ITERATED WREATH PRODUCT OF THE SIMPLEX CATEGORY

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1. Motivation

Grothendieck's conjecture (pursuit of stacks '83)

weak form of n-groupoid $\leftrightarrow$ homotopy n-type

collapsing cells $\leftrightarrow$ truncation

(easy) (difficult)

Example. fundamental groupoid $\leftrightarrow$ 1-type

Notation. $\widehat{\Delta} = \underline{Sets}^{\Delta^{op}}$

$$\begin{tikzcd} \underline{Cat} \arrow[r, shift left=1.5ex, "N_\Delta"] \arrow[r, phantom, "\top"] & \widehat{\Delta} \arrow[l, shift left=1.5ex, "cat_\Delta"] \arrow[r, shift left=1.5ex, "|-|_\Delta"] \arrow[r, phantom, "\perp"] & \underline{Top} \arrow[l, shift left=1.5ex, "S_\Delta"] \end{tikzcd}$$

$\Pi(X) = \gamma\, cat_\Delta S_\Delta(X) =$ fundamental groupoid of X

γ reflector into groupoids

Aim. Describe <u>higher analog</u> of Δ in order to define <u>higher fundamental groupoids</u>.

<u>One possible solution</u>: Joyal's Θ_n.

<u>Shall see</u>: $\Theta_n = \underbrace{\Delta S \Delta S \cdots S \Delta}_{n \text{ times}}$.

1st step   Construct Θ_n with functors

$$n\mathrm{Cat} \xrightarrow{\mathrm{Ner}_n} \hat{\Theta}_n \xrightarrow{|-|_{\Theta_n}} \mathrm{Top}$$

fully faithful nerve functor	left exact, left adjoint realization
$\Updownarrow$	$\Updownarrow$
$\Theta_n \subset n\mathrm{Cat}$	$\Theta_n \hookrightarrow \mathrm{Top}$
"dense embedding"	"flat realization"

2nd step   Construct homotopy theory for $\hat{\Theta}_n$

"Θ_n is a geometric Reedy category".

3rd step   Define homotopy n-types on Θ_n and compare with weak n-groupoids

"Moore-Postnikov sections" + "horn filling".

4th step   Guess what the corresponding weak n-categories are.

"inner horn filling conditions".

Beyond: Define a homotopy theory for weak n-categories.

"Delocalization of the geometric model structure".

2. Inductive description of Joyal's Θ_n

$$\mathrm{Cat} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Grph} \quad \text{(non reflexive, directed)}$$

$$[n] = F\left(\underset{0}{\bullet} \to \underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet} \cdots\cdots \underset{n-1}{\bullet} \to \underset{n}{\bullet}\right)$$

Enriched analog : $(\mathcal{V}, \otimes, I)$ $\left\{\begin{array}{l}\text{closed symmetric} \\ \text{monoidal category}\end{array}\right.$

$$\mathcal{V}\text{-Cat} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{V}\text{-Grph} \qquad (\mathcal{V}\text{-graphs} / \mathcal{V}\text{-matrices})$$

hom-objects in $\mathcal{V}$

for each edge an object in $\mathcal{V}$

$\mathcal{V}$-cats are monoids in $\mathcal{V}$-Grph for the following product of $\mathcal{V}$-graphs with same object set : $(G, H \in \mathrm{Ob}\, \mathcal{V}\text{-Grph})$
$a, b \in \mathrm{Ob}\, G = \mathrm{Ob}\, H$.

$$(G \otimes H)(a,b) = \coprod_c G(a,c) \otimes H(c,b)$$

This tells us what the free $\mathcal{V}$-cats are (namely, the free monoids for this product)

Def. Let $\mathcal{A} \subset \mathcal{V}$ be a full subcategory.
Then, $\Delta S\mathcal{A}$ is the full subcategory of $\mathcal{V}$-Cat spanned by the free $\mathcal{V}$-cats on the $\mathcal{V}$-graphs

$$\underset{0}{\bullet} \xrightarrow{A_1} \underset{1}{\bullet} \xrightarrow{A_2} \underset{2}{\bullet} \xrightarrow{A_3} \underset{3}{\bullet} \cdots\cdots \underset{m-1}{\bullet} \xrightarrow{A_m} \underset{m}{\bullet}$$

Notation: $F(\quad) = ([m]; A_1, A_2, \dots, A_m)$

Rmk. The morphisms of $\Delta \wr \mathcal{A}$ may be described as follows:

$$([m]; A_1, \dots, A_m) \longrightarrow ([n]; B_1, \dots, B_n) \quad \text{is}$$

given by $(\varphi; \varphi_1, \dots, \varphi_m)$ s.th.

$\varphi : [m] \to [n]$ in Δ and for $k = 1, \dots, m$

$$\varphi_k : A_k \xrightarrow{\mathcal{V}} \begin{cases} B_{\varphi(k-1)+1} \otimes \cdots \otimes B_{\varphi(k)} & \text{if } \varphi(k-1) < \varphi(k) \\ I & \text{if } \varphi(k-1) = \varphi(k) \end{cases}$$

with evident composition and units.

Def.

$$\begin{array}{lcl} \Theta_0 = \{*\} & \subset & Set \\ \downarrow & & \\ \Theta_1 = \Delta \wr * = \Delta & \subset & Cat \\ \downarrow & & \downarrow \\ \Theta_2 = \Delta \wr \Delta & \subset & Cat\text{-}Cat = 2Cat \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \Theta_n = \Delta \wr \Theta_{n-1} & \subset & (n-1)Cat\text{-}Cat = nCat \\ \downarrow & & \downarrow \\ \Theta = \varinjlim_n \Theta_n . & \subset & \omega Cat \end{array}$$

Lemma. $\mathcal{V}$ cartesian closed $\Rightarrow$ $\mathcal{V}$-Cat cartesian closed

If moreover $\mathcal{A} \subset \mathcal{V}$ dense,

then $\Delta \wr \mathcal{A} \subset \mathcal{V}$-Cat dense.

Corollary. $\Theta_n \subset nCat$ dense.

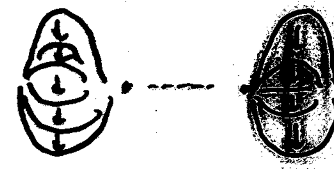
$\mathrm{Cat} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathrm{Grph}$

Alternative descriptions of Θ_n

$$2\mathrm{Cat} = \mathrm{Cat}\text{-}\mathrm{Cat} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Cat}\text{-}\mathrm{Grph} \underset{F_1\text{-}\mathrm{Gr}}{\overset{U_1\text{-}\mathrm{Gr}}{\rightleftarrows}} \mathrm{Grph}\text{-}\mathrm{Grph} = 2\mathrm{Grph}$$

$$2\mathrm{Cat} \xleftarrow{\ F_2\ } 2\mathrm{Grph}$$

$$([m];[k_1],\dots,[k_m]) \longmapsto \left(\underset{0}{\bullet} \xrightarrow{[k_1]} \underset{1}{\bullet} \cdots\cdots \underset{m-1}{\bullet} \xrightarrow{[k_m]} \underset{m}{\bullet} \right) \longmapsto$$

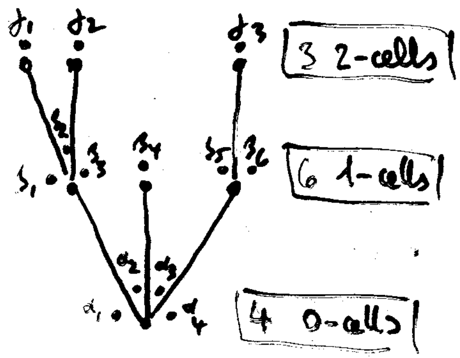


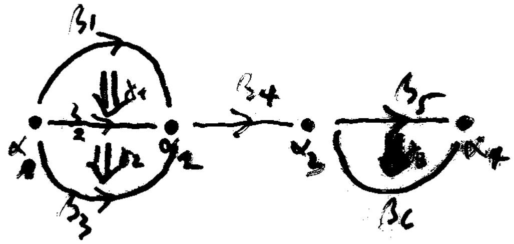
2-globular set

↑ free 2-category on a 2-globular set (2-pasting diagram).

Tree-notation:

$T \leftrightarrow ([3];[2],[0],[1]) = F_2(T_*)$ [redacted]

$T =$ 

$T_* =$ 

$$\mathrm{Ob}\,\Theta_2 = \left\{ \begin{matrix} \text{non deg.} \\ 2\text{-pd} \\ 1\text{-pd} \\ 0\text{-pd} \end{matrix} \right\} = \{ \text{level trees of height} \leq 2 \}$$

$$\Theta_2(S,T) = 2\mathrm{Cat}(F_2(S_*), F_2(T_*))$$

Rmk. Since $\Theta_2 = \Delta \wr \Delta$, we have explicit description of $\Theta_2(S,T)$!

Inductively:

$$\begin{tikzcd} n\mathrm{Cat} = (n-1)\mathrm{Cat}\text{-}\mathrm{Cat} \arrow[r, shift left, "U"] \arrow[rr, bend left, "U_n"] & (n-1)\mathrm{Cat}\text{-}\mathrm{Graph} \arrow[l, shift left, "F"] \arrow[r, shift left, "U_{n-1}\text{-}\mathrm{Gr}"] & (n-1)\mathrm{Graph}\text{-}\mathrm{Graph} = n\mathrm{Graph} \arrow[l, shift left, "F_{n-1}\text{-}\mathrm{Gr}"] \arrow[ll, bend left, "F_n"] \end{tikzcd}$$

$\mathrm{Ob}\, \Theta_n = \{ \text{level trees of height} \leq n \}$

$\Theta_n(S,T) = n\mathrm{Cat}(F_n(S_*), F_n(T_*))$

Theorem (Makkai–Zawadowski, B.)

$\Theta_n^{op} \cong \mathbb{D}_n$ (category of finite "n-disks")

Theorem (Joyal)

$\widehat{\Theta}_n$ is a classifying topos for "n-disks".

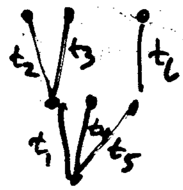
Corollary $\exists$ left adjoint, left exact $|-|_{\Theta_n} : \widehat{\Theta}_n \longrightarrow \mathrm{Top}$

Rmk. It suffices to give this realization functor on representables $\Theta_n(-,S) = \Theta_n[S]$.

$$\begin{tikzcd} \Theta_n \arrow[r] \arrow[d, "y"'] & \mathrm{Top} \\ \widehat{\Theta}_n \arrow[ur, "|-|_{\Theta_n}"'] & \end{tikzcd}$$

This realization functor is compatible with the inclusions $\Theta_n \hookrightarrow \Theta_{n+1}$.

Example.

$$|\Theta[T]| = \Big|$$  $$\Big| = \Big\{ (t_e)_{e \in e(T)} \in [-1,1]^6 \;\Big|\; \begin{array}{l} t_1^2 + t_2^2 \le 1 \\ t_1^2 + t_3^2 \le 1 \\ t_1 \le t_4 \le t_5 \\ t_5^2 + t_6^2 \le 1 \end{array} \Big\}$$

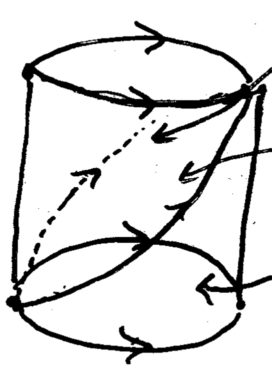
$|\Theta[$  $]| \cong \Delta_m$ $\quad$ $|\Theta[$  $]| = B^m$

Rmk. Left exactness $\Longleftrightarrow$ "shuffle-formula"

$$\Theta[S] \times \Theta[T] = \bigcup_{U \in shuff(S,T)} \Theta[U]$$

generalizes $\Delta[m] \times \Delta[n] = \bigcup_{\frac{(m+n)!}{m!\,n!}} \Delta[m+n]$

$\Theta[$  $] \times \Theta[$  $] = \Theta[$  $] \cup \Theta[$  $] \cup \Theta[$  $]$



3. Geometric Reedy categories

Def. $\mathcal{A}$ is geom. Reedy iff

ⓐ $\mathcal{A}$ skeletal & abs. regular
ⓑ $\mathcal{A}$ spherical CW-structure
$\Big\} \Rightarrow \mathcal{A}$ Reedy with epi/mono factorisation

ⓒ $\mathcal{A}$ CW-complete
ⓓ $\mathcal{A}$ CW-flat

Thm. (Cisinski–B.) If $\mathcal{A}$ is a geom. Reedy category with terminal object, then $\hat{\mathcal{A}}$ admits a Quillen closed model structure s.th.

$$|-|_{\mathcal{A}} : \hat{\mathcal{A}} \rightleftarrows Top : S_{\mathcal{A}}$$

is a Quillen equivalence. The trivial cofibrations are generated by the horn inclusions.

Proposition ⓐ $\mathcal{A}, \mathcal{B}$ geom. Reedy $\Rightarrow \mathcal{A} \times \mathcal{B}$ geom. Reedy
ⓑ Δ geom. Reedy.
ⓒ $\mathcal{A}$ geom. Reedy $\Rightarrow \Delta \wr \mathcal{A}$ geom. Reedy

Corollary. Θ_n is geom. Reedy.

In particular, $\hat{\Theta}_n$ has a homotopy theory for which the weak equivalences are those maps of presheaves which realize to (weak) homotopy equivalences.

4. Higher groupoids and higher categories

<u>Rmk.</u> A gen. Reedy $\Rightarrow$ $\hat{A}$ has "~~combinatorial~~ Moore-Postnikov sections".

Each fibrant presheaf X is the inverse ~~limit~~ of a tower of fibrations $\cdots \to P_n X \to P_{n-1} X \to \cdots$
s.th. $\pi_k X \cong \pi_k P_n X$ for $k \leq n$
$\pi_k X = *$ for $k > n$.

Actually, $P_n X$ is "<u>n-separated</u>"
For $x, y \in (P_n X)(A)$: $x/_{sk_n A} = y/_{sk_n A} \Rightarrow x = y$.

<u>Observe.</u> $n\mathrm{Cat} \subset \mathrm{sep}_n \hat{\Theta}_n \subset \hat{\Theta}_n$

"n-Categorification factors through n-separation".

$$Ho(\mathrm{sep}_n \hat{\Theta}_n) \simeq Ho(n\text{-types})$$

<u>"Def".</u> A weak n-groupoid is a fibrant n-separated object of $\hat{\Theta}_n$.

<u>Proposition.</u> A presheaf $X \in \mathrm{Ob}\,\hat{\Theta}_n$ is the nerve of a strict n-category if and only if every inner horn in X has a unique filler.

<u>"Def."</u> A weak n-category is an n-separated object of $\hat{\Theta}_n$ with fillers for inner horns.