DISCRETE V. COMPUTATIONAL MORSE THEORY

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Extension. The purpose of this note is to place the computational Morse theory of J. Harer in the context of R. Forman's discrete Morse theory. For a thorough discussion of the latter, see [1].

Let K be a finite simplicial complex. Denote a p-simplex of K by $\sigma^{(p)}$ and let K_p be the set of all p-simplices. If τ is a (proper) face of σ, we write $\tau < \sigma$. A discrete Morse function on K is a map

$$f : K \longrightarrow \mathbb{R}$$

such that for every $\sigma^{(p)} \in K_p$

$$\#\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\} \leq 1 \tag{1}$$

$$\#\{\nu^{(p-1)} < \sigma | f(\nu) \geq f(\sigma)\} \leq 1. \tag{2}$$

Essentially, one should think of this as a function that generically increases with the dimension of the cells, in the sense that there is at most one direction in which f decreases as it moves from a p-cell σ to a $(p+1)$-cell τ.

A cell $\sigma^{(p)}$ is critical (with index p) if

- $\#\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\} = 0$, and
- $\#\{\nu^{(p-1)} < \sigma | f(\nu) \geq f(\sigma)\} = 0$.

Discrete Morse functions exist. Indeed, the simplest (and most trivial) example is the following. If σ is a simplex of K, we define $f : K \longrightarrow \mathbb{R}$ by

$$f(\sigma) = \dim \sigma.$$

In this case, every cell is critical.

The set-up for computational Morse theory is much simpler. All that one requires is a function g on the set K_0 of 0-simplices. The first question one asks is the following: is it possible to extend g to a discrete Morse function on K? The answer is provided by the following.

Proposition 0.1. *For any finite simplicial complex K, a function $g : K_0 \longrightarrow \mathbb{R}$ may be extended to a discrete Morse function $\tilde{g} : K \longrightarrow \mathbb{R}$.*

Proof. This holds more generally. According to Lemma 4.2 of [1], any discrete Morse function on any subcomplex N of K may be extended to a discrete Morse function on all of K. In our particular case, the proof goes as follows. Let $c = \max_{v \in K_0} g(v)$. Define a function $\tilde{g} : K \longrightarrow \mathbb{R}$ by

$$\tilde{g}(\sigma) = \begin{cases} g(\sigma) & \sigma \in K_0 \\ c + \dim \sigma & \sigma \notin K_0. \end{cases}$$

It is clear that $\tilde{g}$ satisfies the conditions in the definition of a discrete Morse function. □

Flow lines. Of course, what one really wants is an extension of g that somehow interpolates between the values on K_0, and whose flow lines mimic the flow lines generated by the computational Morse-Smale algorithm.

Problem 0.2. Let g be a function on K_0. Is it possible to extend g to a discrete Morse function $\tilde{g} : K \longrightarrow \mathbb{R}$ such that the flow lines in the sense of discrete Morse theory "equal" the flow lines of g generated by the computational algorithm?

Here is a putative solution to this problem. Assume that K is a 2-dimensional complex in $\mathbb{R}^3$, and let f be a function defined on K_0. We may always perturb f if necessary to assure that it is one-to-one. Given a vertex v in K_0, consider the vertices adjacent to v and label such a u $+$ or $-$ depending on whether $f(u)$ is greater than or less than $f(v)$. We say that v is a *k-fold saddle point* if the number of times the sign changes from $+$ to $-$ is $k+1$. For example, a 0-saddle is one where on one side of the link of v, all the signs are $+$ and on the other side all the signs are $-$ ("side" means a consecutive path of edges in the link); thus, a 0-saddle is a regular point in the usual sense. A 1-saddle has two increasing directions and two decreasing directions and corresponds to a saddle in the usual sense. An *ascending flow line* in K is a path beginning at a k-fold saddle ($k \geq 1$), and flowing in the direction of the steepest edge. Precisely, the first edge goes to the largest value adjacent to v to a vertex v_1. If v_1 is regular, then the path continues along the steepest edge from v_1, etc. The path ends at an ℓ-fold saddle or at the maximum vertex. Note that if v is a k-fold saddle, there are k ascending arcs beginning at v. Similarly, we define the k descending arcs by flowing to the smallest value from each vertex.

In discrete Morse theory, an ascending arc corresponds to a sequence of simplices from a critical edge to a critical 2-simplex, passing through regular cells only. A descending arc is a path from a critical edge to a critical vertex. What we want to do is associate such an ascending arc to a computational ascending arc, and similarly assign a descending arc to a computational descending arc. This requires that we extend our function f on K_0 to a discrete Morse function on all of K in such a way that

(1) we may choose a sequence of 2-simplices "close" to an ascending arc with only the initial edge and final 2-simplex critical; and
(2) we may choose a sequence of edges "close" to a descending arc with only the initial edge and final vertex critical.

Here is an algorithm to do this.

Step 1. Classify the vertices as maxima, minima, k-fold saddles, and regular points.

Step 2. For each regular or maximal vertex, take the edge of steepest descent from it. To those edges assign the average of the values of the endpoints (perhaps $\pm\varepsilon$).

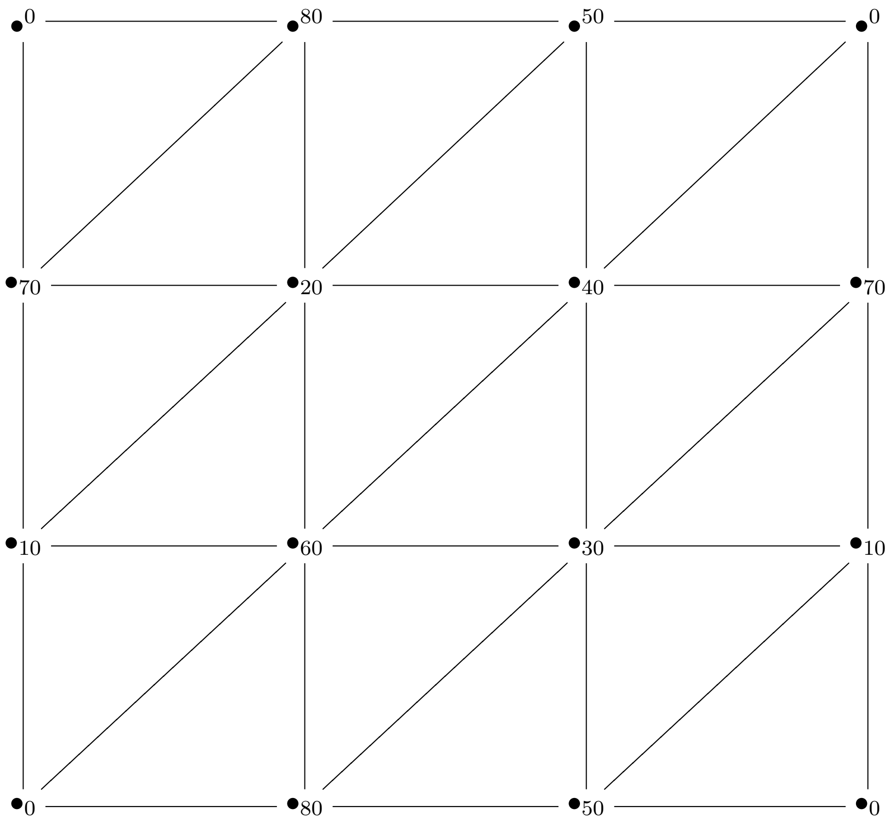
Step 3. For each k-fold saddle v ($k \geq 1$), choose $k+1$ edges of steepest descent, one in each component of the link, emanating from v. We designate k of these as the critical edges that will be the beginning edges of the k descending paths in the discrete sense. Define f on these edges to be $f(v)+\varepsilon_i$, one ε_i for each edge. Since a k-fold saddle should correspond to k regular saddles, we expect to get k descending arcs in the discrete sense.

Step 4. Let R be the union of all of these edges; we call these the *rivers*. These are the descending paths in K.

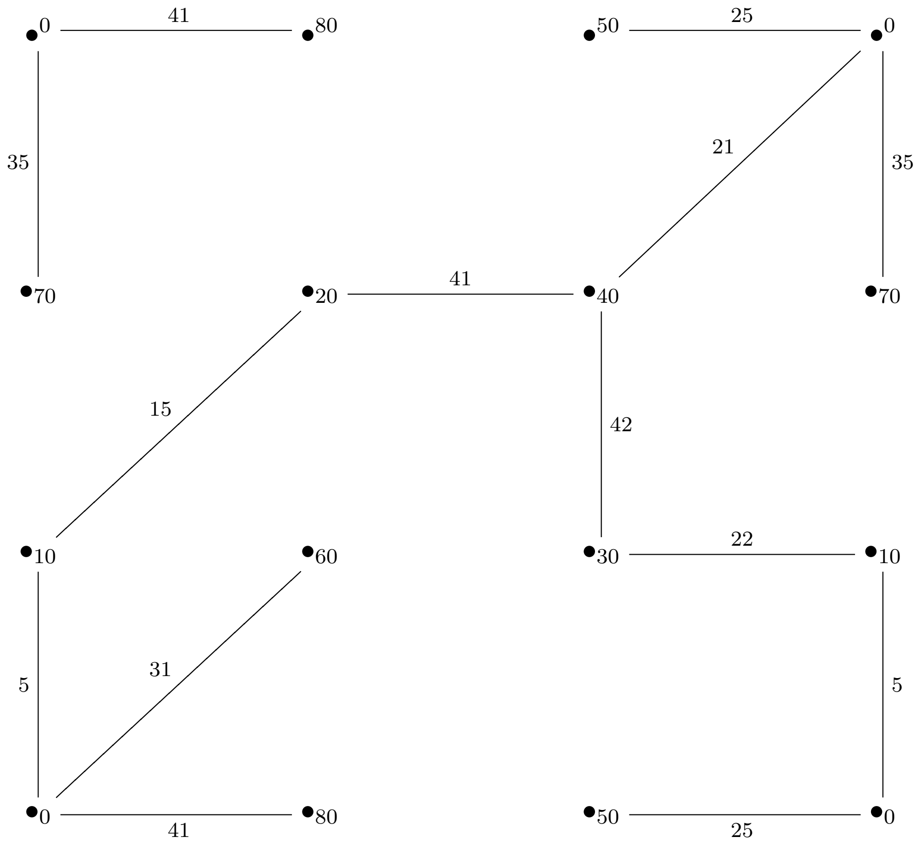
Step 5. Now consider the complement $K-R$. We claim this is contractible. Choose the vertex v of lowest value in the closure of $K-R$. The link in $K-R$ of v consists of a path of vertices of higher value, and so we may collapse this. Then proceed along the vertices in increasing order. At each stage, the link in $K-R$ of a vertex consists only of edges joining higher values and thus is an interval; this can be collapsed. Eventually, we come to the maxima and no further collapsing is possible. Thus, $K-R$ is a union of contractible pieces, one for each maximum.

Step 6. Let D be the dual graph of $K-R$. This is a forest. In each tree, the bottom leaves correspond to 2-simplices having 2 edges in R. These edges have an f value assigned. Define f on this 2-simplex to be $\max+\varepsilon$ where max is the greater of the two edge values. Then by general graph theory, there is a monotonically increasing association of values to each vertex in the tree. At each stage, we must guarantee that the value placed on an edge or vertex is compatible with the definitions of a Morse function, but this is easily accomplished by adding some value to each of the edges and vertices above it (i.e. a set of at most three inequalities must be satisfied, each of which asserts that some quantity must be larger than another). This defines f on the entire complex K, and the paths in the forest D give the ascending arcs in Forman's sense (these are paths leading to a critical 2-simplex).

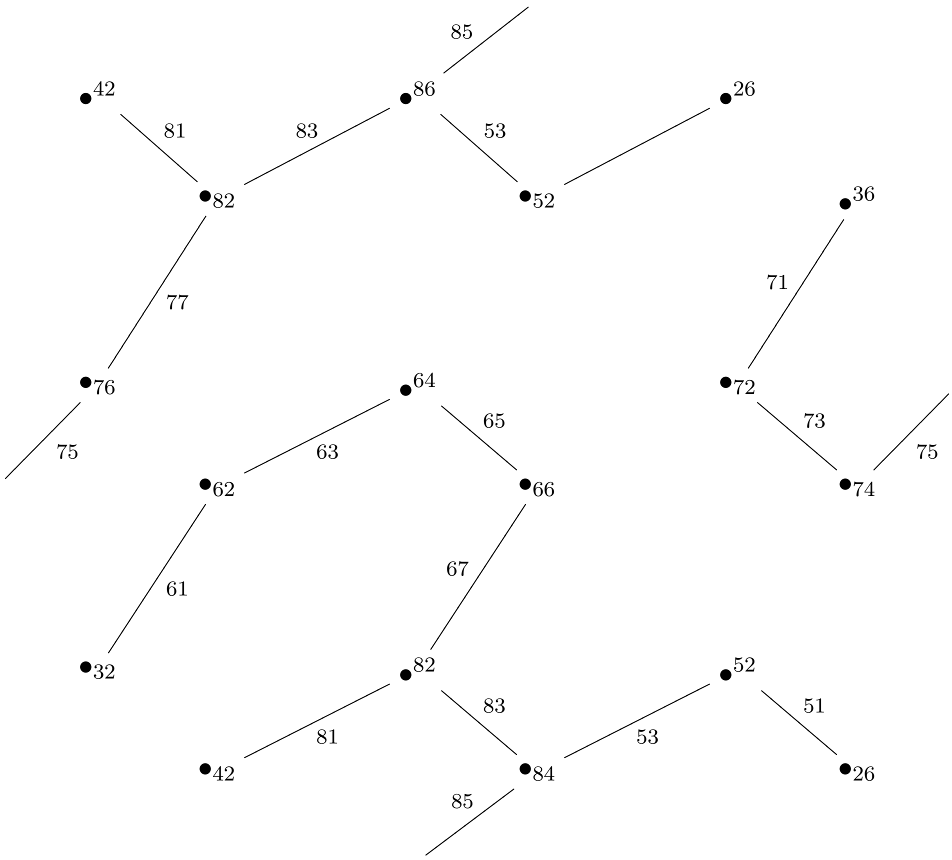
Let us illustrate this with an example. Here is a triangulation of the torus with a function defined on the vertices. Note that the top and bottom edges are to be identified.



Note that the vertex with value 40 is a 2-fold saddle. We choose the three edges of steepest descent in each component of the link, as well as the edges of steepest descent from each regular vertex and maximum. We then compute the rivers. The result is as follows. The values on the edges are chosen to be the average of the endpoints for non-critical edges (perhaps plus some ε). We have chosen the edges from 40 to 20 and from 40 to 30 to be the two critical edges.

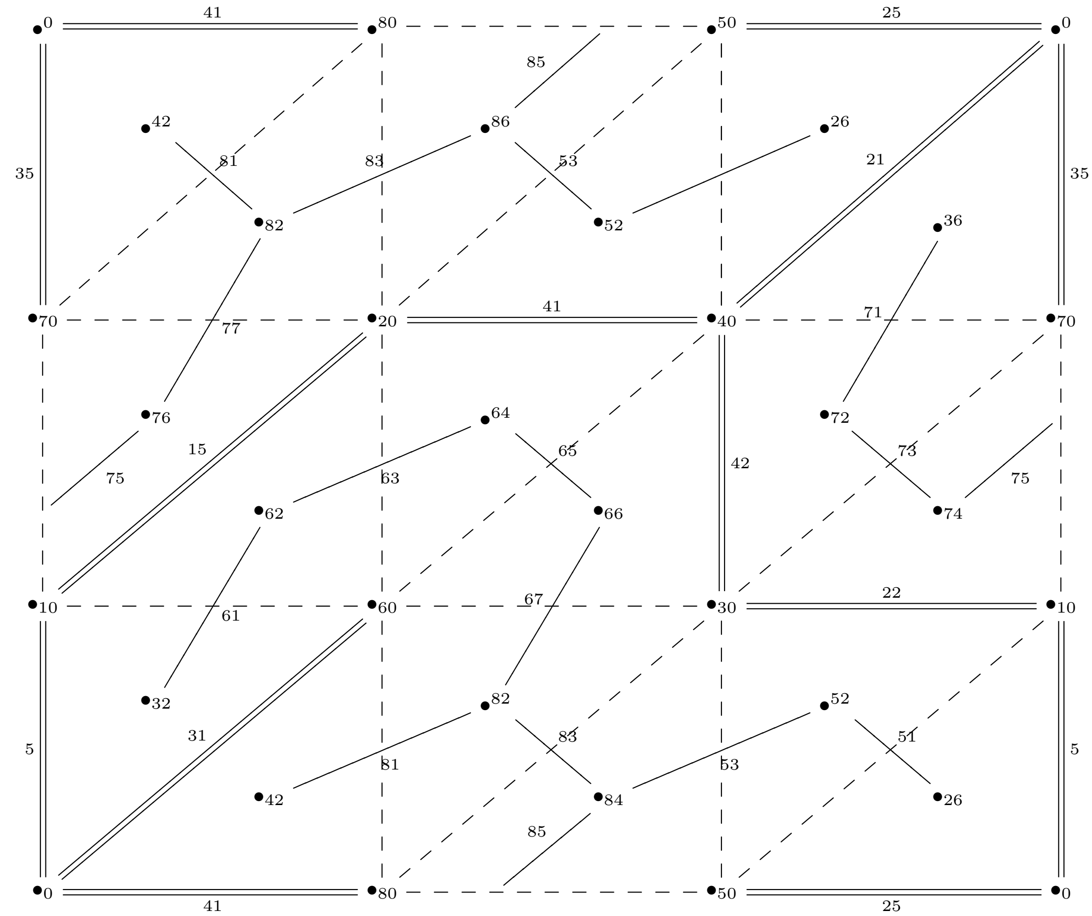


Notice that the complement of the rivers is a disc. To get the ascending arcs, we construct the dual graph in the complement and assign a value to each vertex and edge (this corresponds to assigning a value on each 2-simplex and edge not in a river in the torus). Here is one possible labelling.



From this tree, one can read off paths beginning from minimal 2-simplices proceeding up to the maximal 2-simplex (whose value is 86).

Here is an overlay of the three pictures. The rivers are labelled with doubled edges, and the other edges in the triangulation are labelled with dotted edges.



One may also ask the opposite question.

Problem 0.3. Given a discrete Morse function on a simplicial complex K, is it possible to look at this as a Morse-Smale complex and associate to it a flow in the sense of the computational Morse theory?

The idea is the following. A discrete Morse function on K associates to every simplex of K a certain value, respecting rules (1) and (2). It thus associates a value to every vertex in the first barycentric subdivision $\tilde{K}$ of K, and can be viewed as a Morse function $\tilde{f}$ on the complex $\tilde{K}$ in the sense of computational Morse theory.

Every critical simplex of f corresponds to a critical point of $\tilde{f}$. In 2D, a critical simplex of dimension two will represent a maximum of $\tilde{f}$, a critical simplex of dimension 1 will be a saddle point of $\tilde{f}$ and a critical simplex of dimension 0 will be a minimum of $\tilde{f}$ (this is because of conditions (1) and (2)).

A discrete Morse function f on K defines a flow along the regular simplices starting at any critical simplex of dimension > 0. It points from a given simplex in the direction in which the function descends. This direction is well defined when we go from an edge either to a vertex or to a triangle. From a triangle to an edge, there are two possibilities. We choose the steeper one.

On $\tilde{K}$ this flow determines the descending lines of the flow of $\tilde{f}$. Starting at any critical point of $\tilde{f}$ which is not a minimum, we have a descending line along the 1-skeleton of $\tilde{K}$. In the same way we obtain ascending lines starting at any point which is not a maximum, going in the direction opposite to the flow.

Wrap. A question posed by H. Edelsbrunner is the following. Can the "wrap" algorithm for triangulating data sets be placed inside discrete Morse theory? In other words, given a triangulation K of a point set S (say the Delaunay or weighted Delaunay triangulation), is there a discrete Morse function on K such that the simplices collapsed by wrap are not critical simplices. The answer is provided by the following result of Forman. We first need some terminology. For a discrete Morse function on a simplicial complex M, the complex $M(c)$ is defined to be

$$M(c) = \bigcup_{f(\tau)\leq c} \bigcup_{\sigma \leq \tau} \sigma.$$

Suppose $\sigma^{(p)} < \tau^{(p+1)}$ are two cells of M such that σ is not a face of any other cell, and set $N = M - (\sigma \cup \tau)$. In this case, we say that M *collapses* onto N. In general, we say that M collapses onto N if M can be transformed into N by a finite sequence of such operations. These are precisely the operations performed by the wrap algorithm.

Proposition 0.4 (cf. Lemma 4.3 of [1])**.** *Let M be a simplicial complex and $N \subset M$ a subcomplex such that M collapses onto N. Let f be a Morse function on N and let $c = \max_{\sigma\subseteq N} f(\sigma)$. Then f can be extended to a Morse function on M with $N = M(c)$ and such that there are no critical points in $M - N$.*

Proof. By induction on the number of elementary collapses, it is sufficient to prove this when M collapses onto N by a single collapse. Suppose σ is a simplex of M with a free face $\tau < \sigma$ such that M is a disjoint union

$$M = N \cup \sigma \cup \tau.$$

Define a Morse function $\tilde{f}$ by setting

$$\begin{aligned} \tilde{f}(\nu) &= f(\nu), \quad \nu \neq \sigma, \tau \\ \tilde{f}(\sigma) &= c + 1 \\ \tilde{f}(\tau) &= c + 2. \end{aligned}$$

It is easy to check that $\tilde{f}$ has the required properties. □

In the context of wrap, the proposition implies that we may choose any Morse function on the retracted complex that we want, and then extend it in this way so that the flow gives the retraction.

References

[1] R. Forman, *Morse theory for cell complexes,* Adv. Math. **134** (1998), 90–145.
[2] R. Forman, *A user's guide to discrete Morse theory,* preprint available at http://www.math.rice.edu/∼forman

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