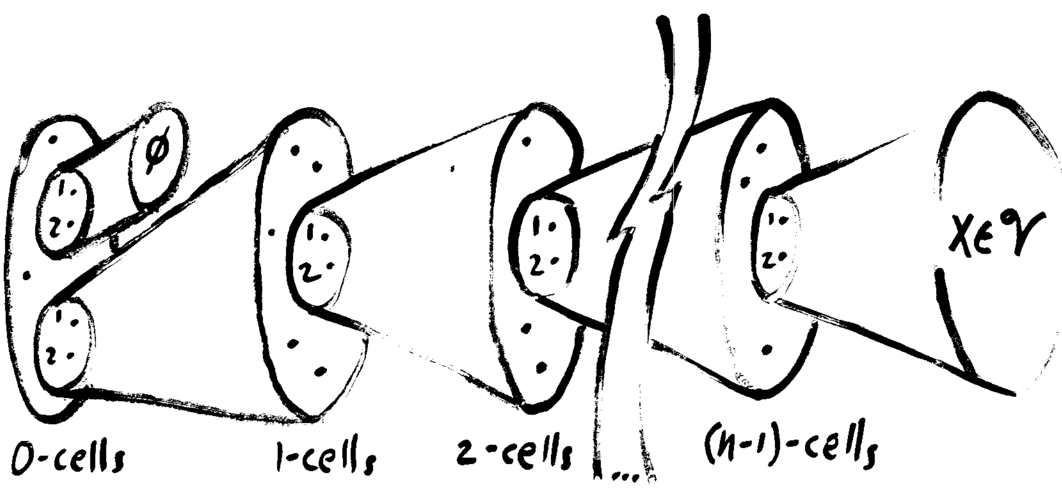
VARIETIES OF ITERATED ENRICHMENT

(as in the recursive def'n of Str-n-Cat)

- Enriching over k-fold monoidal $\mathcal{V}$
  - comparison to delooping
- Operads in k-fold monoidal $\mathcal{V}$
- Weak enrichment and weak morphisms



Enriched n-category over sym. $\mathcal{V}$

Take an n-fold category

$$(\mathcal{V}, \otimes_1, \otimes_2, \ldots, \otimes_n, I)$$

w/ interchange nat. transf's. η^{ij}, $1 \le i < j \le n$

$$\eta^{12}_{ABCD} : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D).$$

Enrich over $(\mathcal{V}, \otimes_1, I)$

Use $\otimes_i$, $i > 1$ to define $n-1$ products on $\mathcal{V}$-Cat.

$\mathcal{A} \otimes_i^{(1)} \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathcal{V}$-Cat

- objects $|\mathcal{A}| \times |\mathcal{B}|$
- hom objects

$$[\mathcal{A} \otimes_i^{(1)} \mathcal{B}]((A,B),(A',B'))$$
$$= \mathcal{A}(A,A') \otimes_{i+1} \mathcal{B}(B,B')$$

- composition M

use $\eta^{1(i+1)}$ (in place of braiding) to accomplish middle interchange

Thus we can iterate

Defn. (strict) $\mathcal{V}$-n-Cat $\equiv$ ($\mathcal{V}$-$(n-1)$-Cat)-Cat

Defn. Unit $\mathcal{V}$-n-category $\mathcal{I}^{(n)}$

- 1 object $\{0\}$
- $\mathcal{I}^{(n)}(0,0) = \mathcal{I}^{(n-1)}$
- $\mathcal{I}^{(0)} = I$

Results :

Thm. $\mathcal{V}$ k-fold monoidal

$\mathcal{V}$-n-Cat is a $(k-n)$-fold monoidal, $(n+1)$-category.

- objects $\mathcal{V}$-n-categories
- 1-cells $\mathcal{V}$-n-functors
- 2-cells $\mathcal{V}$-n-nat. transf's.

$\vdots$

$\mathcal{V}$-n:k-cells

Analogy w/ Delooping:

in ΩX points are paths in X
(objects) (1-cells)

$\Rightarrow$ "lose categorical dimension"

and we can multiply points by concatenating loops.

$\Rightarrow$ "gain a multiplication"

$B(\Omega X)$ reverses this picture.

1 step of enrichment $\mathcal{V}$-n-cat $\to$ $\mathcal{V}$-$(n+1)$-cat

- loses one product
- gains one categorical dimension

Problem:

we would like an explicit relationship between $\mathrm{Nerve}(\mathcal{V})$ and $\Omega^n \mathrm{Nerve}(\mathcal{V}\text{-}n\text{-}\mathrm{Cat})$.

Ex: group G

- $\overline{G}$: objects $\{g \in G\}$, morphisms 1_g only, monoidal $*$,
- a G-torsor P (over $\{\cdot\}$) is a $\overline{G}$-category tensored over $\overline{G}$. equivariant maps $P \to P'$ are enriched functors
- $\mathrm{Nerve}(\mathrm{Tor}\, G) = BG$
- $\mathrm{Nerve}(\overline{G}) = G$

To filter (definitions of) (weak) $\mathcal{V}$-n-Cat, we (often) need operads to live in $\mathcal{V}$ as well.

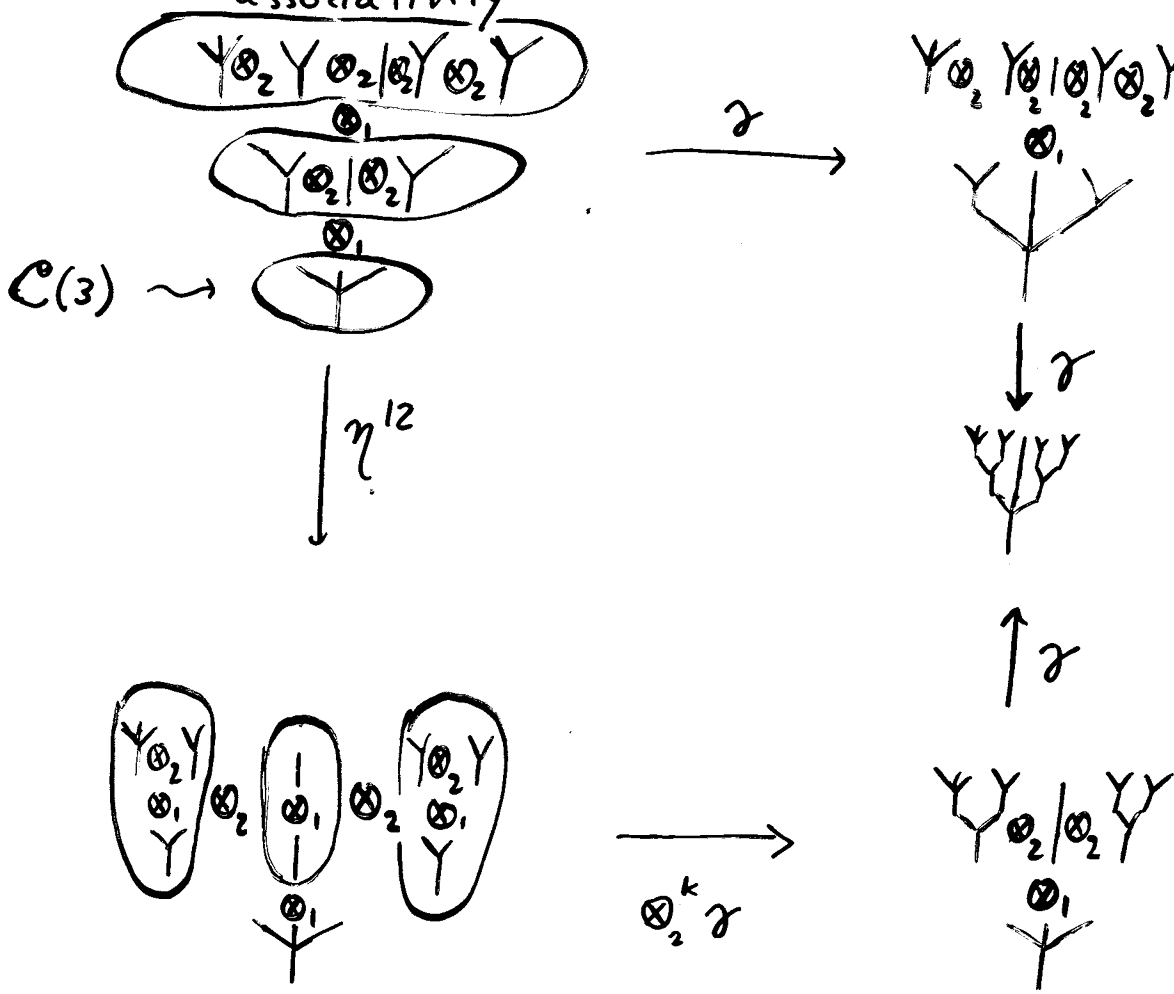
Defn. Operad $\mathcal{C}$ in n-fold $\mathcal{V}$

- objects $\mathcal{C}(j) \in \mathcal{V}$, $j \geq 0$

  Let $\Sigma j_i = j$

- morphisms $\gamma : \mathcal{C}(k) \otimes_1 \mathcal{C}(j_1) \otimes_2 \cdots \otimes_2 \mathcal{C}(j_k) \to \mathcal{C}(j)$

- associativity



(1-weak) - $\mathcal{V}$-(n+1)-Cat :

Defn. a <u>1-weak $\mathcal{V}$-(n+1)-category</u> $\mathcal{A}$

is <u>weakly</u> <u>enriched</u> over (strict) $\mathcal{V}$-n-Cat.

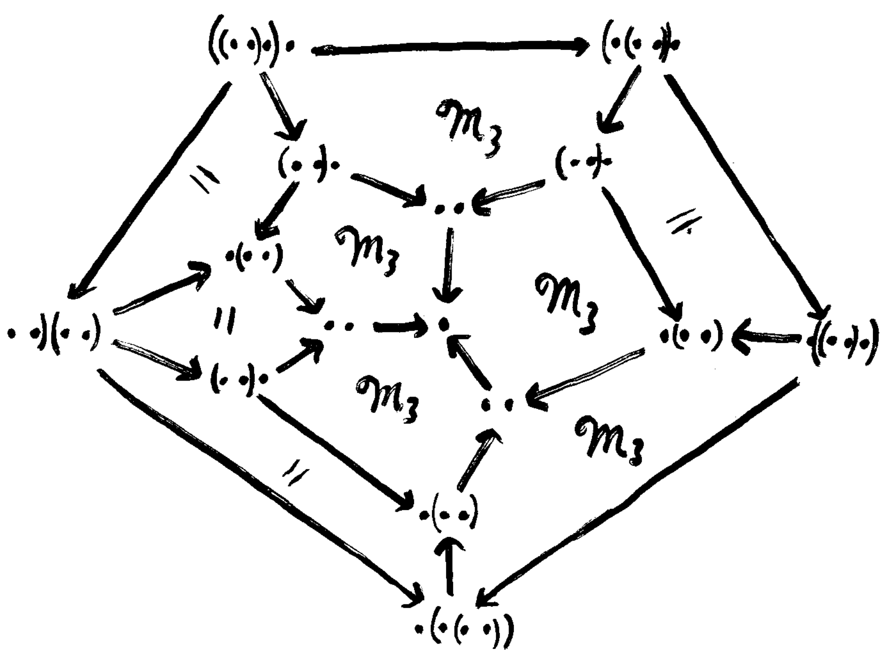
- object set $|\mathcal{A}|$
- for each $A, B \in |\mathcal{A}|$ a $\mathcal{V}$-n-category $\mathcal{A}(A,B)$
- for each $A,B,C \in |\mathcal{A}|$ a $\mathcal{V}$-n-functor (iso)

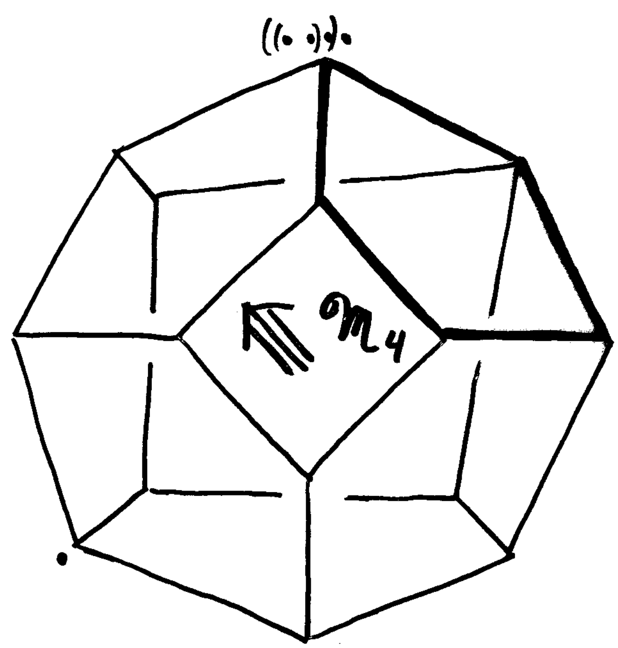
$$m_2 : \mathcal{A}(B,C) \otimes_1^{(n)} \mathcal{A}(A,B) \to \mathcal{A}(A,C) \quad \left(\begin{array}{c} \cdot\cdot \\ \downarrow m_2 \\ \cdot \end{array} \right)$$

- for each $A,B,C,D \in |\mathcal{A}|$ a $\mathcal{V}$-n-nat.transf. m_3 (iso)

$$\begin{tikzcd}
(\cdot\,\cdot)\cdot \arrow[r, "\alpha"] \arrow[d, "m_2 \otimes 1"'] & \cdot(\cdot\,\cdot) \arrow[d, "1 \otimes m_2"] \\
\cdot\,\cdot \arrow[r, phantom, "\Nearrow m_3"] \arrow[dr, "m_2"'] & \cdot\,\cdot \arrow[dl, "m_2"] \\
& \cdot
\end{tikzcd}$$

- for each $A,B,C,D,E \in |\mathcal{A}|$ a $\mathcal{V}$-n-modification m_4 (iso)





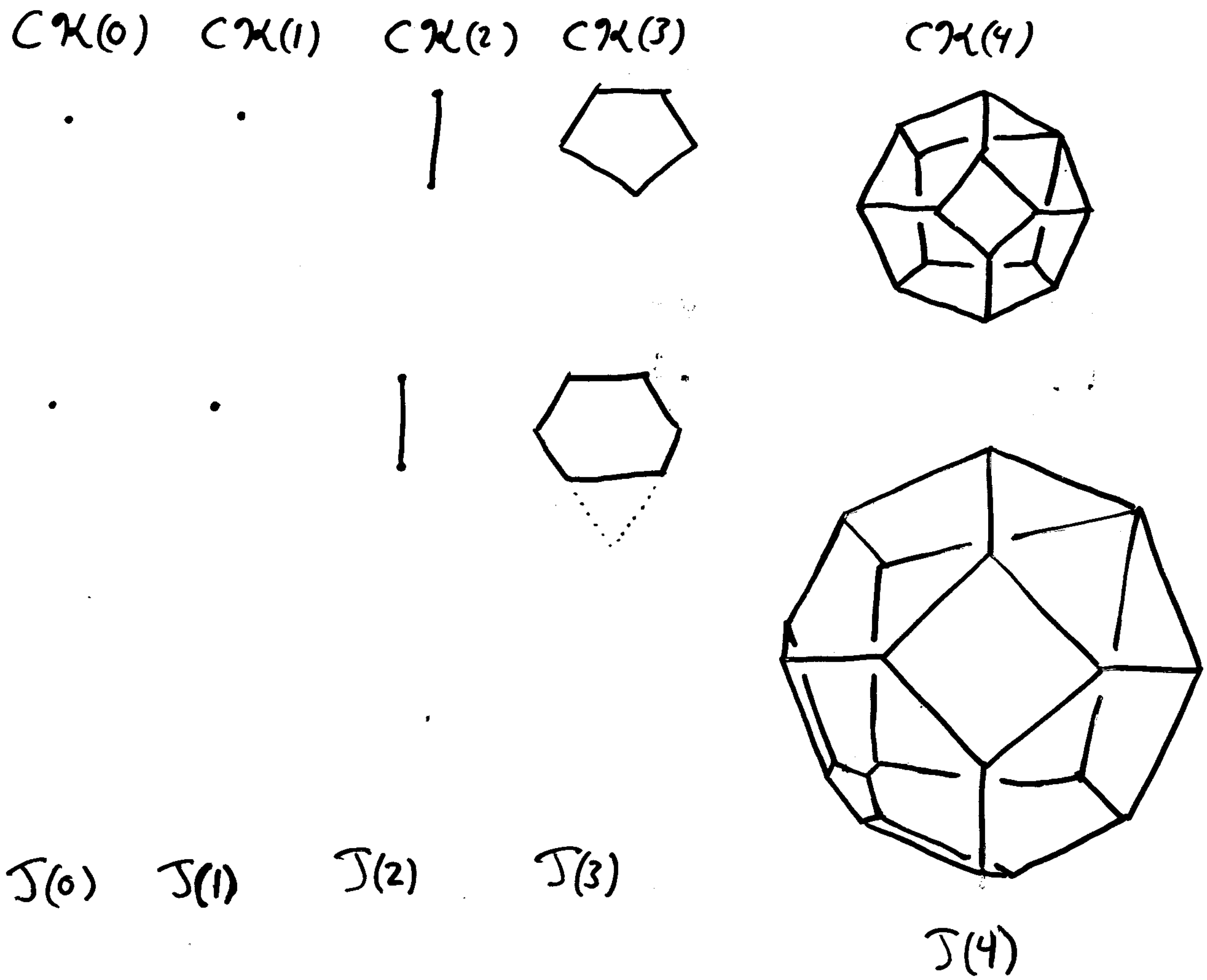
⋮

- for each $\{A_i\}_{i=1}^{n+4} \in |\mathcal{A}|$, m_{n+3} is an identity (n+2)-cell

$s(m_i) \sqcup t(m_i)$ is a polytope denoted $\mathcal{CK}(i)$; composihedra.

$s(\mathcal{M}_i)$ and $t(\mathcal{M}_i)$ glued form a polytope $\mathcal{CK}(i)$
(composihedra)

We can form multiplihedra $\mathcal{J}(i)$ from $\mathcal{CK}(i)$ by truncation.

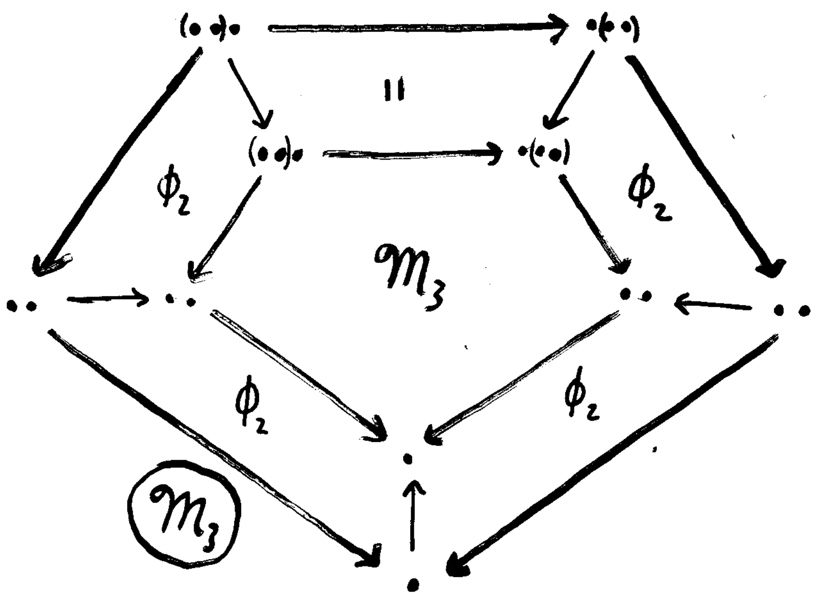


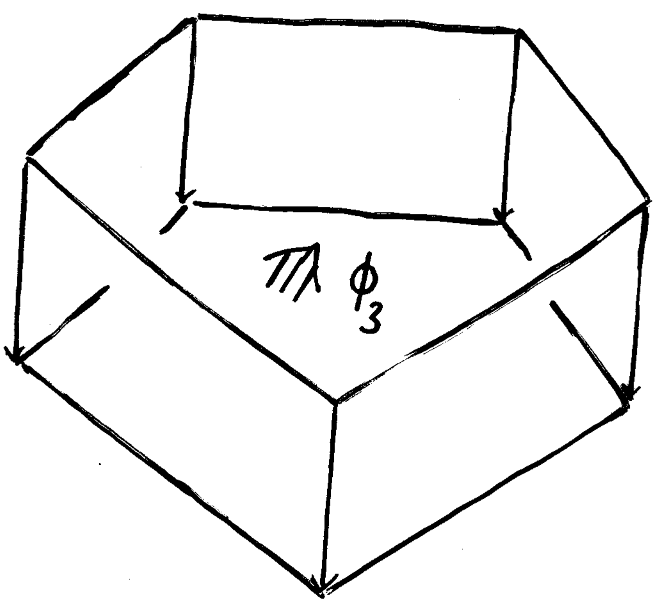
Defn. Weak $\mathcal{V}$-$(n+1)$ functor $F : \mathcal{A} \to \mathcal{B}$

- function $F : |\mathcal{A}| \to |\mathcal{B}|$
- for $A, B \in |\mathcal{A}|$ a $\mathcal{V}$-n-functor $F_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(FA, FB)$
- for $A, B, C \in |\mathcal{A}|$ a $\mathcal{V}$-n-nat. transf. ϕ_2 (iso)

$$\begin{tikzcd}
\bullet\bullet \arrow[r, "\mathcal{M}_2"] \arrow[d, "F \otimes F"'] \arrow[dr, phantom, "\Nearrow \phi_2"] & \bullet \arrow[d, "F"] \\
\bullet\bullet \arrow[r, "\mathcal{M}_2"'] & \bullet
\end{tikzcd}$$

- for $A, B, C, D \in |\mathcal{A}|$ a $\mathcal{V}$-n-mod. ϕ_3 (iso)





⋮

- for $\{A_i\}_{i=1}^{n+2} \in |\mathcal{A}|$ a $\mathcal{V}$-n:$(n+1)$-cell ϕ_{n+1} (iso) which fills a prism on $\mathcal{CK}(n+1)$.
- For $\{A_i\}_{i=1}^{n+3} \in |\mathcal{A}|$, ϕ_{n+2} is the identity.

Defn. Weak $\mathcal{V}$-$(n+1)$:k-cell $\beta : \Psi^{k-1} \to \Phi^{k-1} : \dots : F \to G : \mathcal{U} \to \mathcal{W}$

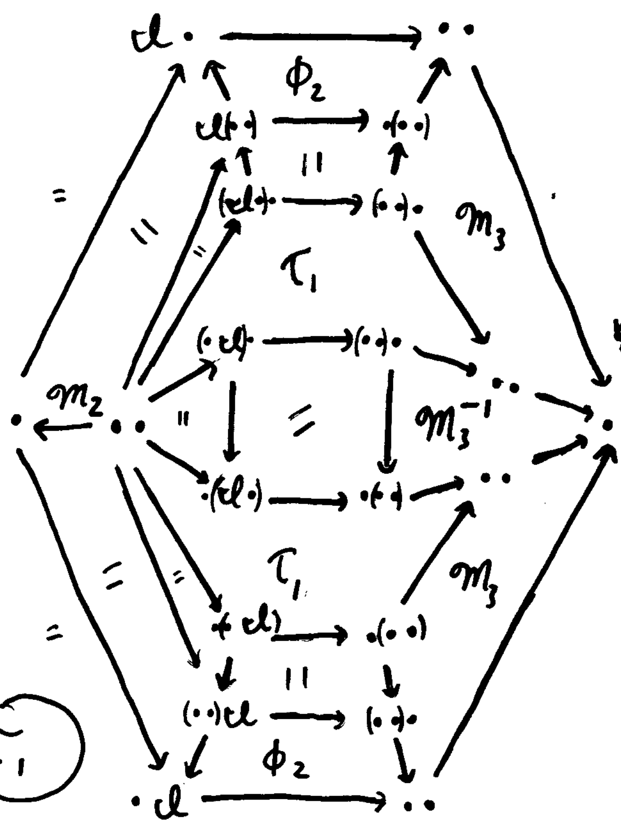
- for $U \in |\mathcal{U}|$ a $\mathcal{V}$-$(n-k+2)$-functor β_U

$$\beta_U : \mathcal{I}^{(n-k+2)} \longrightarrow \mathcal{W}(FU, GU)\dots(\Psi_U^{k-1} 0, \Phi_U^{k-1} 0)$$

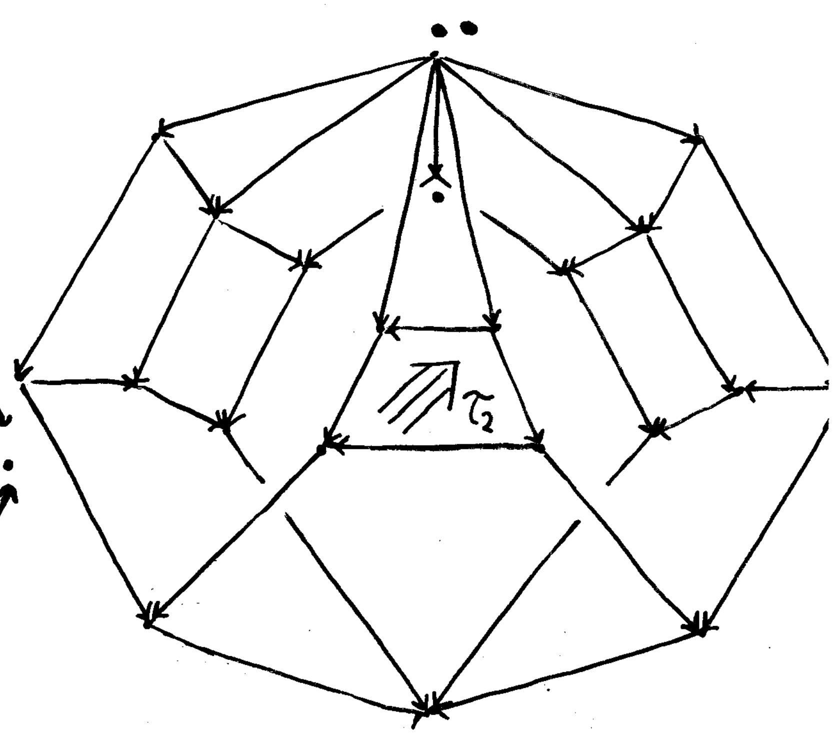
- for $U, U' \in |\mathcal{U}|$ a $\mathcal{V}$-$(n-k+2)$-nat. trans. τ_1 (iso.)

$$\begin{tikzcd}
 & \mathcal{I} \otimes \mathcal{U}(UU')\dots \arrow[r, "\beta \otimes F"] & \mathcal{W}(FU', GU')\dots \otimes \mathcal{W}(FUFU')\dots \arrow[dr, "m_2"] & \\
\mathcal{U}(U,U')\dots \arrow[ur, "="] \arrow[dr, "="'] & & \Uparrow \tau_1 & \mathcal{W}(FU, GU')\dots \\
 & \mathcal{U}(UU')\dots \otimes \mathcal{I} \arrow[r, "G \otimes \beta"'] & \mathcal{W}(GU, GU')\dots \otimes \mathcal{W}(FUGU)\dots \arrow[ur, "m_2"'] &
\end{tikzcd}$$

- for $U, U', U'' \in \mathcal{U}$ a $\mathcal{V}$-$(n-k+2)$-mod. τ_2 (iso.)



τ_1



⋮ for $\{U_i\}_{i=1}^{n-k+3}$, τ_{n-k+2} is the identity.