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Notes on 1- and 2-gerbes

A word of warning in lieu of introduction:

The aim of the following notes is to discuss in an informal manner the construction and some properties of 1- and 2-gerbes. The material pertaining to the construction of the associated cocycles is mainly based on the author's texts [1] and [2]. A notable improvement here is that some diagrams in [2] have been refomulated here as (hyper)cubical diagrams, which mirror in the present context certain diagrams introduced by W. Messing and the author in [3].

Since the concepts discussed are very general, it has at times not been made explicit to precisely which mathematical objects they apply. For example, when we refer to "a space" this might mean a topological space, but also "a scheme" when one prefers to work in an algebro-geometric context. Similarly, in computing cocycles, we will always refer to spaces X endowed with a covering $\mathfrak{U} := (U_i)_{i\in I}$, but the entire discussion remains valid when $\coprod_i U_i$ is replaced by a covering morphism $Y \longrightarrow X$ in an appropriate Grothendieck topology.

Finally, there has been no attempt at a serious bibliography, or at making careful attributions of the results mentioned.

1. **Torsors and bitorsors**

Let G be a bundle of groups on a space X.

Definition 1.1. *A right principal G-bundle (or right G-torsor) on X is a space $P \xrightarrow{\pi} X$ above X, together with a right group action $P \times_X G \longrightarrow P$ of G on P such that the induced morphism*

$$\begin{array}{ccc} P \times_X G & \simeq & P \times_X P \\ (p, g) & \mapsto & (p, pg) \end{array} \tag{1}$$

is an isomorphism. In addition, we require that there exist a family of local sections $s_i : U_i \longrightarrow P$ of π, for some open cover $\mathfrak{U} = (U_i)_{i\in I}$ of X.

Isomorphism classes of G-bundles on X are classified by the degree 1 cohomology set $H^1(X, G)$. This set has a distinguished element, the class of the trivial G-bundle T_G. A principal G-bundle is isomorphic to T_G if and only if it has a global section $s : X \longrightarrow P$.

Definition 1.2. *Let X be a space, and G and H a pair of bundles of groups above X. An (H, G)-bitorsor E on X is an X-space on X, together with a left principal action of H and a right principal action of G on E, which commute with each other.*

We refer to a (G, G)-bitorsor simply as a G-bitorsor.

Examples: *i*) The trivial G-bitorsor on X.

ii) A right G-torsor P on X is an (P^{ad}, G)-bitorsor, where $P^{\mathrm{ad}} := \mathrm{Aut}_G P$ is the gauge group bundle of P.

iii) Let

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1 \tag{2}$$

be a short exact sequence of groups. Then H is a G-bitorsor on K.

In particular, a right G-torsor P is an (H, G)-bitorsor if and only if the gauge group bundle P^{ad} of P is isomorphic to H.

Let Q be an (H, K)-bitorsor, and P be a (G, H)-bitorsor on X. The contracted product

$$P \wedge^{H} Q := \frac{P \times_X Q}{(ph, q) \sim (p, hq)} \tag{3}$$

is a (G, K)-bitorsor on X. In particular, for a given bundle of groups G on X, the category (resp. stack) of G-bitorsors is a grouplike (one also says groupal) monoidal category (resp. stack) on X.

Twisted objects:

Let P be a right G-torsor on X, and E an X-object on which G acts on the left. We say that the X-object ${}^{P}E := P \wedge^{G} E$, defined as in (3), is the P-twisted form of E. The choice of a local section of P above an open set U determines an isomorphism ${}^{P}E_{|U} \simeq E_{|U}$.

Conversely, if E_1 is an X-object for which there exist a open cover $\mathcal{U}$ of X over which E_1 is locally isomorphic to E, then the sheaf $\mathrm{Isom}(E,\, E_1)$ is a right-torsor on X under the X-group $G := \mathrm{Aut}_X E$.

Proposition 1.3. *These two constructions are inverse to each other.*

Example: A rank n vector bundle $\mathcal{V}$ on X is locally isomorphic to the trivial bundle $\mathbb{R}^n_X := X \times \mathbb{R}^n$, whose group of automorphisms is the trivial bundle of groups

$$GL(n, \mathbb{R})_X := GL(n, \mathbb{R}) \times X$$

on X. The associated right $GL(n, \mathbb{R})_X$-torsor on X is the bundle of frames $P_{\mathcal{V}} := \mathrm{Isom}(\mathbb{R}^n_X, \mathcal{V})$ of $\mathcal{V}$, and the vector bundle $\mathcal{V}$ may be recovered from the bundle of frames $P_{\mathcal{V}}$ as

$$\mathcal{V} \simeq P_{\mathcal{V}} \wedge^{GL(n,\mathbb{R})_X} \mathbb{R}^n_X \,,$$

in other words as the $P_{\mathcal{V}}$-twist of the trivial vector bundle $\mathbb{R}^n_X$ on X.

Cocyclic description of a bitorsor:

Consider an (H, G)-bitorsor P on X. Let us **choose** a familly of local sections $s_i : U_i \longrightarrow P$ of P, for some open cover $\mathcal{U} = (U_i)_i \in I$. The associated G-valued 1-cocycles g_{ij} for the underlying right G-torsor, are defined, as usual, by the equations

$$s_j = s_i \, g_{ij} \, .$$

The H-torsor structure on P is then described by the family of U_i isomorphisms $u_i : H_{| \, U_i} \longrightarrow G_{| \, U_i}$ defined by

$$h_i \, s_i = s_i \, u_i(h_i)$$

The pair $(u_i, \; g_{ij})$ satisfies the cocycle conditions

$$\begin{cases} g_{ij} \, g_{jk} & = g_{ik} \\ i_{g_{ij}} \, u_j & = u_i \end{cases} \tag{4}$$

Remark 1.4. When P is a G-bitorsor, the u_i take their values in the group $\mathrm{Aut}(G)$, and the pair of equations (4) define a 0-cocycle of an open cover $\mathcal{U}$ of X, with values in the crossed module $G \longrightarrow \mathrm{Aut}(G)$. The cocycle pair $(u'_i,\ g'_{ij})$ corresponding to the choice of another family of local sections s'_i is cohomologous in an appropriate sense to the pair $(u_i,\ g_{ij})$.

The following proposition is known as the Morita theorem by analogy with the corresponding characterization in terms of bimodules of equivalences between certain categories of modules.

Proposition 1.5. (Giraud[8]) *i) An (H, G)-bitorsor Q on X determines an equivalence*

$$\begin{array}{ccc} Tors(H) & \longrightarrow & Tors(G) \\ P & \mapsto & P \wedge^H Q \end{array}$$

between the corresponding categories of right torsors on X.

ii) Any such equivalence Φ between two categories of torsors is equivalent to one associated in this manner to an (H, G)-bitorsor.

To a given equivalence Φ is associated the right H-torsor $Q := \Phi(T_G)$, where T_G is the trivial right G-torsor. Since $G \simeq \mathrm{Aut}(\mathrm{Tors}(G))$, a section of G induces by functoriality a right G-torsor structure on Q.

Definition 2.1. *A category fibered in groupoids above a space X consists in a family of groupoids $\mathcal{C}_U$, for each open set U in X, together with an inverse image functor*

$$f^* : \mathcal{C}_U \longrightarrow \mathcal{C}_{U_1}$$

associated to every inclusion of open sets $f : U_1 \subset U$ (which is the identity whenever $f = 1_U$), and a natural transfomation

$$\phi_{f,g} : (fg)^* \Longrightarrow g^* f^*$$

for every pair of composable inclusions

$$U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U \; .$$

For each triple of composable inclusions

$$U_3 \overset{h}{\hookrightarrow} U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U,$$

we require that the composite natural transformations

$$\psi_{f,g,h} : (fgh)^* \Longrightarrow h^*\,(fg)^* \Longrightarrow h^*\,(g^* f^*)$$

and

$$\chi_{f,g,h} : (fgh)^* \Longrightarrow (gh)^*\,f^* \Longrightarrow (h^* g^*)\,f^*.$$

coincide.

The following is the analogue for fibered groupoids of the notion of a sheaf of sets, formulated in an informal style:

Definition 2.2. *A stack in groupoids above a space X is a fibered category in groupoids above X such that*

- *("Arrows glue") For every pair of objects $x, y \in \mathcal{C}_U$, the presheaf $\mathrm{Ar}_{\mathcal{C}_U}(x,\, y)$ is a sheaf on U.*
- *("Objects glue") Descent is effective for objects in $\mathcal{C}$.*

The descent condition asserts that we are given, for any open cover $\mathfrak{U} = (U_\alpha)$ of an open set $U \subset X$, a family of objects $x_\alpha \in \mathcal{C}_{U_\alpha}$ and a family of arrows $\phi_{\alpha\beta} : x_{\alpha\,|U_{\alpha\beta}} \longrightarrow x_{\beta\,|U_{\alpha\beta}}$ such that

$$\phi_{\alpha\beta}\ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$$

above $U_{\alpha\beta\gamma}$.

The descent condition $(x_i, \phi_{ij}, \psi_{\alpha\beta\gamma})$ is effective if there exists an object $x \in \mathcal{C}_U$ together with isomorphisms $x_{|U_\alpha} \simeq x_\alpha$ compatible with the morphisms $\phi_{\alpha\beta}$.

3. 1-GERBES

We begin with the global description of the 2-category of gerbes, due to Giraud [8]. For other early discussions of gerbes, see [6], [7].

Definition 3.1. *i) A (1)-gerbe on a space X is a stack in groupoids $\mathcal{G}$ on X which is locally* **non-empty** *and locally* **connected***.*

ii) A morphism of gerbes (resp. a natural transformation between a pair of such morphisms) is a (Cartesian) functor between the underlying stacks (resp. a natural transformation between this pair of cartesian functors).

Example: Let G be a bundle of groups on X. The stack Tors(G) of right G-torsors on X is a gerbe on X.

A gerbe $\mathcal{P}$ on X is called *neutral* (or trivial) when the fiber category $\mathcal{P}_X$ is non empty. In particular, a gerbe Tors(G) is neutral with distinguished object the trivial G-torsor on X.

The choice of a global object $x \in \mathcal{P}_X$ in a neutral gerbe $\mathcal{P}$ determines an equivalence of gerbes

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\sim} & \mathrm{Tors}(G) \\ y & \mapsto & \mathrm{Isom}_{\mathcal{P}}(x,\, y) \end{array}$$

on X, where $G := \mathrm{Aut}_{\mathcal{P}}(x)$.

Let $\mathcal{P}$ be a gerbe on X and $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X.

We now **choose** objects $x_i \in \mathrm{ob}\, \mathcal{P}_{U_i}$ for each $i \in I$. These objects determine corresponding bundles of groups $G_i := \mathrm{Aut}_{\mathcal{P}_{U_i}}(x_i)$ above U_i.

When in addition there exists a bundle of groups G above X, together with U_i-isomorphisms $G_{|\,U_i} \simeq G_i$, for some open cover $\mathcal{U} := (U_i)_{i \in I}$, we say that $\mathcal{P}$ is a G-gerbe on X.

4. Semi-local description of a gerbe

Consider a G-gerbe $\mathcal{P}$ on X, and let us choose a family of local objects $x_i \in \mathcal{P}_{U_i}$. These determine a family of local equivalences

$$\Phi_i : \mathcal{P}_{U_i} \longrightarrow \mathrm{Tors}(G)_{|\,U_i}$$

above U_i. Restricting to U_{ij}, we get an induced family of equivalences

$$\Phi_{ij} := \Phi_{i\,|U_{ij}} \circ \Phi^{-1}_{j\,|U_{ij}} :$$

$$\mathrm{Tors}(G)_{U_{ij}} \longrightarrow \mathcal{P}_{U_{ij}} \longrightarrow \mathrm{Tors}(G)_{U_{ij}}$$

and this corresponds to a G-bitorsor P_{ij} above U_{ij}.

By construction, we have natural transformations

$$\Phi_{ij}\,\Phi_{jk} \Longrightarrow \Phi_{ik}$$

above U_{ijk}, so that there is a corresponding isomorphism of G-bitorsors

$$\psi_{ijk} : P_{ij} \wedge^G P_{jk} \longrightarrow P_{ik} \tag{5}$$

above U_{ijk}. The latter satisfies the corresponding coherence (=2-cocycle) condition above U_{ijkl} described by the commutativity of the following diagram of bitorsors above U_{ijkl}

$$\begin{tikzcd}
P_{ij} \wedge P_{jk} \wedge P_{kl} \arrow[r] \arrow[d] & P_{ik} \wedge P_{kl} \arrow[d] \\
P_{ij} \wedge P_{jl} \arrow[r] & P_{il}
\end{tikzcd} \tag{6}$$

Additional comments:

i) The isomorphism (5), satisfying the coherence condition (6), may be viewed as a 1-cocycle on X with values in the monoidal stack of G-bitorsors on X. We say, following Ulbrich [12] , that the family of bitorsors P_{ij} form a bitorsor cocycle on X.

ii) In the case of *abelian* G-gerbes ([2] definition 2.9), with G an abelian group, the monoidal stack of bitorsors on U_{ij} may be replaced by the symmetric monoidal stack of G-torsors on U_{ij}.

iii) For the multiplicative group $G = GL_1$, the P_{ij} may be viewed as line bundles L_{ij}. This the point of view regarding abelian GL_1 gerbes taken by N. Hitchin in [9].

iv) The semi-local construction extends from G-gerbes to general gerbes, with the stack of G-bitorsors above U_{ij} replaced by the family of stacks of $(G_j,\, G_i)$-bitorsors P_{ij}, and with (5) replaced by the corresponding isomorphism of $(G_k,\, G_i)$-bitorsors

$$\psi_{ijk} : P_{ij} \wedge^{G_j} P_{jk} \longrightarrow P_{ik}$$

satisfying the commutativity condition (6).

v) Let us instead consider generalized open sets on X in the sense of Grothendieck topologies, and replace the given trivializing open cover $\mathcal{U}$ of X by a covering morphism $Y \longrightarrow X$ in this Grothendieck topology. The giving of an object $x \in \mathcal{P}_Y$ determines a Y-group $G := \mathrm{Aut}_{\mathcal{P}_Y}$, together with a (p_2^*G, p_1^*G)-bitorsor P above $Y \times_X Y$ satisfying the coherence condition analogous to (6) above $Y \times_X Y \times_X Y$. A bitorsor P satisfying the coherence condition is also called a bundle gerbe [10] , and this therefore corresponds to the giving of a gerbe $\mathcal{P}$ on X, together with a trivialization of its pullback to Y.

5. Cocyclic description of a gerbe

Let us now **choose** arrows

$$x_j \xrightarrow{\phi_{ij}} x_i \tag{7}$$

above U_{ij}. (Actually, this is a simplification, since the gerbe axioms only allow us to choose such an arrow locally, above each element U_{ij}^{α} of an open cover of U_{ij}. This reflects the fact that while a derived functor H^1 may be described by Čech cocycles, this is not in general the case for a functor H^2, and such an explicit description of H^2 requires the choice of a hypercover of X. For simplicity, we suppose from now on that the topological space X is paracompact. In that case, we may dispense with hypercovers, and carry out the entire discussion in a Čech context)

Conjugation by the arrows ϕ_{ij} induces homomorphisms of group bundles

$$\begin{tikzcd} G_{j\,|U_{ij}} \arrow[r, "\lambda_{ij}"] & G_{i\,|U_{ij}} \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} \gamma \arrow[r, mapsto] & \phi_{ij}\,\gamma\,\phi_{ij}^{-1} \end{tikzcd}$$

above the open sets U_{ij}, in other words homomorphisms characterized by the commutativity of the diagrams

$$\begin{tikzcd} x_j \arrow[r, "\gamma"] \arrow[d, "\phi_{ij}"'] & x_j \arrow[d, "\phi_{ij}"] \\ x_i \arrow[r, "\lambda_{ij}(\gamma)"'] & x_i \end{tikzcd}$$

for all $\gamma \in G_{j\,|U_{ij}}$.

The choice of objects x_i and arrows ϕ_{ij} determines, in addition to the morphisms λ_{ij}, a family of objects $g_{ijk} \in G_{i|U_{ijk}}$ for all (i, j, k), defined by the commutativity of the diagrams

$$\begin{tikzcd}
x_k \arrow[r, "\phi_{jk}"] \arrow[d, "\phi_{ik}"'] & x_j \arrow[d, "\phi_{ij}"] \\
x_i \arrow[r, "g_{ijk}"'] & x_i
\end{tikzcd}$$

above U_{ijk}.

This in turn induces the following commutative diagram of bundles of groups

$$\begin{tikzcd}
G_k \arrow[r, "\lambda_{jk}"] \arrow[d, "\lambda_{ik}"'] & G_j \arrow[d, "\lambda_{ij}"] \\
G_i \arrow[r, "i_{g_{ijk}}"'] & G_i
\end{tikzcd} \tag{9}$$

above U_{ijk}, where i_g is the inner conjugation in $G_{i\,|U_{ijk}}$:

$$\begin{array}{ccc} G & \xrightarrow{i} & \mathrm{Aut}(G) \\ g & \mapsto & (i_g : \gamma \mapsto g\,\gamma\,g^{-1})\,. \end{array}$$

The commutativity of diagram (9) may also be stated algebraically as the equation

$$\lambda_{ij}\,\lambda_{jk} = i_{g_{ijk}}\,\lambda_{ik} \tag{10}$$

Lemma 1. *The elements* g_{ijk} *satisfy the* λ_{ij}*-twisted 2-cocycle equation*

$$\lambda_{ij}(g_{jkl})\,g_{ijl} = g_{ijk}g_{ikl} \tag{11}$$

in $G_{i\,|U_{ijkl}}$.

Note that equation (11) is equivalent to the commutativity of the diagram of groups

$$\begin{tikzcd}
G_i \arrow[r, "g_{ijl}"] \arrow[d, "g_{ikl}"'] & G_i \arrow[d, "\lambda_{ij}(g_{jkl})"] \\
G_i \arrow[r, "g_{ijk}"'] & G_i
\end{tikzcd} \tag{12}$$

above U_{ijkl}.

Proof of lemma 1:

Consider the following cube

$$\begin{tikzcd}
 & x_l \arrow[rr, "\phi_{jl}"] \arrow[dl, "\phi_{il}"'] \arrow[dd, "\phi_{kl}" near end] & & x_j \arrow[dl, "\phi_{ij}"] \arrow[dd, "g_{jkl}"] \\
x_i \arrow[rr, crossing over, "g_{ijl}" near end] \arrow[dd, "g_{ikl}"'] & & x_i \arrow[dd, "\lambda_{ij}(g_{jkl})" near start] & \\
 & x_k \arrow[rr, "\phi_{jk}" near start] \arrow[dl, "\phi_{ik}"'] & & x_j \arrow[dl, "\phi_{ij}"] \\
x_i \arrow[rr, "g_{ijk}"'] & & x_i &
\end{tikzcd}$$

in which the left, back, top and bottom faces are of type (12), and the right-hand one of type (8).

Since these five faces are commutative squares, and all the arrows in the diagram are invertible, the sixth (front) face is also commutative.

□

A pair $(\lambda_{ij},\, g_{ijk})$ satisfying the equations (10) and (11):

$$\begin{cases} \lambda_{ij}\,\lambda_{jk} & = \; i_{g_{ijk}}\,\lambda_{ik} \\ \lambda_{ij}(g_{jkl})\,g_{ijl} & = \; g_{ijk}\,g_{ikl} \end{cases} \tag{13}$$

is called a G_i-valued cocycle pair.

It may be viewed as consisting of a 2-cocycle equation for the elements g_{ijk}, together with auxiliary data attached to the morphisms λ_{ij}.

However, in contrast with the abelian case, these two equations cannot in general be uncoupled.

When $\mathcal{P}$ is a G-bitorsor, the term λ_{ij} in the cocycle pair is a section above U_{ij} of the group $\mathrm{Aut}(G)$, and g_{ijk} is a section of G above U_{ijk}. By analogy with the cocyclic equations (4) attached to a G-bitorsor, the pair of equations (13) now defines a Čech 1-cocycle on X with values in the crossed module $G \longrightarrow \mathrm{Aut}(G)$, and the corresponding element of the cohomology set

$H^1(X, G \longrightarrow \mathrm{Aut}(G))$ is independent of the choices of local objects and arrows in $\mathcal{P}$.

In geometric terms, this can be understood once we introduce the following definition:

Definition 5.1. *Let $\mathcal{G}$ be a monoidal stack on X. A right $\mathcal{G}$-torsor on X is a stack $\mathcal{Q}$ on X together with a right action functor*

$$\mathcal{Q} \times \mathcal{G} \longrightarrow \mathcal{Q}$$

which is coherently associative and satisfies the unit condition, and for which the induced functor

$$\mathcal{Q} \times \mathcal{G} \longrightarrow \mathcal{Q} \times \mathcal{Q}$$

defined as in (1) *is an equivalence. In addition, we require that $\mathcal{Q}$ be locally non-empty.*

Remark 5.2. The three following observations, when put together, explain in more global term why G-gerbes are classified by the set $H^1(X, G \longrightarrow \mathrm{Aut}(G))$.

- To a G-gerbe $\mathcal{P}$ on X is associated its "bundle of frames" $\mathcal{E}q(\mathrm{Tors}(G),\ \mathcal{P})$, and the latter is a torsor under the monoidal stack $\mathcal{E}q(\mathrm{Tors}(G),\ \mathrm{Tors}(G))$

- By the Morita theorem, this monoidal stack is equivalent to the monoidal stack $\mathrm{Bitors}(G)$ of G-bitorsors on X.

- The cocycle computations leading to (4) imply that the monoidal stack $\mathrm{Bitors}(G)$ is the stack associated (by sheafification) to the monoidal prestack defined by the crossed module $G \longrightarrow \mathrm{Aut}(G)$.

Topological interpretation of a G-gerbe

To a group G we attach the topological group-like monoid $\mathrm{Eq}(BG)$ of self-homotopy equivalences of the classifying space of G. The fibre of the evaluation map

$$\mathrm{ev}_* : \mathrm{Eq}(BG) \longrightarrow BG$$

of an equivalence at the distinguished point $*$ of BG is the submonoid $\mathrm{Eq}_*(BG)$ of pointed equivalences, and the latter is homotopy equivalent, by the functor $\pi_1(-, *)$, to the discrete group $\mathrm{Aut}(G)$.

The induced fiber sequence

$$\mathrm{Aut}(G) \longrightarrow \mathrm{Eq}(BG) \longrightarrow BG$$

deloops to a sequence

$$G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Eq}(BG)$$

where the left hand map is the inner conjugation homomorphism. This yields an identification of Eq (BG) with the mapping cone of the map $i : G \longrightarrow \mathrm{Aut}(G)$, which preserves the multiplication.

This proves:

Proposition 5.3. *The simplicial group associated to the crossed module $G \longrightarrow Aut(G)$ is a model for the grouplike topological monoid $Eq(BG)$.*

In this context, the set of 1-cocycles on X with values in the crossed module $G \longrightarrow \mathrm{Aut}(G)$ classify the fibrations on X which are locally homotopy equivalent to the Eilenberg-Maclane space BG.

Example 5.4. (Schreier, circa 1930) The following modernized proof of Schreier's classification of (non-abelian, non-central) group extensions is a strengthened version of the discussion carried out in (2).

Consider a short exact sequence of groups (or bundle of groups) (2). This induces a fibration

$$BG \longrightarrow BH \xrightarrow{\pi} BK$$

above BK, and all the fibers of π are homotopically equivalent to BG. It follows that equivalence classes of such extensions are classified by the pointed set $H^1(BK, G \longrightarrow \mathrm{Aut}(G))$.

6. 2-STACKS AND 2-GERBES

Definition 6.1. *A fibered 2-category in 2-groupoids above a space X consists in a family of 2-groupoids $\mathcal{C}_U$, for each open set U in X, together with an inverse image 2-functor*

$$f^* : \mathcal{C}_U \longrightarrow \mathcal{C}_{U_1}$$

associated to every inclusion of open sets $f : U_1 \subset U$ (which is the identity whenever $f = 1_U$), and a natural transfomation

$$\phi_{f,g} : (fg)^* \Longrightarrow g^* \, f^*$$

for every pair of composable inclusions

$$U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U\ .$$

For each triple of composable inclusions

$$U_3 \overset{h}{\hookrightarrow} U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U,$$

we require a modification

$$\begin{tikzcd}(fgh)^* \arrow[r, bend left, "\psi_{f,g,h}"] \arrow[r, bend right, "\chi_{f,g,h}"'] \arrow[r, phantom, "\Downarrow\alpha_{f,g,h}"] & h^*g^*f^*\end{tikzcd}$$

betweeen the composite natural transfomations

$$\psi_{f,g,h} : (fgh)^* \Longrightarrow h^*\,(fg)^* \Longrightarrow h^*\,(g^*f^*)$$

and

$$\chi_{f,g,h} : (fgh)^* \Longrightarrow (gh)^*\,f^* \Longrightarrow (h^*g^*)\,f^*.$$

Finally, for any $U_4 \hookrightarrow U_3$*, the two methods by which the modifications* α *compare the composite 2-arrows*

$(fghk)^* \Longrightarrow (ghk)^*f^* \Longrightarrow ((hk)^*g^*f^* \Longrightarrow k^*h^*g^*f^*$
and

$(fghk)^* \Longrightarrow k^*(fgh)^* \Longrightarrow k^*(h^*(fg)^*) \Longrightarrow k^*h^*g^*f^*$

must coincide.

Definition 6.2. *A 2-stack in 2-groupoids above a space X is a fibered 2-category in 2-groupoids above X such that*

- *For every pair of objects $X, Y \in \mathcal{C}_U$, the fibered category $\mathrm{Ar}_{\mathcal{C}_U}(X, Y)$ is a stack on U.*
- *2-descent is effective for objects in $\mathcal{C}$.*

The 2-descent condition asserts that we are given, for an open cover U_α of an open set $U \subset X$, a family of objects $x_i \in \mathcal{C}_{U_i}$, of 1-arrows $\phi_{\alpha\beta} : x_\alpha \longrightarrow x_\beta$ between the restrictions to $\mathcal{C}_{U_{\alpha\beta}}$ of the objects x_α and x_β and a family of 2-arrow

$$\begin{tikzcd}
& x_\beta \arrow[dl, "\phi_{\beta\gamma}"'] \arrow[dr, "\phi_{\alpha\beta}"] \arrow[d, phantom, "\psi_{\alpha\beta\gamma}\Downarrow"] & \\
x_\gamma \arrow[rr, "\phi_{\alpha\gamma}"'] & {} & x_\alpha
\end{tikzcd}$$

for which the tetrahedral diagram of 2-arrows induced by ψ in $\mathcal{C}_{U_{\alpha\beta\gamma\delta}}$ commutes:

$$\begin{tikzcd}
 & x_\delta \arrow[dl] \arrow[dr] \arrow[dd] & \\
x_\gamma \arrow[rr, dashed] \arrow[dr] & & x_\alpha \\
 & x_\beta \arrow[ur] &
\end{tikzcd}$$

The descent condition $(x_i,\, \phi_{ij},\, \psi_{\alpha\beta\gamma})$ is effective if there exists an object $x \in \mathcal{C}_U$ together with isomorphisms $x_{|U_\alpha} \simeq x_\alpha$ compatible with the ϕ_{ij} and $\psi_{\alpha\beta\gamma}$.

Definition 6.3. *A 2-gerbe $\mathcal{P}$ is a 2-stack in 2-groupoids on X which is locally non-empty and locally connected.*

To each object x in $\mathcal{P}_U$ is associated a grouplike monoidal stack (= *gr*-stack) $\mathcal{G}_x := \mathcal{A}r_U(x,\, x)$ above U.

Definition 6.4. *Let $\mathcal{G}$ be a grouplike monoidal stack on X. We say that a 2-gerbe $\mathcal{P}$ is a $\mathcal{G}$-2-gerbe if there exists an open cover $\mathcal{U} := (U_i)_{i\in I}$ of X, a family of objects $x_i \in \mathcal{P}_{U_i}$, and U_i-equivalences $\mathcal{G}_{U_i} \simeq \mathcal{G}_{x_i}$.*

The choice of a path $\phi_{ij} : x_j \longrightarrow x_i$ above U_{ij} determines, as in (7), a monoidal equivalence $\lambda_{ij} : \mathcal{G}_{|U_{ij}} \longrightarrow \mathcal{G}_{|U_{ij}}$. In addition, the paths ϕ_{ij} allow us to choose an object $g_{ijk} \in \mathcal{G}_{U_{ijk}}$ together with a 2-arrow m_{ijk}:

$$\begin{tikzcd}
x_k \arrow[r, "\phi_{jk}"] \arrow[d, "\phi_{ik}"'] & x_j \arrow[d, "\phi_{ij}"] \\
x_i \arrow[r, "g_{ijk}"'] \arrow[ru, phantom, "\overset{m_{ijk}}{\Nearrow}"] & x_i
\end{tikzcd} \tag{14}$$

These in turn determine a 2-arrow ν_{ijkl} above U_{ijkl}

$$\begin{tikzcd}
x_i \arrow[r, "g_{ijl}"] \arrow[d, "g_{ikl}"'] & x_i \arrow[d, "\lambda_{ij}(g_{jkl})"] \\
x_i \arrow[r, "g_{ijk}"'] \arrow[ru, phantom, "\overset{\nu_{ijkl}}{\Swarrow}"] & x_i
\end{tikzcd} \tag{15}$$

as the unique 2-arrow such that the following diagram of 2-arrows with front face ν_{ijkl} commutes:

$$\begin{tikzcd}
 & x_l \arrow[rr, "\phi_{jl}"] \arrow[dl, "\phi_{il}"'] \arrow[dd, "\phi_{kl}" near end] \arrow[dr, phantom, "\Swarrow m_{jkl}"] \arrow[rrdd, phantom, ""] & & x_j \arrow[dl, "\phi_{ij}"'] \arrow[dd, "g_{jkl}"] \arrow[dd, phantom, "\Searrow" near start, shift right=3ex] \\
x_i \arrow[rr, "g_{ijl}" near end] \arrow[dd, "g_{ikl}"'] \arrow[dr, phantom, "\Downarrow m_{ikl}" near end] & & x_i \arrow[dd, "\lambda_{ij}(g_{jkl})" near start] \arrow[ul, phantom, "m_{ijl} \Rightarrow" near start] & \\
 & x_k \arrow[rr, "\phi_{jk}" near start] \arrow[dl, "\phi_{ik}"] \arrow[dr, phantom, "\Nearrow m_{ijk}" near start] & & x_j \arrow[dl, "\phi_{ij}"] \\
x_i \arrow[rr, "g_{ijk}"'] & & x_i \arrow[ul, phantom, "\nu_{ijkl} \Swarrow" near start] &
\end{tikzcd}$$

The 3-cocycle condition for ν_{ijkl}:

$$\begin{tikzcd}
 & x_i & \Swarrow \nu_{ijkl} & x_i \\
x_i & \Nwarrow \nu_{ijlm} & x_i & \Nearrow \lambda_{ij}(\nu_{jklm}) \\
\nu_{iklm} \Nearrow & x_i & & x_i \\
x_i & \Nwarrow \{\widetilde{m}_{ijk}, g_{klm}\}^{-1} \quad \nu_{ijkm} \Swarrow & x_i &
\arrow[from=1-2, to=1-4, "g_{ijl}"]
\arrow[from=2-1, to=1-2, "g_{ilm}"]
\arrow[from=2-3, to=1-4, "\lambda_{ij}(g_{jlm})"]
\arrow[from=1-2, to=3-2, "g_{ikl}"]
\arrow[from=2-1, to=2-3, "g_{ijm}", crossing over]
\arrow[from=1-4, to=3-4, "\lambda_{ij}(g_{jkl})"]
\arrow[from=2-1, to=4-1, "g_{ikm}"']
\arrow[from=3-2, to=3-4, "g_{ijk}"]
\arrow[from=2-3, to=4-3, "\lambda_{ij}(g_{jkm})", crossing over]
\arrow[from=4-1, to=3-2, "\lambda_{ik}(g_{klm})"']
\arrow[from=4-3, to=3-4, "\lambda_{ij}\lambda_{jk}(g_{klm})"']
\arrow[from=4-1, to=4-3, "g_{ijk}"']
\end{tikzcd}$$

The bottom 2-arrow $\{\widetilde{m}_{ijk},\, g_{klm}\}$ is essentially the 2-arrow obtained by evaluating the natural transformation

$$\widetilde{m}_{ijk} : i_{g_{ijk}}\lambda_{ik} \Rightarrow \lambda_{ij}\,\lambda_{jk}$$

induced by the 2-arrow m_{ijk} (14) on the object $g_{ijk} \in G_i := \mathrm{Aut}_{\mathcal{P}}(x_i)$:

$$\begin{tikzcd}
x_i \arrow[r, "g_{ijk}"] & x_i \\
x_i \arrow[u, "\lambda_{ik}(g_{klm})"] \arrow[r, "g_{ijk}"'] \arrow[ur, phantom, "\Searrow"] & x_i \arrow[u, bend left=40, "i_{g_{ijk}}\lambda_{ik}(g_{klm})", ""{name=A, right}] \arrow[u, bend right=40, "\lambda_{ij}\lambda_{jk}(g_{klm})"', ""{name=B, left}] \arrow[from=A, to=B, Rightarrow, "\widetilde{m}(g_{klm})"]
\end{tikzcd}$$

Proof of the 3-cocycle condition:

$$\begin{tikzcd}
 & x_i \arrow[rrrrrr, "g_{ijm}"] \arrow[dddl, "g_{ilm}"'] \arrow[dddd, "g_{ikm}"] & & & & & & x_i \arrow[dddl, "\lambda_{ij}(g_{jlm})"'] \arrow[dddd, "\lambda_{ij}(g_{jkm})"] \\
 & & & x_m \arrow[ull, "\phi_{im}"'] \arrow[rr, "\phi_{jm}"] \arrow[dl, "\phi_{lm}"'] \arrow[dddd, "\phi_{km}"] & & x_j \arrow[urr, "\phi_{ij}"] \arrow[dl, "g_{jm}"'] \arrow[dddd, "g_{jkm}"'] & & \\
 & & x_l \arrow[dll, "\phi_{il}"'] \arrow[rr, "\phi_{jl}"] \arrow[dddd, "\phi_{lk}"] & & x_j \arrow[drr, "\phi_{ij}"] \arrow[dddd, "g_{jkl}"'] & & & \\
x_i \arrow[rrrrrr, "g_{ijl}"] \arrow[dddd, "\phi_{ikl}"'] & & & & & & x_i \arrow[dddd, "\lambda_{ij}(g_{jkl})"] & \\
 & x_i \arrow[rrrrrr, "g_{ijk}"'] \arrow[dddl, "\lambda_{ik}(g_{klm})"] & & & & & & x_i \arrow[dddl, "\lambda_{ij}\lambda_{jk}(g_{klm})"] \\
 & & & x_k \arrow[ull, "\phi_{ik}"] \arrow[rr, "\phi_{jk}"'] \arrow[dl, "g_{klm}"] & & x_j \arrow[urr, "\phi_{ij}"'] \arrow[dl, "\lambda_{jk}(g_{klm})"] & & \\
 & & x_k \arrow[dll, "\phi_{ik}"'] \arrow[rr, "\phi_{jk}"] & & x_j \arrow[drr, "\phi_{ij}"'] & & & \\
x_i \arrow[rrrrrr, "g_{ijk}"'] & & & & & & x_i &
\end{tikzcd}$$

inner	left	right	top	bottom	front	back	outer
C_{jklm}	C_{iklm}	Conj(ϕ_{ij})	C_{ijlm}	$\{\ ,\ \}$	C_{ijkl}	C_{ijkm}	3-cocyle
	m_{klm}	ν_{jklm}	m_{jlm}	$M_{jk}(m_{klm})$	m_{jkl}	m_{jkm}	

Table 1

Here is the description of the bottom cube, labelled $\{\,,\,\}$, in table 1:

$$\begin{tikzcd}
& x_k \arrow[rr, "\phi_{jk}"] \arrow[dl, "\phi_{ik}"'] \arrow[dd, "g_{klm}"] \arrow[dr, phantom, "\overset{m_{ijk}}{\Rrightarrow}"] & & x_j \arrow[dl, "\phi_{ij}"'] \arrow[dd, "\lambda_{jk}(g_{klm})"] \\
x_i \arrow[rr, "g_{ijk}"] \arrow[dd, "\kappa_{024}"'] & & x_i \arrow[dd, "\lambda_{ij}\lambda_{jk}(g_{klm})"] & \\
& x_k \arrow[rr, "\phi_{jk}"] \arrow[dl, "\phi_{ik}"] \arrow[dr, phantom, "\underset{m_{ijk}}{\Rrightarrow}"] & & x_j \arrow[dl, "\phi_{ij}"] \\
x_i \arrow[rr, "g_{ijk}"'] & & x_i &
\end{tikzcd}$$

Algebraic description of the 3-cocycle condition:

The commutativity of the 3-cocycle cube translates to a very twisted algebraic 3-cocycle condition:

$$\lambda_{ij}(\nu_{jklm})\,\left({}^{\lambda_{ij}(g_{jkl})}\nu_{ijlm}\right)\nu_{ijkl} = \left({}^{\lambda_{ij}\lambda_{jk}(g_{klm})}\nu_{ijkm}\right)\left[\widetilde{m}_{ijk}, g_{klm}\right]\ {}^{g_{ijk}}\nu_{iklm}$$

This is an equation satisfied by elements with values in the fibre of Ar $(\mathcal{G})$ above U_{ijklm}.

Similar non-abelian 3-cocycle relations first appeared in the work of Dedecker [5] , in the context of cohomology of groups, rather than as here for Čech-cohomology.

The definition of ν_{ijkl} as the front face of the cube (15) induces by conjugation the following equation among elements of the fibre of Ar $\mathcal{E}q(\mathcal{G})$ above U_{ijkl}:

$$\widetilde{m}_{ijl}\,({}^{g_{ijk}}\widetilde{m}_{ikl})\,i_\nu = (\lambda_{ij}(\widetilde{m}_{jkl}))\,({}^{\lambda_{ij}(g_{jkl})}\widetilde{m}_{ijl}) \tag{16}$$

Observe finally that g_{ijk} is an object of $\mathcal{G}$ above U_{ijk} and λ_{ij} an object of $\mathrm{ob}\,\mathcal{E}q(\mathcal{G})$ above U_{ij}.

The previous discussion may be summarized as the assertion that the quadruple $(\lambda_{ij},\, \widetilde{m}_{ijk},\, g_{ijk},\, \nu_{ijkl})$ associated to a $\mathcal{G}$-2-gerbe $\mathcal{P}$ on X determines a Čech 1-cocycle on X, with values in the "crossed module of *gr*-stacks"

$$\mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G})$$

Remark 6.5. *i*) When $\mathcal{G}$ is the *gr*-stack associated to a crossed module $\delta : G \longrightarrow \Pi$, this coefficient crossed module of *gr*-stacks is a stackified version of the crossed square associated by K.J.Norrie (see [11], [4]) to the crossed module $G \longrightarrow \pi$:

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\delta"'] & \mathrm{Der}^*(\pi, G) \arrow[d] \\ \pi \arrow[r] & \mathrm{Aut}(G \to \pi) \end{tikzcd} \tag{17}$$

It is however less restrictive than Norrie's version, since the latter would correspond to the diagram of gr-stacks

$$\mathcal{G} \longrightarrow \mathrm{Isom}(\mathcal{G})$$

whereas we really need to consider, as in (17), equivalences of the monoidal stack $\mathcal{G}$ with itself, rather than simply isomorphisms.

ii) It was important for us to choose in (14) a cubical filler for the edges defined by the arrows ϕ_{ij}. If we had postulated in (14) that $g_{ijk} = 1_{x_i}$, the 3-cocycle ν_{ijkl} (15) would have taken its values in the abelian group of 2-arrows from the 1-arrow 1_{x_i} to itself. More precisely, one would only obtain in this manner the 3-cocycles ν with values in the center ZG of the group G.

References

[1] L. Breen, *Bitorseurs et cohomologie non abélienne* in *The Grothendieck Festschrift I*, Progress in Mathematics **86**, Birkhaüser, 401-476 (1990).

[2] L. Breen, *Classification of 2-gerbes and 2-stacks*, Astérisque **225**, Société Mathématique de France (1994).

[3] L. Breen and W. Messing, *Differential Geometry of Gerbes*, arXiv:math.AG/0106083.

[4] R. Brown and N.D. Gilbert, *Algebraic Models of 3-types and Automorphism structures for Crossed modules*, Proc. London Math. Soc **59**, 51-73 (1989).

[5] P. Dedecker, *A 3-dimensional Non-Abelian Cohomology of Groups with Applications to Homotopy Classification of Continuous Maps*, in Lecture Notes in Mathematics **92**, (1969).

[6] J.-C. Douai, *2-cohomologie Galoisienne des groupes semi-simples*, doctoral thesis, Lille (1976).

[7] J. Duskin, *An outline of non-abelian cohomology in a topos (1):the theory of bouquets and gerbes*, Cahiers de topologie et éométrie différentielle XXIII (1982).

[8] J. Giraud, *Cohomologie non abélienne*, Grundlehren **179**, Springer-Verlag (1971).

[9] N. Hitchin, *Lectures on special Lagrangians on submanifolds*, arXiv:math.DG/9907034 (1999).

[10] M. K. Murray, *Bundle gerbes*, J. of the London Math. Society **54**, 403-416 (1996).

[11] K. Norrie *Actions and automorphisms*, Bull. Soc. Math. de France **118**, 109-119 (1989).

[12] K.-H. Ulbrich, *On cocycle bitorsors and gerbes over a Grothendieck topos*, Math. Proc. Cambridge Phil. Soc. **110**, 49-55 (1991).