# MORE ON DIRECTED TOPOLOGY AND CONCURRENCY TALK AT IMA, MINNEAPOLIS, JUNE 17 2004

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This talk was the second part of an introduction to geometric models in concurrency. The first part was given by Eric Goubault.

The slides were not meant to stand alone, so here is a short introduction/explanation to them.

## 1. Higher Dimensional Automata, HDA

These were introduced in 1991 by Vaughan Pratt. They model *true concurrency*, i.e., they distinguish the random choice between "a goes before b" and "b goes before a" from "a and b can go together - they are independent. The actions of one program are edges in a graph. A concurrent program is a directed semi cubical complex, where the interior of an $n$-cube models that the actions on the edges of the cube are independent.

This is both an algebraic and a geometric model.

## 2. Local po-spaces

There are several ways of modelling a topological space with a local "direction". If there are no loops, a po-space is the natural choice. In the case with loops, one model is a local po-space. Other models are d-spaces (Marco Gandis), which are topological spaces where a subset of the set of paths is chosen as the directed paths. This subset should be closed under concatenaton and subdivision and also contain the constant paths.

The geometric realization of a higher dimensional automaton is a local po-space and also a d-space in a natural way.

Lcoal po-spaces have been studied in general; and as with topological spaces, it turns out, that there are too many pathologies to give interesting results, so one has to restrict to subcategories with nicer properties.

Studying the subcategory of HDA's and sometimes even the subcategory of loop free HDA's is also meaningful from the point of view of applications to computer science.

There are many examples of local po-spaces. For instance projective spaces are geometric realizations of a semi cubical complex with an induced po-space structure. In particular, having a local po-structure does not induce an orientation.

## 3. From discrete to continuous and back again

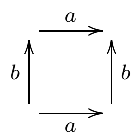
Translating from the algebraic HDA to the geometric realization introduces infinitely many states. This is of course not useful from a computer science point of view. We want to return to the discrete/algebraic setting via invariants and then hopefully have a smaller number of states or execution paths in the new model.

Reducing the state space is via diconnected components. Reducing the number of execution paths is via $\vec{\pi}_1$, dihomotopy classes of dipaths. An execution is a dipath in the HDA, and executions are equivalent if the corresponding dipaths are dihomotopic.

## 4. Dihomotopy in the special case of an HDA

There are several notions of equivalence of dipaths. Let $\gamma_1$ and $\gamma_2$ be dipaths in $X$ from $p$ to $q$. Then they are equivalent if

- There is a dimap $H : I \times \vec{I} \to X$ which is a usual homotopy with fixed endpoints between $\gamma_1$ and $\gamma_2$
- They are equivalent in the equivalence relation generated by homotopies through dimaps $H : \vec{I} \times \vec{I} \to X$
- If $p$ and $q$ are vertices in the cubical complex and $\gamma_i$ are dipaths on the 1-skeleton, then they are equivalent in the equivalence relation generated by elementary dihomotopies: In a 2-cube




$ab = ba$

In the geometric realization of an HDA, all these notions are the same.

This is not a general fact for (local) po-spaces: Consider a globe

$$\vec{I} \times Y/(0, y) \sim *_1, (1, y) \sim *_2$$

i.e, an unreduced suspension with direction up the suspension coordinate. If $Y$ is connected, then there is only one dihomotopy class of dipaths from $*_1$ to $*_2$ in the first definition. In the second definition, there are as many dihomotopy classes as there are points in $Y$.

## 5. Higher dimensional dihomotopy groups

We define for a pair of basepoints $x_0, x_1 \in X$

$$\vec{\pi}_n (X, x_0, x_1) = \vec{\pi}_1 (X^{S^{n-1}}, x_0, x_1)$$

That is, dipaths in the free $n-1$ loop space; from the constant $n-1$ loop $x_0$ to the constant $n-1$ loop $x_1$. The partial order on the loop space is induced from the po on $X$ - see the slide.

We get a composition by composing dipaths in the loopspace.

There is another composition - "perpendicularly" to the first one, when we pick a base point $*$ in $S^{n-1}$ (but still study the free loopspace). Then the elements of $\vec{\pi}_n (X, x_0, x_1)$, $f_t : S^{n-1} \to X$, for which the base point of $S^{n-1}$ traverses a fixed dipath $\alpha : \vec{I} \to X$, i.e., $f_t(*) = \alpha(t)$ can be composed by $[f_t] \cdot [g_t] = [f_t \vee g_t]$ using the usual product in $\pi_{n-1}(X, \alpha(t))$.

There are some natural questions arising:

- How do we keep track of reparametrizations of $\alpha$? (An operad structure?)
- What about other representatives of the dihomotopy class of $\alpha$? (Another operad?)
- The two compositions interact like the globular pasting diagrams. Can (weak) $n$-categories give information here?
- Is there an $\omega$-category underneath? As in the non directed case?