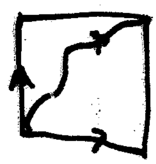
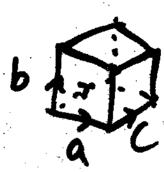
HDA, higher dimensional automata.
(V. Pratt, 91)

An algebraic <u>and</u> geometric model for concurrency

An n-cube represents n actions which can be performed concurrently

a and b can be executed "at the same time"





if the cube is "filled in", a, b, c can all go together

if it is <u>not</u> the full cube, at most 2 can go together

Glueing cubes gives

A semi-cubical complex $\xrightarrow[\text{Realization}]{\text{Geo.}}$ A local po-space

Definition: A po-space is a topological space, X, with a partial order $\leq$, s.t.

$\{(x,y) \mid x \leq y\}$ is closed in $X \times X$

Definition A local po-space is a topological space, X, which is Hausdorff. Together with an open covering $\mathcal{U} = \{(U_i, \leq_{U_i}), i \in J\}$ of po-spaces s.t.

For all $x \in \cancel{X}$ there is a nonempty open po-neighborhood $(W_x, \leq_{W_x})$ s.t.

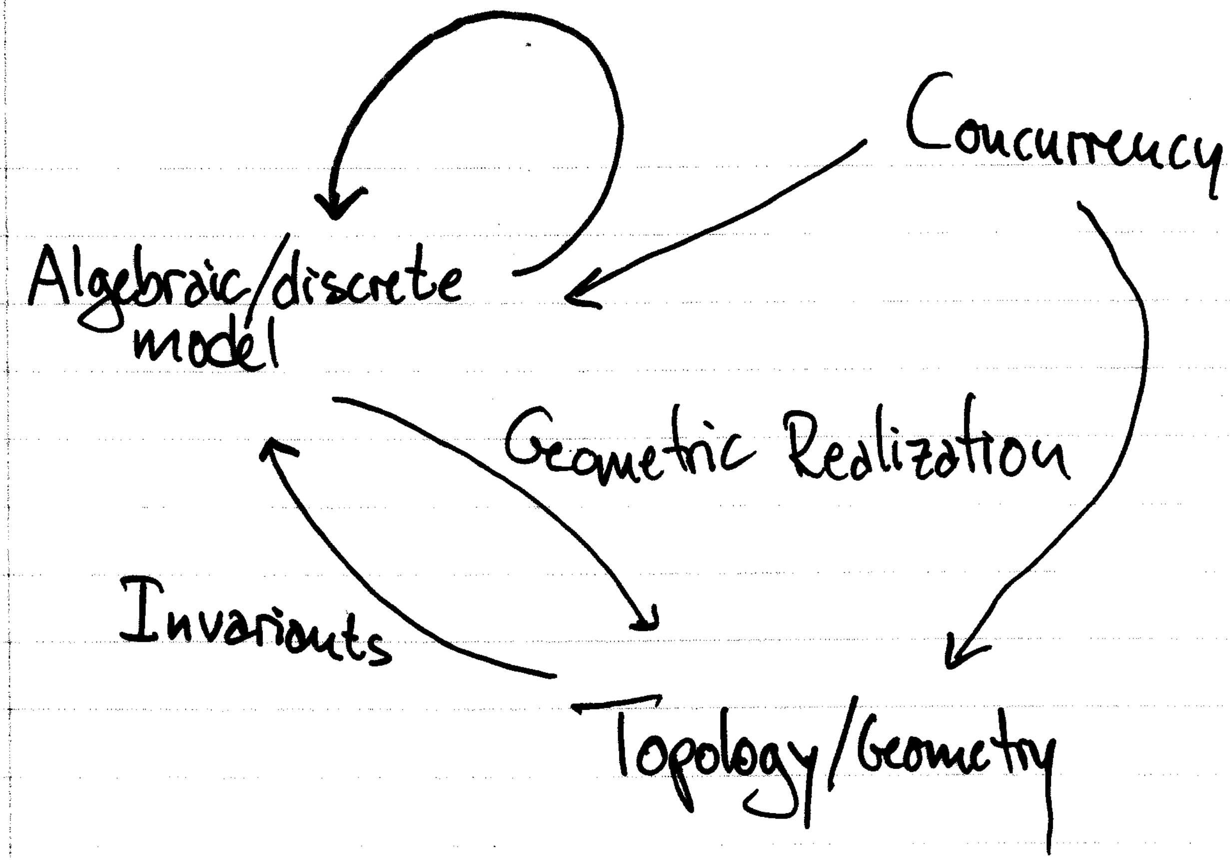
$$x, y, z \in U_i \cap W_x \Rightarrow (y \leq_{U_i} z \Leftrightarrow y \leq_{W_x} z)$$

Two local partial orders $\mathcal{U}, \mathcal{V}$ are equivalent if their union is a local partial order

Definition A map $f: X \to Y$ of local po-spaces is a dimap if

- f is continuous
- $\forall x \in X \; \exists \, W_x, W_{f(x)}$, po-neighborhoods s.t.

$$z, w \in f^{-1}(W_{f(x)} \cap W_x), \; z \leq_{W_x} w \Rightarrow f(z) \leq_{W_{f(x)}} f(w)$$


Concurrency
Algebraic/discrete model
Geometric Realization
Invariants
Topology/Geometry

Definition: A semi-cubical complex, M, is a sequence of sets and maps

$$\Rightarrow M_n \Rightarrow M_{n-1} \Rightarrow \cdots \Rightarrow M_1 \Rightarrow M_0$$

each arrow $\Rightarrow$ represents $2n$ face maps

$$\partial_i^k : M_n \to M_{n-1}, \quad i \in \{1, \_, n\}, \ k \in \{0,1\}$$

satisfying

$$\partial_i^k \partial_j^l = \partial_{j-1}^l \partial_i^k \quad (i<j)$$

A geometric semicubical cx. has the following property:

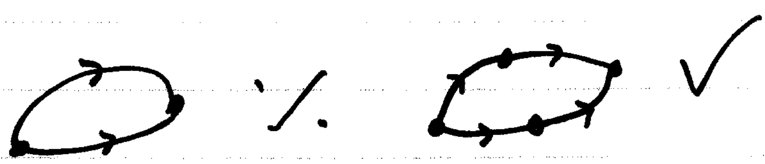
For each pair of "cubes"

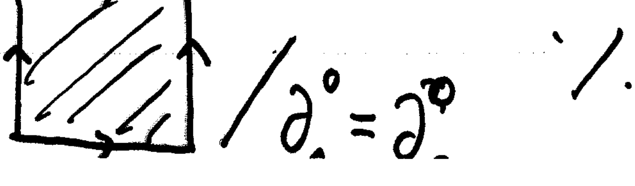
$$L_n \in M_n, \quad K_m \in M_m$$

there is a unique maximal common face $F_k \in M_k$ s.t.

$$\partial L_n \cap \partial K_m = \partial F_k$$

set of iterated faces.





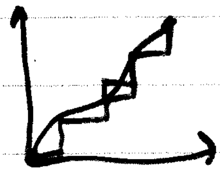
$/ \partial_1^0 = \partial_2^0$

Combinatorial $\longleftrightarrow$ Geometric

<u>Theorem</u>: Let X be the geometric realization of a semi cubical complex. (locally finite)

$$\gamma : \vec{I} \longrightarrow X \qquad \gamma(0) = v_0 \in X_0$$
$$\gamma(1) = v_1 \in X_0$$

Then γ is dihomotopic to a combinatorial dipath (a dipath in $X_0 \cup X_1$)

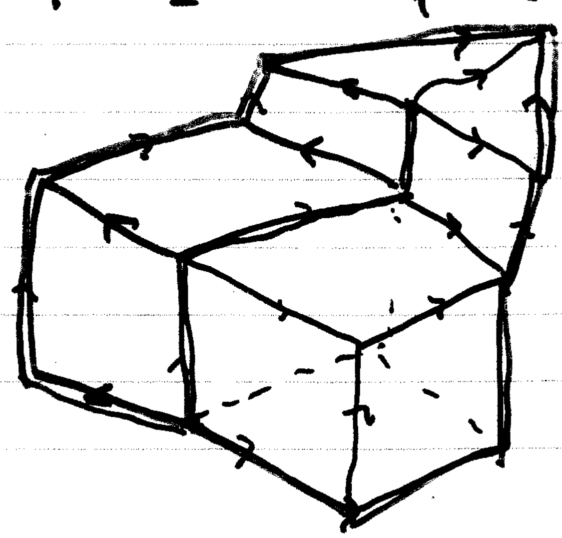


<u>Theorem</u>: Let X be the geometric realization of a geometric semi cubical cx. Then

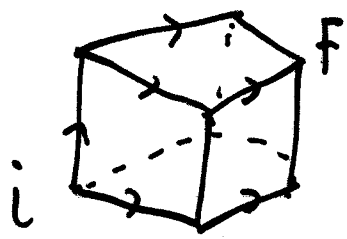
$$\gamma_i : \vec{I} \to X \quad \gamma_i(0) = v_0 \,,\ \gamma_i(1) = v_1 \quad i \in \{1,2\}$$

- γ_1 is d-equiv to $\gamma_2 \Leftrightarrow \gamma_1$ is dihomotopic to γ_2 $(\gamma_1 \vec{\simeq} \gamma_2)$
- If γ_1 and γ_2 are combinatorial, then

$$\gamma_1 \vec{\simeq} \gamma_2 \Leftrightarrow \gamma_1 \text{ is comb. equiv to } \gamma_2$$



Higher $\vec{\pi}_n$



$\vec{\pi}_1(X, i, f) = \{*\}$

Surface of a cube

$\vec{\pi}_2$ should be nontrivial

constant maps from S^{n-1}

$$\vec{\pi}_n(X, x_0, x_1) = \vec{\pi}_1(X^{S^{n-1}}, x_0, x_1)$$

Two compositions

$f, g : S^{n-1} \to X$

$f \le g \iff \forall x \in S^{n-1}\ f(x) \le g(x)$

$$\vec{\pi}_n(X, x_0, x_1) \times \vec{\pi}_n(X, x_1, x_2) \to \vec{\pi}_n(X, x_0, x_2)$$

$* \in S^{n-1}$, basepoint, $\alpha : \vec{I} \to X$, $\alpha(0) = x_0$
$\alpha(1) = x_1$

$$\vec{\pi}_n(X, \alpha, x_0, x_1) = \vec{\pi}_1\left((X, \alpha)^{(S^{n-1}, *)}, x_0, x_1\right)$$

$f_t(x) = \alpha(t)$

$$\vec{\pi}_n(X, \alpha, x_0, x_1) \times \vec{\pi}_n(X, \alpha, x_0, x_1) \to \vec{\pi}_n(X, \alpha, x_0, x_1)$$

$$[f_t], \quad [g_t] \longrightarrow [\underbrace{f_t * g_t}_{\text{as in } \pi_{n-1}}]$$

GROUPSTRUCTURE!