow{\scriptstyle \alpha^i} & & \downarrow{\scriptstyle \eta^{ij}_{(U \otimes_i W)(V \otimes_i X)YZ}} \\
(U \otimes_j V) \otimes_i ((W \otimes_j X) \otimes_i (Y \otimes_j Z)) & & ((U \otimes_i W) \otimes_i Y) \otimes_j ((V \otimes_i X) \otimes_i Z) \\
\downarrow{\scriptstyle 1_{U \otimes_j V} \otimes_i \eta^{ij}_{WXYZ}} & & \downarrow{\scriptstyle \alpha^i \otimes_j \alpha^i} \\
(U \otimes_j V) \otimes_i \big((W \otimes_i Y) \otimes_j (X \otimes_i Z)\big) & \xrightarrow{\eta^{ij}_{UV(W \otimes_i Y)(X \otimes_i Z)}} & (U \otimes_i (W \otimes_i Y)) \otimes_j (V \otimes_i (X \otimes_i Z))
\end{array}$$

(d) External associativity condition: The following diagram commutes

$$
\begin{array}{ccc}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) & \xrightarrow{\eta^{ij}_{(U\otimes_j V)W(X\otimes_j Y)Z}} & \big((U \otimes_j V) \otimes_i (X \otimes_j Y)\big) \otimes_j (W \otimes_i Z) \\
\downarrow{\scriptstyle \alpha^j \otimes_i \alpha^j} & & \downarrow{\scriptstyle \eta^{ij}_{UVXY} \otimes_j 1_{W \otimes_i Z}} \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) & & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \\
\downarrow{\scriptstyle \eta^{ij}_{U(V\otimes_j W)X(Y\otimes_j Z)}} & & \downarrow{\scriptstyle \alpha^j} \\
(U \otimes_i X) \otimes_j \big((V \otimes_j W) \otimes_i (Y \otimes_j Z)\big) & \xrightarrow{1_{U\otimes_i X} \otimes_j \eta^{ij}_{VWYZ}} & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{array}
$$

(e) Finally it is required for each triple (i, j, k) satisfying $1 \leq i < j < k \leq n$ that the giant hexagonal interchange diagram commutes.

$$
\begin{array}{ccc}
& ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) & \\
\swarrow{\scriptstyle \eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'}} & & \searrow{\scriptstyle \eta^{ij}_{(A\otimes_k A')(B\otimes_k B')(C\otimes_k C')(D\otimes_k D')}} \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) & & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \\
\downarrow{\scriptstyle \eta^{ik}_{(A\otimes_j B)(A'\otimes_j B')(C\otimes_j D)(C'\otimes_j D')}} & & \downarrow{\scriptstyle \eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}} \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \\
\searrow{\scriptstyle \eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'}} & & \swarrow{\scriptstyle \eta^{jk}_{(A\otimes_i C)(A'\otimes_i C')(B\otimes_i D)(B'\otimes_i D')}} \\
& ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{array}
$$

The authors of [Balteanu et.al, 2003] remark that a symmetric monoidal category is n-fold monoidal for all n. This they demonstrate by letting

$$\otimes_1 = \otimes_2 = \ldots = \otimes_n = \otimes$$

and defining (associators added by myself)

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$. Here $c_{BC} : B \otimes C \to C \otimes B$ is the symmetry natural transformation.

3. Categories Enriched over a k-fold Monoidal Category

The correct theory for enriching over a k-fold monoidal category $\mathcal{V}$ may depend somewhat upon the point of view of the theorist. Here we are biased by the knowledge of research that reveals $\mathcal{V}$ to be precisely analogous to a k-fold loop space, as well as by the observation that forming the category of categories enriched over $\mathcal{V}$ is something akin to delooping especially in the cases of braided and symmetric monoidal categories. It turns out that if we let ourselves be guided by that intuition, what works quite well is to simply consider categories enriched over the monoidal category given by $\mathcal{V}$ with $\otimes_1$. Of course the extra structure of $\mathcal{V}$ is very important – precisely when it comes to describing $\mathcal{V}$-Cat. We are ready to state the initial result.

3.1. Theorem. *For $\mathcal{V}$ a k-fold monoidal category $\mathcal{V}$-Cat is a $(k-1)$-fold monoidal 2-category.*

We begin by describing the $k=2$ case. $\mathcal{V}$ is 2-fold monoidal with products $\otimes_1, \otimes_2$. $\mathcal{V}$-categories (which are the objects of $\mathcal{V}$-Cat) are defined as being enriched over $(\mathcal{V}, \otimes_1, \alpha^1, I)$. Here $\otimes_1$ plays the role of the product given by $\otimes$ in the axioms of section 1. We need to show that $\mathcal{V}$-Cat has a product.

The unit object in $\mathcal{V}$-Cat is the enriched category $\mathcal{I}$ where $|\mathcal{I}| = \{0\}$ and $\mathcal{I}(0,0) = I$. Of course $M_{000} = 1 = j_0$. The objects of the tensor $\mathcal{A} \otimes_1^{(1)} \mathcal{B}$ of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are simply pairs of objects, that is, elements in $|\mathcal{A}| \times |\mathcal{B}|$. The hom-objects in $\mathcal{V}$ are given by $(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes_2 \mathcal{B}(B,B')$. The composition morphisms that make $\mathcal{A} \otimes_1^{(1)} \mathcal{B}$ into a $\mathcal{V}$-category are immediately apparent as generalizations of the braided case. Recall that we are describing $\mathcal{A} \otimes_1^{(1)} \mathcal{B}$ as a category enriched over $\mathcal{V}$ with product $\otimes_1$. Thus

$$M_{(A,B)(A',B')(A'',B'')} : (\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A',B')) \to (\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A'',B''))$$

is given by

$$\begin{array}{c}
(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A',B')) \\
\Vert \\
(\mathcal{A}(A',A'') \otimes_2 \mathcal{B}(B',B'')) \otimes_1 (\mathcal{A}(A,A') \otimes_2 \mathcal{B}(B,B')) \\
\downarrow \eta^{1,2} \\
(\mathcal{A}(A',A'') \otimes_1 \mathcal{A}(A,A')) \otimes_2 (\mathcal{B}(B',B'') \otimes_1 \mathcal{B}(B,B')) \\
\downarrow M_{AA'A''} \otimes_2 M_{BB'B''} \\
(\mathcal{A}(A,A'') \otimes_2 \mathcal{B}(B,B'')) \\
\Vert \\
(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A,B),(A'',B''))
\end{array}$$

Next we describe the $k = 3$ case. $\mathcal{V}$ is 3-fold monoidal with products $\otimes_1, \otimes_2$ and $\otimes_3$. $\mathcal{V}$-categories are defined as being enriched over $(\mathcal{V}, \otimes_1, \alpha^1, I)$. Now $\mathcal{V}$-Cat has two products. The objects of both possible tensors $\mathcal{A} \otimes_1^{(1)} \mathcal{B}$ and $\mathcal{A} \otimes_2^{(1)} \mathcal{B}$ of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are elements in $|\mathcal{A}| \times |\mathcal{B}|$. The hom-objects in $\mathcal{V}$ are given by

$$(\mathcal{A} \otimes_1^{(1)} \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes_2 \mathcal{B}(B, B')$$

just as in the previous case, and by

$$(\mathcal{A} \otimes_2^{(1)} \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes_3 \mathcal{B}(B, B').$$

The composition that makes $(\mathcal{A} \otimes_2^{(1)} \mathcal{B})$ into a $\mathcal{V}$-category is analogous to that for $(\mathcal{A} \otimes_1^{(1)} \mathcal{B})$ but uses $\eta^{1,3}$ as its middle exchange morphism.

Now we need an interchange 2-natural transformation $\eta^{(1)1,2}$ for $\mathcal{V}$-Cat. The family of morphisms $\eta^{(1)1,2}_{\mathcal{ABCD}}$ that make up a 2-natural transformation between the 2-functors $\times^4\mathcal{V}$-Cat $:\to \mathcal{V}$-Cat in question is a family of enriched functors. Their action on objects is to send

$$((A, B), (C, D)) \in \left|(\mathcal{A} \otimes_2^{(1)} \mathcal{B}) \otimes_1^{(1)} (\mathcal{C} \otimes_2^{(1)} \mathcal{D})\right| \text{ to } ((A, C), (B, D)) \in \left|(\mathcal{A} \otimes_1^{(1)} \mathcal{C}) \otimes_2^{(1)} (\mathcal{B} \otimes_1^{(1)} \mathcal{D})\right|.$$

The correct construction of the family of hom-object morphisms in $\mathcal{V}$-Cat for each of these functors is also clear. Noting that

$$\begin{aligned}
&[(\mathcal{A} \otimes_2^{(1)} \mathcal{B}) \otimes_1^{(1)} (\mathcal{C} \otimes_2^{(1)} \mathcal{D})](((A, B), (C, D)), ((A', B'), (C', D'))) \\
&= (\mathcal{A} \otimes_2^{(1)} \mathcal{B})((A, B), (A', B')) \otimes_2 (\mathcal{C} \otimes_2^{(1)} \mathcal{D})((C, D), (C', D')) \\
&= (\mathcal{A}(A, A') \otimes_3 \mathcal{B}(B, B')) \otimes_2 (\mathcal{C}(C, C') \otimes_3 \mathcal{D}(D, D'))
\end{aligned}$$

and similarly

$$\begin{aligned}
&[(\mathcal{A} \otimes_1^{(1)} \mathcal{C}) \otimes_2^{(1)} (\mathcal{B} \otimes_1^{(1)} \mathcal{D})](((A, C), (B, D)), ((A', C'), (B', D'))) \\
&= (\mathcal{A}(A, A') \otimes_2 \mathcal{C}(C, C')) \otimes_3 (\mathcal{B}(B, B') \otimes_2 \mathcal{D}(D, D'))
\end{aligned}$$

we make the obvious identification, where by obvious we mean based upon the corresponding structure in $\mathcal{V}$. Thus we write:

$$\eta^{(1)1,2}_{\mathcal{ABCD}_{(ABCD)(A'B'C'D')}} = \eta^{2,3}_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')\mathcal{D}(D,D')}.$$

4. Category of $\mathcal{V}$-n-Categories

The definition of a category enriched over $\mathcal{V}$-$(n-1)$-Cat is simply stated by describing the process as enriching over $\mathcal{V}$-$(n-1)$-Cat with the first of the $k-n$ ordered products. In detail this means that:

4.1. Definition. *A (small, strict) $\mathcal{V}$-n-category $\boldsymbol{\mathcal{U}}$ consists of*

1. *A set of objects $|\boldsymbol{\mathcal{U}}|$*
2. *For each pair of objects $A, B \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$-$(n-1)$-category $\boldsymbol{\mathcal{U}}(A, B)$.*
3. *For each triple of objects $A, B, C \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$-$(n-1)$-functor*

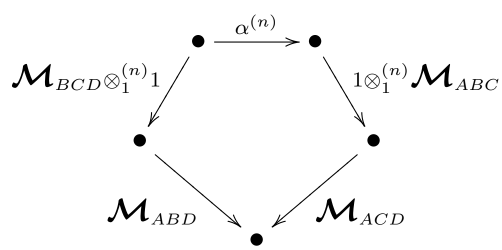
$$\boldsymbol{\mathcal{M}}_{ABC} : \boldsymbol{\mathcal{U}}(B, C) \otimes_1^{(n-1)} \boldsymbol{\mathcal{U}}(A, B) \to \boldsymbol{\mathcal{U}}(A, C)$$

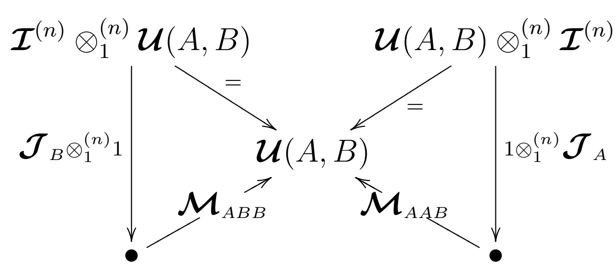
4. *For each object $A \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$-$(n-1)$-functor*

$$\boldsymbol{\mathcal{J}}_A : \boldsymbol{\mathcal{I}}^{(n-1)} \to \boldsymbol{\mathcal{U}}(A, A)$$

   Henceforth we let the dimensions of domain for and particular instances of $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{J}}$ largely be determined by context.

5. *Axioms: The $\mathcal{V}$-$(n-1)$-functors that play the role of composition and identity obey commutativity of a pentagonal diagram (associativity axiom) and of two triangular diagrams (unit axioms). This amounts to saying that the functors given by the two legs of each diagram are equal.*





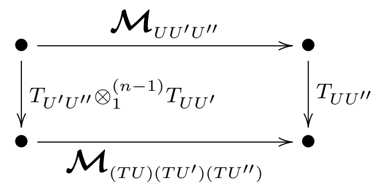
This definition requires that there be definitions of the unit $\boldsymbol{\mathcal{I}}^{(n)}$ and of $\mathcal{V}$-n-functors in place. First, from the proof of monoidal structure on $\mathcal{V}$-n-Cat, we can infer a recursively defined unit $\mathcal{V}$-n-category.

4.2. Definition. *The unit object in $\mathcal{V}$-n-Cat is the $\mathcal{V}$-n-category $\boldsymbol{\mathcal{I}}^{(n)}$ with one object $\mathbf{0}$ and with $\boldsymbol{\mathcal{I}}^{(n)}(\mathbf{0},\mathbf{0}) = \boldsymbol{\mathcal{I}}^{(n-1)}$, where $\boldsymbol{\mathcal{I}}^{(n-1)}$ is the unit object in $\mathcal{V}$-$(n-1)$-Cat. Of course we let $\boldsymbol{\mathcal{I}}^{(0)}$ be I in $\mathcal{V}$. Also $\boldsymbol{\mathcal{M}}_{000} = \boldsymbol{\mathcal{J}}_0 = 1_{\boldsymbol{\mathcal{I}}^{(n)}}$.*

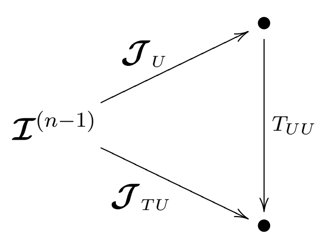
Now we can define the functors:

4.3. Definition. *For two $\mathcal{V}$-n-categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a $\mathcal{V}$-n-functor $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$-$(n-1)$-functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U,U') \to \boldsymbol{\mathcal{W}}(TU,TU')$. These latter obey commutativity of the usual diagrams.*

1. *For $U, U', U'' \in |\boldsymbol{\mathcal{U}}|$*



2.



Here a $\mathcal{V}$-0-functor is just a morphism in $\mathcal{V}$.

$\mathcal{V}$-n-categories and $\mathcal{V}$-n-functors form a category.

4.4. Definition. *A $\mathcal{V}$-n:k-cell α between $(k-1)$-cells ψ^{k-1} and ϕ^{k-1}, written*

$$\alpha : \psi^{k-1} \to \phi^{k-1} : \psi^{k-2} \to \phi^{k-2} : \ldots : \psi^2 \to \phi^2 : F \to G : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$$

where F and G are $\mathcal{V}$-n-functors and where the superscripts denote cell dimension, is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a $\mathcal{V}$-$((n-k)+1)$-functor

$$\alpha_U : \boldsymbol{\mathcal{I}}^{((n-k)+1)} \to \boldsymbol{\mathcal{W}}(FU,GU)(\psi_U^2\mathbf{0}, \phi_U^2\mathbf{0})\ldots(\psi_U^{k-1}\mathbf{0}, \phi_U^{k-1}\mathbf{0})$$

in such a way that we have commutativity of the following diagram. Note that the final (curved) equal sign is implied recursively by the diagram for the $(k-1)$-cells.

$$\boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)\dots(x_{k-1},y_{k-1}) \xrightarrow{=} \boldsymbol{\mathcal{I}}^{((n-k)+1)} \otimes_{k-1}^{((n-k)+1)} \boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)\dots(x_{k-1},y_{k-1})$$

$$\xrightarrow{\alpha_{U'} \otimes_{k-1}^{((n-k)+1)} F} \boldsymbol{\mathcal{W}}(FU',GU')(\psi_{U'}^2\mathbf{0},\phi_{U'}^2\mathbf{0})\dots(\psi_{U'}^{k-1}\mathbf{0},\phi_{U'}^{k-1}\mathbf{0}) \otimes_{k-1}^{((n-k)+1)} \boldsymbol{\mathcal{W}}(FU,FU')(F(x_2),F(y_2))\dots(F(x_{k-1}),F(y_{k-1}))$$

$$\xrightarrow{\boldsymbol{\mathcal{M}}} \boldsymbol{\mathcal{W}}(FU,GU')(\psi_{U'}^2\mathbf{0}F(x_2),\phi_{U'}^2\mathbf{0}F(y_2))\dots(\psi_{U'}^{k-1}\mathbf{0}F(x_{k-1}),\phi_{U'}^{k-1}\mathbf{0}F(y_{k-1}))$$

$$=\!=$$

$$\boldsymbol{\mathcal{W}}(FU,GU')(G(x_2)\psi_U^2\mathbf{0},G(y_2)\phi_U^2\mathbf{0})\dots(G(x_{k-1})\psi_U^{k-1}\mathbf{0},G(y_{k-1})\phi_U^{k-1}\mathbf{0})$$

$$\boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)\dots(x_{k-1},y_{k-1}) \xrightarrow{=} \boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)\dots(x_{k-1},y_{k-1}) \otimes_{k-1}^{((n-k)+1)} \boldsymbol{\mathcal{I}}^{((n-k)+1)}$$

$$\xrightarrow{G \otimes_{k-1}^{((n-k)+1)} \alpha_U} \boldsymbol{\mathcal{W}}(GU,GU')(G(x_2),G(y_2))\dots(G(x_{k-1}),G(y_{k-1})) \otimes_{k-1}^{((n-k)+1)} \boldsymbol{\mathcal{W}}(FU,GU)(\psi_U^2\mathbf{0},\phi_U^2\mathbf{0})\dots(\psi_U^{k-1}\mathbf{0},\phi_U^{k-1}\mathbf{0})$$

$$\xrightarrow{\boldsymbol{\mathcal{M}}} \boldsymbol{\mathcal{W}}(FU,GU')(G(x_2)\psi_U^2\mathbf{0},G(y_2)\phi_U^2\mathbf{0})\dots(G(x_{k-1})\psi_U^{k-1}\mathbf{0},G(y_{k-1})\phi_U^{k-1}\mathbf{0})$$

5. Delooping: group torsors

We show that for $\mathcal{V}$ k-fold monoidal the structure of a $(k-n)$-fold monoidal strict $(n+1)$-category is possessed by $\mathcal{V}$-n-Cat. At each stage of successive enrichments, the number of monoidal products decreases and the categorical dimension increases, both by one. This is suggestive of topology. When we consider the loop space of a topological space, we see that paths (or 1-cells) in the original are now points (or objects) in the derived space. There is also now automatically a product structure on the points in the derived space, where multiplication is given by concatenation of loops. Delooping is the inverse functor here, and thus involves shifting objects to the status of 1-cells and decreasing the number of ways to multiply.

An example that sheds more light on the delooping picture is as follows (thanks to J. Baez.)

Given a group G:

1. Let $\underline{G}$ be the category whose objects are elements of G and whose only morphisms are identity arrows. $\underline{G}$ is monoidal with $\otimes$ given by the group operation and $I = e$.

2. Let Tor(G) be the category of G-torsors and G-equivariant maps (respect action.) A G-torsor $\mathcal{B}$ is a set $\mathcal{B}$ with effective G-action; that is for all $x, y \in \mathcal{B}$ there exists a unique $g_{xy} \in G$ such that $g_{xy}x = y$. First we notice that $\mathcal{B}$ is a $\underline{G}$-category. $\mathcal{B}(x, y) = g_{xy}$ and equivariant maps are enriched functors. Secondly we notice that G itself is a G-torsor; i.e. $\underline{G}$ is closed. Enriched functors $G \to G$ are the elements of G. We denote by $\overline{G}$ this single category subcategory of Tor(G). We also note that a $\underline{G}$-category tensored over $\underline{G}$ is precisely a G-torsor, since letting $x \otimes g = gx$ gives $\mathcal{B}(x \otimes g, z) = \underline{G}(g, \mathcal{B}(x, z))$.

3. We use the fact every G-torsor is isomorphic to G. Thus $\overline{G}$ is a skeleton of Tor(G). Recall that $Nerve\text{Tor}(G) = Nerve\overline{G} = BG = B(Nerve\underline{G})$ and thus we have that $B(Nerve\underline{G}) \subseteq Nerve(\underline{G}\text{-Cat})$. The subset becomes an equality when we restrict to $\underline{G}$-categories tensored over $\underline{G}$.

Further study of the general case should attempt to elucidate the relationship of the nerves of the n-categories in question. For instance, we would like to know the relationship between $\Omega Nerve(\mathcal{V}\text{-Cat})$ and $Nerve(\mathcal{V})$. This would even be quite interesting in the case of symmetric $\mathcal{V}$ where the nerve is an infinite loop space. It would be nice to know if there are symmetric monoidal categories whose nerves exhibit periodicity under the vertically iterated enrichment functor.

In [Street, 1987] Street defines the nerve of a strict n-category. Recently Duskin in [Duskin, 2002] has worked out the description of the nerve of a bicategory. This allows us to ask whether these nerves will prove to be the logical link to loop spaces for higher dimensional iterated monoidal categories.

Passing to the category of enriched categories basically reduces the number of products so that for $\mathcal{V}$ a k-fold monoidal n-category, $\mathcal{V}$-Cat becomes a $(k-1)$-fold monoidal $(n +$

1)-category. This picture was anticipated by Baez and Dolan [Baez and Dolan, 1998] in the context where the k-fold monoidal n-category is specifically a (weak) $(n + k)$-category with only one object, one 1-cell, etc. up to only one k-cell. Their version of categorical delooping simply consists of creating from a monoidal category $\mathcal{V}$ the one object bicategory that has its morphisms the objects of $\mathcal{V}$. Relating the two versions of delooping is important to an understanding of how categories model spaces.

6. n-fold operads

Let $\mathcal{V}$ be an n-fold monoidal category as defined in the last section.

6.1. Definition. *An operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, a right action by the symmetric group Σ_j on $\mathcal{C}(j)$ for each j and composition maps in $\mathcal{V}$*

$$\gamma : \mathcal{C}(k) \otimes_1 \mathcal{C}(j_1) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k) \to \mathcal{C}(j)$$

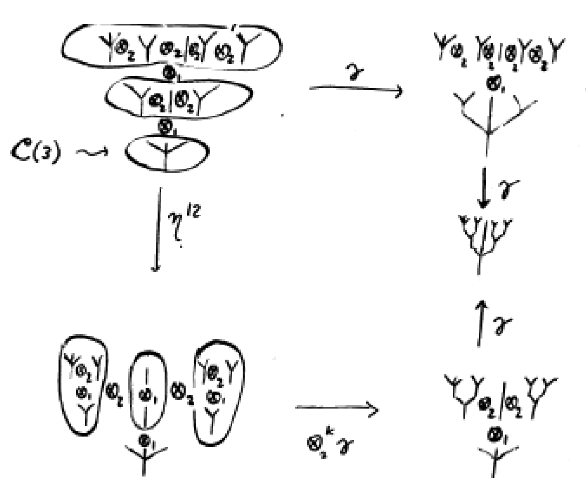
for $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms

Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $m_s = \sum_{n=1}^{s} j_n$ and let $h_s = \sum_{n=1+g_{s-1}}^{g_s} i_n$.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2 \mathcal{C}(j_s) \right) \otimes_1 \left(\bigotimes_{r=1}^{j} {}_2 \mathcal{C}(i_r) \right) & \xrightarrow{\gamma \otimes_1 \mathrm{id}} & \mathcal{C}(j) \otimes_1 \left(\bigotimes_{r=1}^{j} {}_2 \mathcal{C}(i_r) \right) \\
\Big\downarrow \mathrm{id} \otimes_1 \eta^{12} & & \Big\downarrow \gamma \\
 & & \mathcal{C}(i) \\
 & & \Big\uparrow \gamma \\
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2 \mathcal{C}(j_s) \otimes_1 \left(\bigotimes_{q=1}^{j_s} {}_2 \mathcal{C}(i_{q+g_{s-1}}) \right) \right) & \xrightarrow[\mathrm{id} \otimes_1 (\otimes_2^k \gamma)]{} & \mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2 \mathcal{C}(h_s) \right)
\end{array}$$

Equivariance and respect of units are required just as in the symmetric case.

Here is a pictorial version of the associativity axiom:



A similar translation defines algebras over an n-fold operad. The product of two operads is given by

$$(\mathcal{C} \otimes_i^{(1)} \mathcal{D})(j) = \mathcal{C}(j) \otimes_{i+2} \mathcal{D}(j).$$

If A is an algebra of $\mathcal{C}$ and B is an algebra of $\mathcal{D}$ then, for example, $A \otimes_3 B$ is an algebra for $\mathcal{C} \otimes_1^{(1)} \mathcal{D}$.

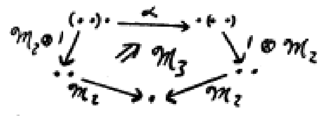
7. Weak Enrichment

The chief difficulty that arises in any naive attempt to define a weak n-category is that the legs of axiomatic diagrams that would commute in the strict case but instead form the domain and range of a higher cell are themselves not well defined pasting diagrams. Indeed all the ways of expressing the full composition of a higher dimensional pasting diagram in a weak n-category are not equal. Instead there exist coherent isomorphisms between them. To avoid a special choice of composition order we can make use of coherence theorems about the entities in which the undefined compositions take place. For instance, any diagram in a bicategory can be defined as having the value of the preimage of the composition of its image taken in the equivalent strict 2-category. The difficulty then truly arises at level three. In [Gordon, Power and Street, 1993] Gordon, Power, and Street demonstrate that their tricategories are not all equivalent to corresponding strict versions. As examples of such pathological tricategories they offer braided monoidal categories (not symmetric and the braiding not the identity on the diagonal) as single-object, single-arrow tricategories. Conceptually then, the problem of not having an equivalent strict arena in which to paste is connected to the existence of nontrivial embeddings of codimension two, and assumed to persist as n increases. Approaches to circumventing this problem include defining universal contractions or fillers to mediate between interpretations of pasting diagrams, and an adjoint approach. By the latter we refer to definitions that use operads, or more generally multicategories, where the problem of coherence is relegated to the level of the monoidal category. A first check of the validity of these definitions as is performed in [Leinster, 2002] is that they agree with the classical definitions of category and bicategory. The goal of this paper is in part to provide an analogous series of higher checks by which full definitions of n-category may be gauged and compared. We choose to base our definition of 1-weak $\mathcal{V}$-n-Cat on an arbitrary symmetric (or k-fold) monoidal category $\mathcal{V}$.

The means of our construction is to assume that we are defining a (1-weak) $\mathcal{V}$-n-category as being weakly enriched over an object equivalent to a strict $(n-1)$-category. In other words, we consider a special case of n-category for which the hom-categories are objects of a strict $(n-1)$-category. A strict n-category is a general n-category for which all of the canonical isomorphisms are identities. In general only the top dimensional cells being composed along the next highest dimension cells is an operation that is associative. In our construction all the higher cells will compose associatively except when composing along 0-cells. Thus this is a study of weak n-category theory restricted to horizontal compositions. Several of the following descriptions are incomplete in that prrofs are needed to ensure well-defined-ness of pasting diagrams.

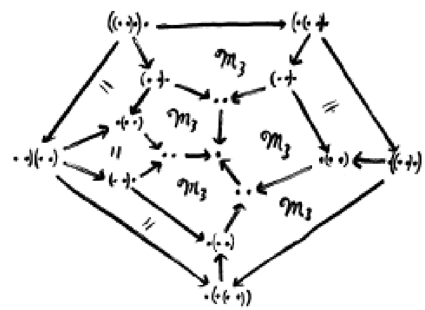
In an enriched category $\mathcal{A}$ (over $\mathcal{V}$) the role of composition is taken over by special morphisms in the monoidal category $\mathcal{V}$. I'll call a string of these hom-objects (such as the string of length two in the domain of the composition morphism $M : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C)$) composable if they can be reduced to a single hom-object by repeated uses of M. Of course the parenthization of the original string matters. Keep in mind then also the associator in $\mathcal{V}$, α, used to get from one parenthization to another. For a string of length n we can draw the associahedron $\mathcal{K}_n$ and put the various parenthizations at the vertexes, and the associators on the edges.

Since, when enriching over $\mathcal{V}$-n-Cat, composition morphisms $\boldsymbol{\mathcal{M}}_2 : \boldsymbol{\mathcal{U}}(B,C)\otimes\boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$ are enriched n-functors, then the pentagon they are usually required to satisfy exactly can instead be filled in with an (invertible) enriched n-natural transformation $\boldsymbol{\mathcal{M}}_3$.

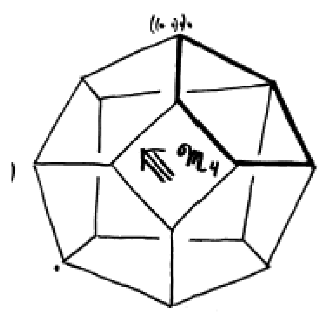


where we represent $\boldsymbol{\mathcal{M}}_2 : \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$ by "$\bullet\bullet \to \bullet$" to save space.

Then we draw the pentagonal diagram of $\mathcal{K}_4$ with vertices labeled by parenthesized strings of composable hom-categories, and compose each vertice by use of $\boldsymbol{\mathcal{M}}_2$ to a common final hom-category. What this does is to subdivide $\mathcal{K}_4$ into two-cells that we fill in again with instances of $\boldsymbol{\mathcal{M}}_3$. We are now looking at a three dimensional polyhedron with one of the pentagonal faces being $\mathcal{K}_4$.

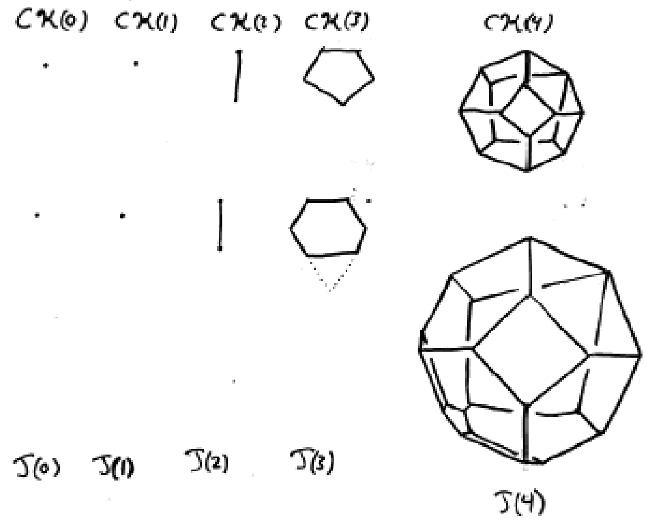


There are exactly two 2-dimensional paths that make up the front and back of the polyhedron. Between the latter there should now exist an enriched modification $\boldsymbol{\mathcal{M}}_4$.



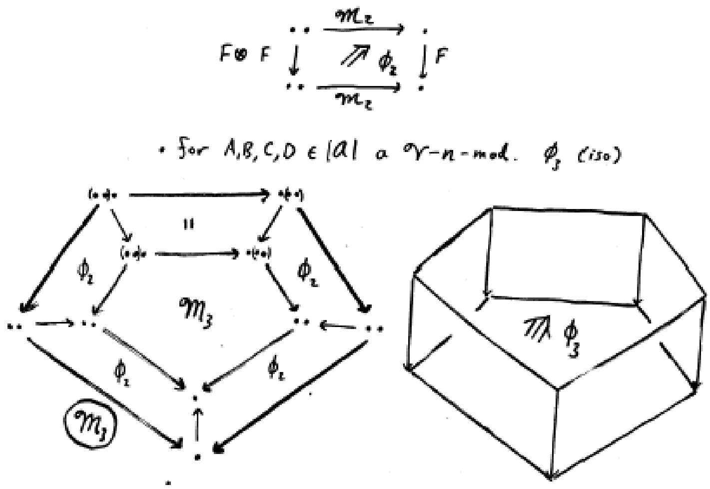
In general we continue subdividing associahedra with vertices labeled by composable hom-objects and filling in with a k-cell $\boldsymbol{\mathcal{M}}_{k+1}$ until at last $\boldsymbol{\mathcal{M}}_{n+3}$ is an identity morphism – i.e. the last diagram made by composing the vertices of the last associahedron and filling in with $\boldsymbol{\mathcal{M}}_{n+2}$ is required to commute.

The series of enriched k-cells $\boldsymbol{\mathcal{M}}_{k+1}$ fill in diagrams that are in the form of polytopes. We call these composihedra, and denote them by $\mathcal{CK}_i = s(\boldsymbol{\mathcal{M}}_i) \sqcup t(\boldsymbol{\mathcal{M}}_i)$ to symbolize their derivation from $\mathcal{K}_i$. A truncation of the polytope $\mathcal{CK}_i$ gives us the well known polytope $\mathcal{J}_i$ of the family that goes by the name of multiplihedra. The first few of the truncations needed to get from $\mathcal{CK}_i$ to $\mathcal{J}_i$ are shown below.



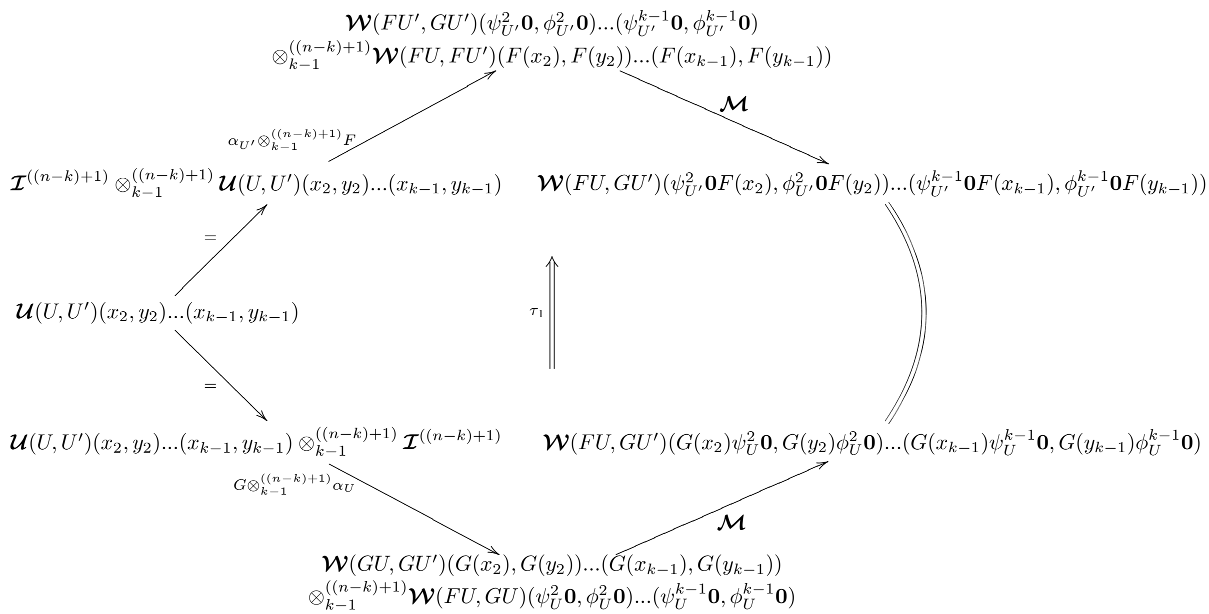
Before relating this description to those using operads, we describe the polytopes that are utilized in the axioms governing morphisms of 1-weak $\mathcal{V}$-n-Cat. For two 1-weak $\mathcal{V}$-n-categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a lax *$\mathcal{V}$-n-functor* $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$-$(n-1)$-functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U, U') \to \boldsymbol{\mathcal{W}}(TU, TU')$. We define a series of enriched k-cells ϕ_k that fill in a polytope diagram made by taking a right prism of the polytope $\mathcal{CK}_k$.

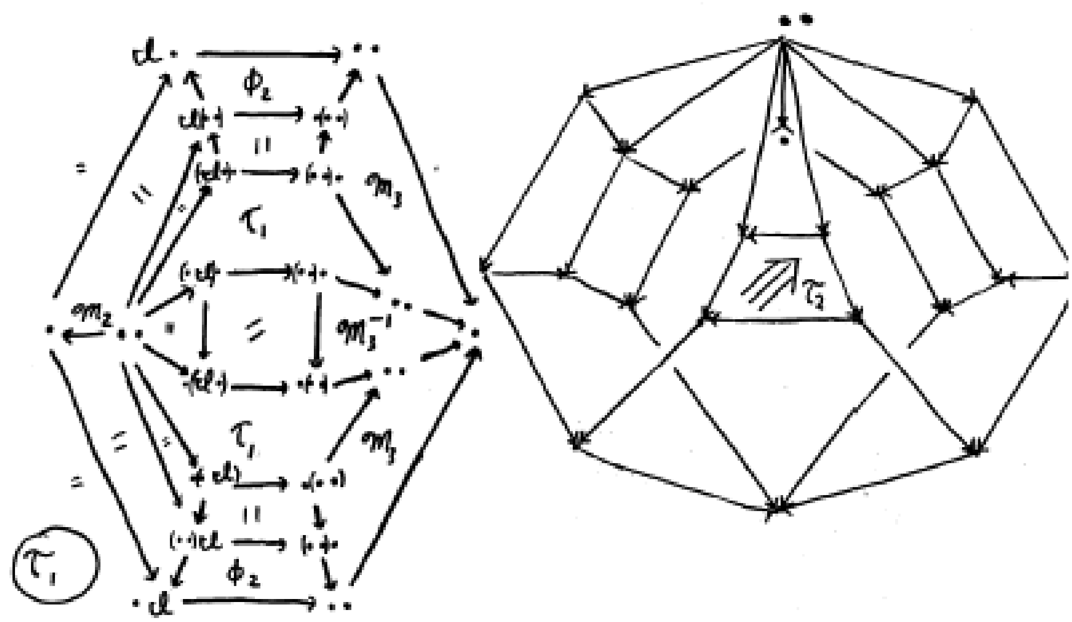
Since the $T_{UU'}$ are enriched functors the square they are usually required to satisfy exactly can instead be filled in with an (invertible) enriched n-natural transformation ϕ_2. Then we draw the pentagonal prism of $\mathcal{CK}_3$, faces filled in with $\boldsymbol{\mathcal{M}}_3$ on both pentagonal ends and with ϕ_2 on the sides. Then the prism itself is filled in with the enriched modification ϕ_3, and the process continues until ϕ_{n+1} is the identity enriched $(n+1)$-cell.



Domain and range drawings for ϕ_2 and ϕ_3. The second row consists of two drawings of the same polytope, the first flattened for clearer labelling. The circled copy of $\boldsymbol{\mathcal{M}}_3$ has domain and range on the perimeter of the planar figure.

All higher morphisms have a shared form of their axiomatic commuting diagram, as seen above in Definition 4.4. Thus only a single new sequence of polytopes is require to describe them. We will call this family the naturahedra, $\mathcal{N}_n$ and the enriched $n+1$-cell that fills the polytope diagram described by $\mathcal{N}_n$ will be called τ_n. $\mathcal{N}_n$ is formed by taking $\mathcal{CK}_{n-1}$ and using each hom-category in its vertices as a starting point, or 0-source, for τ_1. Here is shown the domain and range for τ_1 and τ_2.





Two pictures of the naturahedra that is formed from the domain and range of τ_2. The circled copy of τ_1 has domain and range on the perimeter of the planar figure. It reappears on the back side of the three dimensional drawing.

Since the multiplihedra are completely defined in [Iwase and Mimura, 1986] (see also [Markl, Shnider, and Stasheff, 2002]), we only need to prove that there is a parity structure on the composihedra following from the known such structure on the associahedra to ensure that the diagrams can be split into domain and range halves. This requirement is bypassed by an operad action definition to be introduced shortly, but may be needed in order to provide the basis for the weak enriched functors. The latter use a prism on the composihedra, which once a parity structure for the composihedra is established will have its own parity structure by a construction of Street. The majority of work that needs to be done to make these definitions complete is on the last definition of weak enriched k-cell between weakly enriched $(k-1)$-cells between ... weakly enriched n-categories. The naturahedra are not yet well understood at all. It is a major future project to completely describe their combinatorics and establish a parity structure.

Now we introduce an operad theoretic way to define the weak composition equivalent to though not as flexible as the description above using the polytopes $\mathcal{CK}_n$.

We define an operad of $\mathcal{V}$-n-categories (in $\mathcal{V}$-n-Cat) which we denote by $\overline{\mathcal{K}}$. This notation is due to the fact that we define $\overline{\mathcal{K}}(j)$ to be the $\mathcal{V}$-n-category generated by the (directed) Stasheff associahedra $\mathcal{K}_j$. Vertices are objects, and given two vertices there is a hom-$\mathcal{V}$-$(n-1)$-category of paths along edges. Since we are in $\mathcal{V}$-n-Cat the $n-3$ dimensional faces (including degeneracies) of $\mathcal{K}_j$ will correspond to hom-objects in $\mathcal{V}$ and here we require that those hom-objects be isomorphic to the unit $I \in \mathcal{V}$.

This actually forms a pseudo-operad, and the composition is inclusion just as for the operad of associahedra.

Then requiring that there be an operad action on composable strings of hom-categories as in:

$$\overline{\mathcal{K}}(j) \otimes^{(n)} \boldsymbol{\mathcal{U}}(X_{j-1}, X_j) \otimes^{(n)} \ldots \otimes^{(n)} \boldsymbol{\mathcal{U}}(X_0, X_1) \to \boldsymbol{\mathcal{U}}(X_0, X_j)$$

is equivalent to the previous description of filled polytope diagrams.

This operadic description should allow for comparisons – such as determining whether 1-weak $\mathcal{V}$-n-Cat is a subcategory of May's n-category, and for $\mathcal{V} = \mathbf{Set}$ whether 1-weak $\mathcal{V}$-n-Cat is a subcategory of Trimble's (floppy) n-category. Also closely related is Batanin's defintion of weak n-category based on internal and n-operads. It may be that his ideas include the concept of weak enrichment as an enriched version or special case.

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