Some of the basic tools we are discussing here first arose in topology in 1960's: operads, coherence diagrams Stasheff, MacLane, Boardman & Vogt, May

Later in 1970's, same tools were exploited in the development of Quillen's algebraic K-theory

Dictionary

Monoidal categories ⟷ 1-fold loop spaces

Symmetric monoidal categories ⟷ Ω^∞-spaces

Braided monoidal categories $\longleftrightarrow \Omega^2$ spaces

Problem

$? \longleftrightarrow \Omega^n$-spaces

category $2 < n < \infty$

Pretty clear that morally speaking the requisite structure is some notion of iterated monoidal category

Unless one is careful however, n-fold monoidal categories degenerate into symmetric monoidal categories for $n > 2$

Balteanu, F, Schwänzl & Vogt

To simplify matters, considered strictly monoidal categories

Crucial idea: consider weakly monoidal functors

$$\eta_{A,B} : F(A \Box B) \longrightarrow F(A) \Box F(B)$$

natural transformation

<u>NOT</u> isomorphism

$\eta_{A,B}$ satisfies associativity & unit condition

Resulting category of strictly monoidal categories & weakly monoidal functors called MonCat

·has products

Now start iterating:

(n+1)-fold monoidal category $\overset{\text{def}}{=}$ (strict) monoid in category of n-fold monoidal categories & weakly monoida functors

Unraveling this we get the following explicit definition

An n-fold monoidal category is a category $\mathcal{C}$ with

$$\Box_1, \Box_2, \ldots, \Box_n : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

strictly associative, having a common unit object 0

Also for each $1 \leq i < j \leq n$ there is an interchange transformation

$$\eta^{ij}_{ABCD} : (A \Box_j B) \Box_i (C \Box_j D) \longrightarrow (A \Box_i C) \Box_j (B \Box_i D)$$

Satisfy two associative, unit laws

η_{ij} - natural transformation
which makes $\Box_j$ weakly
monoidal with respect
to $\Box_i$

one associative, unital law
comes from conditions
nat. transformation for
weakly monoidal functor $\Box_j$

other associative, unital law
comes from requirement
that $\Box_j$ is strictly associative
functor in (j-1)-MonCat

Finally for each $1 \le i \le j \le k \le n$ commutative hexagon

$$\begin{tikzcd}
 & {[(1 \square_k 2) \square_j (3 \square_k 4)] \square_i [(5 \square_k 6) \square_j (7 \square_k 8)]} \arrow[dl, "\eta_{jk}"'] \arrow[dr, "\eta_{ij}"] & \\
{} \arrow[d, "\eta_{ik}"] & & {} \arrow[d, "\eta_{ik}"] \\
{} \arrow[dr, "\eta_{ij}"] & & {} \arrow[dl, "\eta_{jk}"] \\
 & {[(1 \square_i 5) \square_j (3 \square_i 7)] \square_k [(2 \square_i 6) \square_j (4 \square_i 8)]} &
\end{tikzcd}$$

Remark If we had insisted that η_{ij} be natural isomorphisms then Nerve $(\mathcal{M}_n(\,))$ would have wrong homotopy type: Nerve $\left(\begin{matrix}\text{category of}\\ \text{isos}\end{matrix}\right)$

$$= \coprod K(\pi, 1)$$

$\mathcal{C}_n$ has this homotopy type only for $n = 1, 2, \infty$

$\mathcal{M}_n(k) \subseteq$ free n-fold monoidal category on $\{1, 2, \ldots, k\}$

- full subcategory whose objects are algebraic expressions in $\{1, 2, \ldots, k\}$, $\{\Box_1, \Box_2, \ldots, \Box_n\}$ where each $1, 2, \ldots, k$ appears exactly once.

E.g. $[(3 \Box_2 2) \Box_1 (1 \Box_3 4) \Box_2 6] \Box_3 5$

$\in \mathcal{M}_3(6)$

$(1 \Box_3 2) \Box_2 (1 \Box_1 3) \notin \mathcal{M}_3(3)$

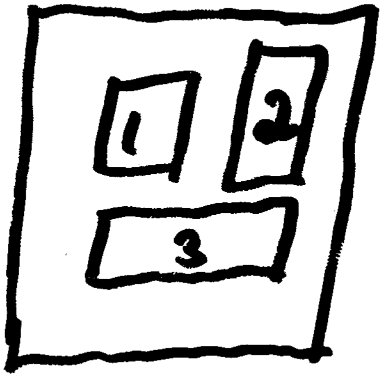
morphisms - built out of

η_{ij}

We would like to show that operad $\mathcal{M}_n = \{\mathcal{M}_n(k)\}_{k \geq 0}$ which characterizes n-fold monoidal categories has a nerve which is a topological operad of the same homotopy type as $\mathcal{C}_n = \{\mathcal{C}_n(k)\}_{k \geq 0}$ little n-cubes operad which acts on n-fold loop spaces

Little n-Cubes Operad

$\mathcal{C}_n(k)$ – space of all k-fold configurations of subcubes of $\underline{I}^n$, interiors of subcubes are disjoint



"subcubes" are labelled with $\{1, 2, \dots, k\}$

$$\mathcal{C}_n(k) \times (\Omega^n X)^k \longrightarrow \Omega^n X$$

$$(c, w_1, w_2, \dots, w_k) \longmapsto \lambda$$

$\lambda|_{i\text{-th subcube of } c} = w_i$ $\quad \lambda|_{\text{exterior}} = *$

First observe $\mathcal{M}_n(2)$ has right homotopy type

$\mathcal{M}_3(2)$:

$$\begin{tikzcd}
 & & 1\square_3 2 & & \\
 & & 1\square_2 2 \arrow[u] \arrow[dd, dash] & & \\
1\square_1 2 \arrow[uurr] \arrow[urr] \arrow[drr] \arrow[ddrr] & & & & 2\square_1 1 \arrow[uull] \arrow[ull] \arrow[dll] \arrow[ddll] \\
 & & 2\square_2 1 \arrow[uuu, bend left] \arrow[d] \arrow[d, dotted, dash, shift left] & & \\
 & & 2\square_3 1 & &
\end{tikzcd}$$

$\mu^{ij}_{A,B,C,D}$

$: (A\square_j B)\square_i(C\square_j D)$

$\to (A\square_i C)\square_j(B\square_i D)$

$\cong S^2 \simeq \mathcal{C}_3(2)$

$\mathcal{M}_n(2)\cong S^{n-1}\simeq \mathcal{C}_n(2)$

First need a better description of $\mathcal{M}_n(k)$ "Coherence Theorem"

If $\{a,b\} \subseteq \{1,2,\dots,k\}$

there is a natural functor

$$\mathcal{M}_n(k) \longrightarrow \mathcal{M}_n(\{a,b\}) \cong \mathcal{M}_n(2)$$

$$i \longmapsto \begin{cases} i & \text{if } i=a \text{ or } i=b \\ 0 & \text{otherwise} \end{cases}$$

Coherence Theorem

$$\mathcal{M}_n(k) \longrightarrow \prod_{\{a,b\}} \mathcal{M}_n(2)$$

is an imbedding of posets

Basic Idea to show $\mathcal{M}_n$ & $\mathcal{C}_n$ have same homotopy type: "cellular decomposition" (Berger)

Definition X top. space has a "cellular" decomposition based on a finite poset $\mathcal{P}$ if to each $P \in \mathcal{P}$, there is associated a contractible closed subspace $F(P) \subseteq X$

(1) $P \leq Q \Rightarrow F(P) \subseteq F(Q)$ cofibration

(2) $F(P) \cap F(Q) = \bigcup_{\substack{R \leq P \\ R \leq Q}} F(R)$

(3) $X = \bigcup F(P)$

Theorem

$$\begin{tikzcd}
X \cong \operatorname{colim}_{\mathcal{P}} F & \operatorname{hocolim}_{\mathcal{P}} F \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \operatorname{nerve}(\mathcal{P}) \arrow[d, equal] \\
 & & \operatorname{hocolim}_{\mathcal{P}} *
\end{tikzcd}$$

Theorem: $\operatorname{nerve}(\mathcal{M}_n)$ is a top. operad of same homotopy type as $\mathcal{C}_n$

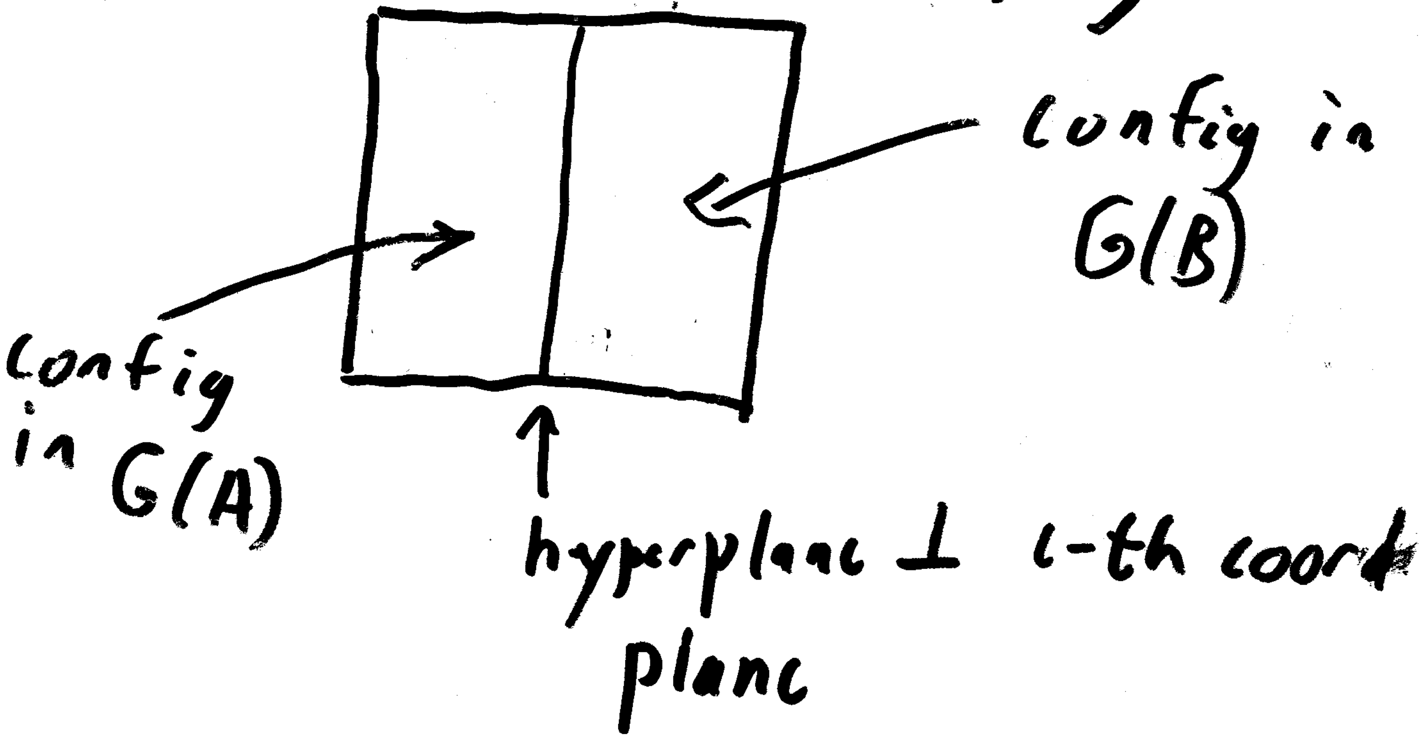
Proof Sketch Find a cellular decomposition of $\mathcal{C}_n(k)$ based on poset $\mathcal{M}_n(k)$

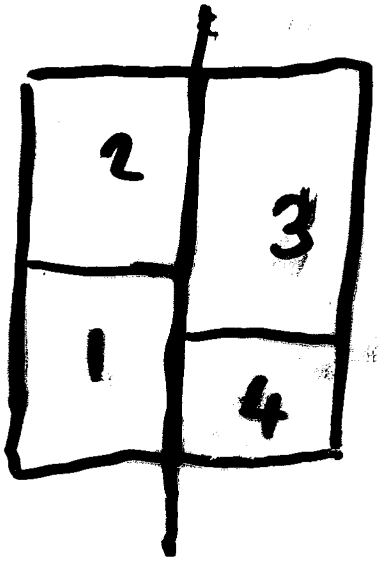
First define a preliminary decomposition

$$G : \mathrm{obj}\, \mathcal{M}_n(k) \longrightarrow \text{closed subsets of } \mathcal{C}_n(k)$$

defined recursively

$G(A \,\square_i\, B) =$ configurations of little n-cubes satisfying



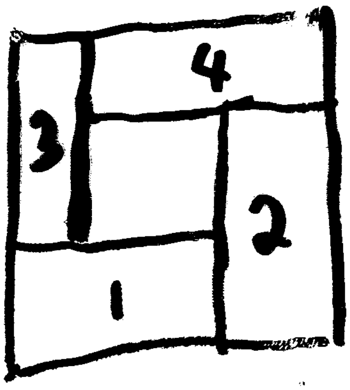


$$\in G\Big(\overset{\overset{\mathcal{M}_2(4)}{\downarrow}}{(}1 \square_2 2) \square_1 (4 \square_2 3)\Big)$$

Then define

$$F(A) = \bigcup_{B \leq A} G(B)$$

$$\bigvee_{A \in \mathcal{M}_n(k)} F(A) = \underset{\overset{\uparrow}{\text{decomposable configuration}}}{\mathcal{D}_n(k)} \overset{\sim}{\subseteq} \mathcal{C}_n(k)$$



indecomposable

$$\begin{tikzcd}
\mathrm{nerve}\, \mathcal{M}_n & \operatorname{hocolim}_{\mathcal{M}_n} F \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
 & \operatorname{colim}_{\mathcal{M}_n} F \arrow[d, equal] \\
 & \mathcal{D}_n \overset{\simeq}{\subseteq} \mathcal{C}_n
\end{tikzcd}$$

chain of equivalences of operads