Surface Registration via Umbilics

Thomas Grandine* Jung-ha An† Viktoria Averina‡ Giulio Ciraolo§
Wondimagegnehu Geremew¶ Derek Hansen|| Guo Luo**
Todd Moeller††

8/18/04

Abstract

In this paper, we implement and test a surface registration algorithm [Ko] which aligns surfaces in different coordinate systems by locating and matching their generic umbilics. Since, by the very definition, generic umbilics are isolated, are invariant under rigid motion/scaling and are stable under small perturbations, they provide an ideal way to capture intrinsic geometric properties of smooth surfaces. In this paper we discuss the implementation of such an algorithm by first examining how to extract and classify umbilics, and then providing details on the registration process. Experimental results on both artificial and real world test cases are also presented and discussed.

1 Introduction

Surface matching, also known as surface registration, is an important industrial application. For example, in the manufacturing process a surface is first designed in a CAD system, then manufactured, and then shipped to a destination. During manufacturing and shipping some deformation (such as sagging due to the weight of the surface) may occur and could affect the performance of the final product. For example, minor deformations of an airplane wing can lead to an airplane crash. Therefore, it is important to develop a method to measure the deviation between the designed and manufactured surfaces. A major obstacle is that the designed surface and the (digitally scanned) manufactured surface are given in different coordinate systems. Thus, we seek a method that allows us to place these surfaces in one coordinate system.

*The Boeing Company (thomas.a.grandine@pss.Boeing.com)
†University of Florida (jungha@math.ufl.edu)
‡University of Minnesota (averina@math.umn.edu)
§Università degli Studi di Firenze (ciraolo@math.unifi.it)
¶Wayne State University (gwondi@math.wayne.edu)
||Rutgers University (djhansen@math.rutgers.edu)
**Ohio State University (gluo@math.ohio-state.edu)
††Georgia Institute of Technology (moeller@math.gatech.edu)

There are several methods developed for this purpose. One of the most recent was proposed by [Ko] and [Maek]. This method is based on indentifying certain geometric “fingerprints” of a surface – generic umbilical points. An umbilical point (or umbilic) is a point on a surface at which the principal curvatures are equal [doC]. What makes these points especially attractive is that they are independent of the parametrization of the surface and stable with respect to small perterbations of the parameterization.

Our goal is to implement this method for some simulated examples and some real-world examples from the airframe industry and to test the robustness of the resulting matching. We assume the surfaces to be matched are C^2 and represented by cubic B-splines.

In this report we explain the details of the implementation of the chosen matching method. We follow three major steps. First, we find the umbilics on each surface and classify them according to their types. Second,we find the correct correspondence between the umbilics of the two surfaces. This is done by constructing a cost function that includes penalty terms for matching umbilics of different types and also accounts for the possibility of different orientations of the surface parameterizations. Finally, we solve the Procrustes problem to find the orthogonal matrix and translation vector which minimize the distance between the corresponding umbilics of the two surfaces.

Our results suggest the above method is not as accurate as desired. However, it produces a very good initial guess for further improvement of surface registration.

This paper is organized as follows. Section 2 provides the necessary background on B-splines and differential geometry. In Section 3, 4, and 5, respectively, we describe in detail the process of locating, classifying and matching umbilics on two given surfaces. The algorithm for solving the Procrustes problem of aligning the matched umbilics is presented in Section 6, which is followed by experimental results, and the paper is concluded by a section of brief discussion.

2 Background

2.1 B-Splines

The surfaces discussed in this paper are modeled using *B-splines*. To define a B-spline, we first introduce the notion of *knots* and *B-spline basis functions*.

Definition 2.1 *Let* $U = \{u_0, u_1, \ldots, u_n\}$ *with* $u_0 \le u_1 \le \ldots \le u_n$. *Each* u_i *is a* **knot** *and* U *is a* **knot vector**.

Definition 2.2 *Given a knot vector* U, *the* **B-spline basis functions of degree p** *corresponding to* U *are given by*

$$B_{i,0}(u) = \begin{cases} 1, & \textit{if } u_i \le u < u_{i+1} \\ 0, & \textit{otherwise} \end{cases} ,$$

$$B_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} B_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} B_{i+1,p-1}(u).$$

We are now ready for the definition of B-spline (curves):

Definition 2.3 *Given U, a* **B-spline curve** *(or simply a* **B-spline***) in $\mathbb{R}^3$* **of degree p** *is a function $f: [u_0, u_n] \to \mathbb{R}^3$ which takes the form:*

$$f(u) = \sum_i \alpha_i B_{i,p}(u).$$

The vectors α_i are called **control points**.

B-spline surfaces can be defined via the tensor product of two B-spline curves. Specifically, a B-spline surface of degree $(p+q)$ has the form:

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{i,j} B_{i,p}(u) C_{j,q}(v)$$

where $B_{i,p}$ and $C_{j,q}$ are B-spline basis functions of degree p and q. B-splines have many nice properties that make them important in many applications such as data modeling/fitting, computer aided design (CAD), computer graphics and computer animation. We refer interested readers to [deB] for further information.

2.2 Principal Curvatures and Principal Directions

Consider a smooth surface Σ in $\mathbb{R}^3$ with explicit representation $z = h(x, y)$. Since the curvature of a surface is defined in terms of second derivatives, the Hessian of $h(x, y)$ is the fundamental tool for deriving local geometric information of Σ.

Definition 2.4 *The* **Hessian** *of Σ at a point $p \in \mathbb{R}^3$ is the bilinear form:*

$$\mathrm{Hess}_p = \begin{pmatrix} h_{xx}(p) & h_{xy}(p) \\ h_{yx}(p) & h_{yy}(p) \end{pmatrix} : T_p\Sigma \times T_p\Sigma \to \mathbb{R},$$

where T_p is the tangent plane of Σ at p.

Note that the smoothness of the surface ensures the symmetry of the Hessian. Therefore, all eigenvalues of Hess_p are real and its eigenvectors form a complete orthogonal set in $\mathbb{R}^3$. As a result we are able to make the following definitions.

Definition 2.5 *The unit eigenvectors of* Hess_p *are referred to as the* **principal directions** *of Σ at p. The corresponding eigenvalues are called the* **principal curvatures** *of Σ at p. The principal curvatures are denoted κ_1 and κ_2 where $\kappa_1 \geq \kappa_2$.*

Geometrically, the principal curvatures κ_1 and κ_2 determine the maximum and minimum amount of bending of Σ at p. Several important quantities that characterize the local geometric properties of the surface can be defined based on κ_1 and κ_2.

Definition 2.6 *The* **Gauss curvature** *of Σ at p is defined to be $K = \kappa_1\kappa_2$.*

Definition 2.7 *The* **mean curvature** *of Σ at p is defined to be $H = (\kappa_1 + \kappa_2)/2$.*

The point at which $\kappa_1 = \kappa_2$ is of special importance to us.

Definition 2.8 *A point $p \in \Sigma$ is said to be an* **umbilic** *of Σ if $\kappa_1 = \kappa_2$ at p. The set of umbilics on Σ is denoted by* $\mathrm{Umb}(\Sigma)$.

As pointed out in [Ko], umbilics can be isolated or form lines or regions. Isolated umbilics can be classified into two types based on the stability with respect to small perturbations: *generic* and *non-generic*. Generic umbilics are stable with respect to small perturbations to surfaces and they are the ones we will utilize to perform surface registration.

3 Numerical Extraction of Umbilics

The first step in aligning two surfaces is to find all their (generic) umbilics. In this section we describe the numerical methods for locating umbilics, and for this purpose a brief review of some basic concepts from differential geometry is necessary.

In the sequal we assume that the surface Σ has the *parametric* representation

$$S(u, v) = [x(u, v), y(u, v), z(u, v)],$$

where $(u, v) \in [0, 1] \times [0, 1]$. The first fundamental form **I** and the second fundamental form **II** of Σ are defined as:

$$\begin{aligned} I &= dS \cdot dS = E \cdot du^2 + 2F \cdot dudv + G \cdot dv^2, \\ II &= -dS \cdot dN = L \cdot du^2 + 2M \cdot dudv + N \cdot dv^2, \end{aligned}$$

where

$$N = \frac{S_u \times S_v}{\|S_u \times S_v\|},$$
$$E = S_u \cdot S_u, \qquad F = S_u \cdot S_v, \qquad G = S_v \cdot S_v,$$
$$e = N \cdot S_{uu}, \qquad f = N \cdot S_{uv}, \qquad g = N \cdot S_{vv}.$$

It can be shown that [doC]:

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)},$$

and the principal curvatures can be calculated by:

$$\kappa = H \pm \sqrt{H^2 - K}.$$

So, if we let $W = H^2 - K$, the necessary and sufficient condition of a point $S(u, v)$ on Σ being an umbilic would be $W = 0$. Since W is always non-negative, at an umbilic W must attain its global minimum. Consequently the gradient of W must also vanish at an umbilic, i.e.

$$\frac{\partial W}{\partial u} = 0, \qquad \frac{\partial W}{\partial v} = 0,$$

must hold at an umbilic.

Since we are restricting our attention to B-spline surfaces, $W = \frac{P_N}{P_D}$ where P_N is a piecewise polynomial. Hence solving for umbilics reduces to finding the roots of polynomial equations:

$$P_N(u,v) = 0, \qquad \frac{\partial P_N(u,v)}{\partial u} = 0, \qquad \frac{\partial P_N(u,v)}{\partial v} = 0, \tag{3.1}$$

where $P_N = (Eg + Ge - 2Ff)^2 - 4(eg - f^2)(EG - F^2)$. It can be shown that the degree of P_N is bounded by $(10(n-1) - 2)$, so, since we are using cubic B-splines, the system we have to solve consists of piecewise polynomials of degree 18.

The solution procedure is divided into two steps. First, we compute the critical points of P_N, i.e. the solutions of the system:

$$\begin{cases} \dfrac{\partial P_N(u,v)}{\partial u} = 0 \\ \dfrac{\partial P_N(u,v)}{\partial v} = 0 \end{cases}.$$

Next, for each critical point (u, v), we check whether it is an umbilic by considering following two inequalities:

$$|\kappa_2(u,v) - \kappa_1(u,v)| \;<\; \tau_1 \cdot |H(u,v)|, \tag{3.2}$$

$$|P_N(u,v)| \;<\; \tau_2 \cdot \max_{u,v\in[0,1]} P_N(u,v), \tag{3.3}$$

where τ_1, τ_2 are user-defined positive constants (tolerance). (u, v) is considered an umbilic if both inequalities are satisfied.

4 Classification of Umbilics

As previously discussed, the generic umbilics of a surface are of special interest since they are stable with respect to small perturbations. We present three different classication schemes for generic umbilics: *catastrophe* classification, *pattern* classification, and *index* classification [Ber].

To understand the different classifications of generic umbilics, consider a smooth surface Σ in $\mathbb{R}^3$ with explicit representation $z = h(x, y)$. Suppose U is an umbilic on Σ, then, after a suitable rigid transformation, h can be brought into the Monge form [Maekawa96]:

$$\begin{aligned} h(x,y) &= -\frac{\kappa(0,0)}{2}(x^2+y^2) \\ &\quad + \frac{1}{6}\left[\alpha x^3 + 3\beta x^2 y + 3\gamma x y^2 + \delta y^3\right] + O((\sqrt{x^2+y^2})^4), \end{aligned} \tag{4.4}$$

where

$$\alpha = h_{xxx}(0,0), \qquad \beta = h_{xxy}(0,0), \qquad \gamma = h_{xyy}(0,0), \qquad \delta = h_{yyy}(0,0).$$

In polar coordinates, the above equation can be written as:

$$\begin{aligned} h(r,\theta) &= -\frac{\kappa(0,0)}{2} r^2 \\ &\quad + \frac{r^3}{6}\left[\alpha\cos^3\theta + 3\beta\cos^2\theta\sin\theta + 3\gamma\cos\theta\sin^2\theta + \delta\sin^3\theta\right] + O(r^4). \end{aligned} \tag{4.5}$$

For the catastrophe classification, the umbilic is classified as either elliptic or hyperbolic, which is determined by the coefficients of the third order term of h [Ber]:

$$\text{if } C(\alpha,\beta,\gamma,\delta) \equiv 4(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2)-(\alpha\delta-\beta\gamma)^2 \begin{cases} >0, & \text{then elliptic} \\ <0, & \text{then hyperbolic} \end{cases}.$$

For the pattern classification, U falls into three categories: lemon, star, and monstar, depending on the number of lines of curvature passing through the umbilic. There are three lines of curvature passing through a star (monstar) while there is only one passing through a lemon. It can be shown that the direction angles of these lines are the extrema of the cubic term h_c in (4.5) [Ber]:

$$h_c(\theta) = \frac{r^3}{6}\left[\alpha\cos^3\theta + 3\beta\cos^2\theta\sin\theta + 3\gamma\cos\theta\sin^2\theta + \delta\sin^3\theta\right]. \tag{4.6}$$

So, to determine the number of lines of curvature passing through U, it suffices to find all critical points of h_c. In other words, we only need to solve the equation:

$$\frac{dh_c(\theta)}{d\theta} = \frac{r^3}{2}\left(\beta\cos^3\theta - a_1\sin\theta\cos^2\theta + a_2\sin^2\theta\cos\theta - \gamma\sin^3\theta\right) = 0 \tag{4.7}$$

where

$$a_1 = \alpha - 2\gamma, \qquad a_2 = \delta - 2\beta.$$

For the index classification, U has index $\frac{1}{2}$ if it is a lemon (monstar) or $-\frac{1}{2}$ if it is a star. Here the index is defined to be the amount of rotation the principal direction experiences when moving (in a counterclockwise direction) along a closed curve enclosing the umbilic. Two approaches can be used to calculate the index of an umbilic. The first approach [Mae] evaluates the angle ψ_i of the principal direction at $n+1$ points along a closed curve surrounding U. The change of the angle is accumulated and the index Ind is computed by

$$\text{Ind} = \frac{1}{2\pi}\sum_{i=0}^{n}(\psi_{(i+1)\bmod n} - \psi_i), \qquad \text{where } \psi_{(i+1)\bmod n} - \psi_i \in [-\pi/2, \pi/2]. \tag{4.8}$$

In the second approach [Ber], the index Ind is determined by using the following criterion:

$$\text{if } J(\alpha,\beta,\gamma,\delta) \equiv \alpha\gamma - \gamma^2 + \beta\delta - \beta^2 \begin{cases} >0, & \text{then Ind} = \frac{1}{2} \\ <0, & \text{then Ind} = -\frac{1}{2} \end{cases}. \tag{4.9}$$

When U is an interior point on Σ, both methods give the same result and either one of them can be used. When U is on the boundary, it is impossible to find a closed curve to enclose the umbilic, so the second method should be used. In our work, we employ the second method to calculate the index of a generic umbilic. But if desired, the first method can also be used to calculate the index of a non-generic umbilic. In fact, we have applied both methods to a paraboloid which has a non-generic umbilic at its vertex, and the first method found the index 1 successfully while the second one failed (remember that non-generic umbilic is sensative to small perturbations, so the discriminant J can be either positive or negative due to round-off errors).

By combining all three classification schemes (catastrophe, pattern and index), we are able to classify all generic umbilics by using the following algorithm (see text for explanation):

```
# Pattern classification
if |beta| < tol and |gamma| < tol
  # theta = 0, theta = pi/2 are two real roots

  if |alpha| < tol
    if |delta| < tol
      # All theta are solutions
      type = 'non-generic'
    else
      # theta = 0 is another real root (double root)
      type = 'non-generic'
  else
    if |delta| < tol
      # theta = pi/2 is another real root (double root)
      type = 'non-generic'
    else
      # theta = atan(alpha/delta) is a real root
      # May be star or monstar
      type = 'undetermined'
else
  # The discriminant
  dscr = q**2 + p**3
  if dscr > tol
    # Only one real root
    type = 'lemon'
  elseif |dscr| < tol
    # Three real roots, but at least two of which are equal
    type = 'non-generic'
  else
    # Three distinct real roots. Can be star or monstar
    type = 'undetermined'

# Index classification
if type != 'non-generic'
  calculate its index using the second method

if type == 'undetermined'
  if index == 0.5
    type = 'monstar'
  else
    type = 'star'

# Catastrophe classification
if type == 'star'
  compute C
```

```
if C > 0
  type = 'starE'
else
  type = 'starH'
```

Several remarks are in order.

- When $\beta = \gamma = 0$, $\theta = 0$, $\theta = \frac{\pi}{2}$ are two real roots of $dh_c/d\theta$ and (4.6) essentially reduces to a linear equation. If α, δ are not both zero and the third real root of (4.6) is different from 0 and $\frac{\pi}{2}$, then the umbilic might be a star or monstar. If both α, δ vanish (in which case there are infinitely many solutions) or the third real root is 0 or $\frac{\pi}{2}$, a small perturbation will change the behavior of the umbilic[1] and in this case it is a non-generic one.

- When $\beta \neq 0$, θ cannot be 0 or π, so we divide the equation by $\beta \sin^3 \theta$ to obtain:

$$t^3 - \frac{a_1}{\beta} t^2 + \frac{a_2}{\beta} t - \frac{\gamma}{\beta} = 0,$$

  where $t = \cot \theta$. Similarly, when $\gamma \neq 0$, θ cannot be $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, so we divide the equation by $\gamma \cos^3 \theta$ to obtain:

$$\hat{t}^3 - \frac{a_2}{\gamma} \hat{t}^2 + \frac{a_1}{\gamma} \hat{t} - \frac{\beta}{\gamma} = 0,$$

  where $\hat{t} = \tan \theta$. After a substitution

$$t = s + \frac{a_1}{3\beta}, \qquad \hat{t} = s + \frac{a_2}{3\gamma},$$

  the equation is brought into the normal form:

$$s^3 + 3ps + 2q = 0$$

  where[2]

$$\begin{aligned} \text{when } \beta \neq 0 \qquad p &= \frac{3\beta a_2 - a_1^2}{9\beta^2}, \\ q &= \frac{-a_1[2a_1^2 - 9a_2\beta] - 27\beta^2\gamma}{54\beta^3}, \\ \text{when } \gamma \neq 0 \qquad p &= \frac{3\gamma a_1 - a_2^2}{9\gamma^2}, \\ q &= \frac{-a_2[2a_2^2 - 9a_1\gamma] - 27\beta\gamma^2}{54\gamma^3}. \end{aligned}$$

- The general criteria used to differentiate between lemon and star (monstar) are as follows [Mae]:

[1] As mentioned in [Mae], the three direction angles of lines of curvature passing through a monstar are contained in a right angle, which is not true for star. If the three angles are given by, say, 0, 0, $\frac{\pi}{2}$, then a small perturbation may result in three distinct angles that are either contained or not contained in a right angle.

[2] The equations given in [Mae] are wrong.

– If $q^2+p^3 > 0$, there is only one real root for (4.7) and it corresponds to an extremum of h_c. Thus the umbilic is a lemon.

– If $q^2 + p^3 = 0$, there are three real roots at least two of which are equal. Since the discriminant may become either positive or negative after a small perturbation, in this case we have a non-generic umbilic.

– If $q^2 + p^3 < 0$, there are three distinct real roots all of which correspond to the extrema of h_c. So the umbilic is a star or monstar.

- In the implementation of above algorithm, we compare the absolute value of α, β, γ, and δ with a fixed positive number (`tol`) to determine whether it is close to zero or not. We take `tol` $= 10^{-6}$ for all our experiments.

5 The Assignment Problem

Once the umblical points of the two surfaces have been indentified, we need to find the "best match" between them. The match is determined by a linear assignment problem (LAP).

In a classical LAP, N elements of one set are bijectively assigned to N elements of a second set. The number of umbilics on Σ_1, however, is not necessarily the same as that on Σ_2. Consider the case where $\mathrm{Umb}(\Sigma_1) = \{p_1, p_2, \ldots, p_m\}$ and $\mathrm{Umb}(\Sigma_2) = \{q_1, q_2, \ldots, q_n\}$ with $m \leq n$. We reduce to the classical setting by adding $(n-m)$ "phantom" umbilics to $\mathrm{Umb}(\Sigma_1)$. We adopt the notation $p_i \sim q_j$ to mean p_i is assigned q_j.

Our LAP takes the form:

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{i,j} d_{ij} x_{ij}, && (5.1)\\ \text{s.t.} \quad & \textstyle\sum_j x_{ij} = 1, && (5.2)\\ & \textstyle\sum_i x_{ij} = 1, && (5.3)\\ & x_{ij} \in \{0, 1\}. && (5.4) \end{aligned}$$

The decision variables, x_{ij}, when optimized, encode the following information:

$$x_{ij} = \begin{cases} 1, & \text{if } p_i \sim q_j \\ 0, & \text{otherwise} \end{cases}.$$

Let

$$d_{ij}^{\pm} = \begin{cases} \left| 1 \pm \dfrac{H_{p_i}}{H_{q_j}} \right| + \omega * g_{ij}, & \text{if } |H_{p_i}| \leq |H_{p_j}|, \quad i = 1, \ldots, m \\ \left| 1 \pm \dfrac{H_{q_j}}{H_{p_i}} \right| + \omega * g_{ij}, & \text{if } |H_{p_i}| > |H_{p_j}|, \quad i = 1, \ldots, m \\ 0, & \text{if } i = m+1, \ldots, n \end{cases},$$

where

$$g_{ij} = \begin{cases} 0, & \text{if } p_i \text{ and } q_j \text{ have the same type} \\ 1, & \text{otherwise} \end{cases}$$

and ω is a weight parameter. When the orientations of Σ_1 and Σ_2 coincide we set $d_{ij} = d_{ij}^{-}$. Therefore the cost function favors assigning p_i to q_j if they share the same classification and have approximately the same mean curvature. If Σ_1 switches orientation, $d_{ij} = d_{ij}^+$ since reversing orientation reverses the sign of the mean curvature.

The Hungarian Method is used to solve the LAP. The output is a permutation (assignment) matrix $(x_{ij})_{i,j=1,\ldots,n}$ that minimizes the cost (5.1). We restrict our interest to the assignments made to the subset of umbilics, $\text{Umb}^{'}(\Sigma_2) \subset \text{Umb}(\Sigma_2)$, that have *not* been assigned to a phantom umbilic.

6 Procrustes Problem

Given the assignment matrix modulo the phantom assignments we align Σ_1 with Σ_2. We use the orthogonal Procrustes algorithm to generate the alignment. A review of Procrustes problem is presented before the algorithm.

6.1 Formulation of Procrustes problem

Suppose we are given $A_{m\times p}$ and $B_{m\times p}$, where each row of which contains the coordinates of a point in $\mathbb{R}^p$. Procrustes problem is formulated as follows:

$$\text{minimize} \quad \|A - BQ\|_F \tag{6.1}$$

$$\text{s.t.} \quad Q^TQ = I \tag{6.2}$$

where $\|\cdot\|_F$ is the Frobenius norm.

If $Q_{p\times p}$ is an orthogonal matrix, then

$$\|A - BQ\|_F^2 = \text{tr}(A^TA) + \text{tr}(B^TB) - 2\cdot\text{tr}(Q^TB^TA), \tag{6.3}$$

where $\text{tr}(\cdot)$ is the trace operator. Hence minimizing $\|A - BQ\|_F^2$ over Q is equivalent to maximizing $\text{tr}(Q^TB^TA)$ over Q. We maximize $\text{tr}(Q^TB^TA)$ by calculating the singular value decomposition (SVD) of the matrix B^TA. If $U^TB^TAV = D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$ is the SVD of B^TA and if we define the orthogonal matrix $Z = V^TQ^TU$, then

$$\text{tr}(Q^TB^TA) = \text{tr}(Q^TUDV^T) = \text{tr}(ZD) = \sum_{i=1}^{p} z_{ii}\sigma_i \leq \sum_{i=1}^{p} \sigma_i. \tag{6.4}$$

Equality is achieved and the maximum is attained when $Q = UV^T$. When (6.1) is zero, the rows of A are rotations ($\det(Q) = 1$) or reflections ($\det(Q) = -1$) of the rows of B. If (6.1) is non-zero, Q is the rotation or reflection matrix that best aligns the rows of A to the rows of B.

6.2 Algorithm

The LAP of Section 5 returns $\text{Umb}(\Sigma_1) = \{p_1, p_2, \ldots, p_m\}$, $\text{Umb}^{'}(\Sigma_2) = \{q_1, q_2, \ldots, q_m\}$, and the assignment. We use this information to construct two $m \times 3$ matrices, A and B, where $[A^T]_i = p_i$ and $[B^T]_i = q_i$. Without loss of generality assume $p_i \sim q_i$.

Algorithm 1 PROCRUSTES (A,B)

```
1: ā_ij = Σ_{k=1}^m (1/m) a_kj ; b̄_ij = Σ_{k=1}^m (1/m) b_kj
2: Ā = (ā_ij), B̄ = (b̄_ij)
3: C = (B − B̄)^T (A − Ā)
4: (U, V, D) ← SVD(C)
5: Q ← UV^T ; T ← Ā − B̄Q
6: RETURN Q, T
```

1: $\bar{a}_{ij} = \sum_{k=1}^{m} \frac{1}{m} a_{kj}$; $\bar{b}_{ij} = \sum_{k=1}^{m} \frac{1}{m} b_{kj}$
2: $\bar{A} = (\bar{a}_{ij})$, $\bar{B} = (\bar{b}_{ij})$
3: $C = (B - \bar{B})^T (A - \bar{A})$
4: $(U, V, D) \leftarrow \mathrm{SVD}(C)$
5: $Q \leftarrow UV^T$; $T \leftarrow \bar{A} - \bar{B}Q$
6: **RETURN** Q, T

The discussion on the mechanics of the registration algorithm concludes with pseudo-code for **PROCRUSTES**.

Two $m \times p$ matrices form the input of **PROCRUSTES**. Line 1 find the centroids of the data sets A and B. Line 2 translates the origin to the centroids of A and B. Line 3 and 4 solve the Procrustes problem for the augmented matrices. The orthogonal matrix and translation vector that best fit B to A are returned in line 6.

7 Results

We have applied the algorithm to six test cases, however here we will present only the three most interesting ones, which are referred to as Test A (Fig.1), Test B (Fig.2) and Test C (Fig.3), respectively. Each test case is comprised of two surfaces. In the first two test cases one surface is a slight perturbation of the other; in the last test case one surface is half of a "flying wing" aircraft while the other is its unperturbed reflection.

We start by extracting umbilical points as shown in Section 3 with $\tau_1 = 10^{-3}$ and $\tau_2 = 10^{-8}$ in (3.2). The (u, v) coordinates of the umbilics are shown in Tables 1 and 2. In Test C we found 100 umbilics on each surface; due to the size we have not included the corresponding table.

The second step is to classify the umbilical points using methods described in Section 4. The results are shown in Tables 1 and 2. *Hyperbolic star* umbilics are labelled "starH" while "starE" denotes *elliptic star* umbilics. Non-generic umbilics were detected only in Test C. The umbilics for one of the surfaces in each case are shown in Figures 4 through 6. Here we have color coded the umbilics according to their type: yellow – lemon, green – hyperbolic star, purple – elliptic star, and black – monstar. On one of the surfaces we have also plotted some of the lines of curvature.

In the third step we solved the assignment problem (see Section 5) to match the umbilics on the two surfaces. In the assignment problem we do not consider the non-generic points since they are not stable under surface perturbations. Results are listed in Tables 3-4. The LAP returns unique minima for Test A and Test C. In Test B there are two possible solutions that minimize the cost. One assignment insists that the two surfaces differ by a rotation and a translation. The other insists they differ by a rotation, reflection and a translation. The cost function d was not sensitive enough to differentiate between these two assignments.

Visual evidence for Test B assures us that the surfaces are not reflections of each other. Therefore we chose the assignment for which **PROCRUSTES** returns a rotation matrix ($\det(Q) = 1$).

Finally we solve the Procrustes problem and apply the resulting rotation matrix and translation vector to map the second surface into the vicinity of the first. See the attached mpeg file for illustration of the process for Test A. In Test C we note that the rotation matrix has determinant -1, as expected, since the surfaces are mirror images of each other.

As a conclusion to the surface registration algorithm we determine the maximal distance between points on two matched surfaces (see Table 5). On Figure 7 the reader can see how far the match for Test A is from the "original" difference between two surfaces (Surface 2 was constructed by first slightly perturbing the parameterization of surface one then applying a rigid motion. The purple surface in Figure 7 is Surface 2 *before* applying the rigid motion).

We have also included the VRML-files with some of the surfaces mentioned above. These files can be downloaded from the IMA website.

8 Discussion

Our implementation suggests the possibility of a robust-umbilic based surface registration method. To be of any industrial significance, however, the following three issues must be addressed.

1. Some interesting surfaces do not possess the minimum number of umbilics (three) required to generate a reliable surface match.

2. Experimentation suggests that small surface perturbations can result in a large migration of the umbilics. Such drifts jeopardize the reliability of the matching process.

3. Detecting umbilics on a surface modeled by B-splines of degree p requires solving a system of polynomial equations each of which has degree $\approx 10p$. Therefore the extraction process is computationally expensive for complicated surfaces.

Geometric solutions to the first two problems require insight into the way surface perturbations create and destroy umbilics. Some intuition would be gained by measuring the robustness of a generic umbilic under surface perturbations. Such a procedure is not immediately evident.

Furthermore the surface topology dictates, via Gauss-Bonnet Theorem, that umbilics are created and destroyed in a way that preserves the total surface curvature. This implies that all generic, and many non-generic, umbilics are created and destroyed in pairs. Determining the pairings might improve the sensitivity of the cost function and increase the reliability of the assignment process.

Acknowledgement

We would like to extend our sincere thanks to Dr. Thomas Grandine for his encouragement, insight and excellent project. We would also like to thank the IMA for the providing the opportunity to pursue this project. Finally, we would like to thank Niclas Börlin at the Department of Computing Science, Umea University, Sweden, for his efforts in implementing the matlab function of the Hungarian method. This function has been used in our program.

References

[1] [Ber] M.V. Berry and J.H. Hannay. Umbilic points on Gaussian Random surfaces. *Journal of Physics A.,* 10(11):1809-1821,1977.

[2] [Chua] C.S. Chua and R. Jarvis. 3d free-form surface registration and object recognition. *Internation Journal of Computer Vision,* 17(1):77-99, 1996.

[3] [deB] C.De Boor. A Practical Guide to Splines. Spinger-Verlag, New York, 1978.

[4] [Fri] D. Fritsch The Multiscale Medial Axis and Its Applications to Image Registration. *Pattern Recognition Letters* , 1994.

[5] [doC] P.M. Do Carmo, Differential Geometry of Curves and Surfaces. Prentice Hall; New Jersey, 1976

[6] [Gol] H. Golub and C.F. Van Loan, Matrix Computations, second edition. The Johns Hopkins University Press; Maryland, 1989.

[7] [Hil] M. Hilaga, et.al. Topology Matching for Fully Automatic Similarity Estimation of 3D Shapes. *SIGGRAPH 2001 Proceedings* , 2001.

[8] [Ko] Ko et. al, Shape Intrinsic Fingerprints for Free-From Object Matching.

[9] [Maek] T. Maekawa, F.-E. Wolter, and N.M. Patrikalakis. Umbilics and lines of curvature for shape iterrogation. *Computer Aided Geometric Design,* 12(2):133-161, March 1996.

[10] [Pot] H. Pottman, et. al. Registration without ICP. *Computer Vision and Image Understanding* , 2004.

Table 1: Umbilics for Test A

	SURFACE 1			SURFACE 2			
	u coord.,v coord.	mean curv.	type	u coord.,v coord.	mean curv.	type	
1	0.0355 , 0.7126	0.606734	starH	0.0248 , 0.2791	0.576719	starH	1
2	0.0688 , 0.0954	-4.06309	starE	0.0827 , 0.7973	-4.56627	starH	2
3	0.1571 , 0.6906	3.47718	lemon	0.2485 , 0.3115	5.10935	lemon	3
4	0.2451 , 0.9754	0.575938	starH	0.2541 , 0.7507	-2.54166	lemon	4
5	0.3176 , 0.7374	4.94267	lemon	0.2579 , 0.6094	-0.144636	starH	5
6	0.4366 , 0.1853	0.316987	starH	0.2878 , 0.0785	0.662057	starH	6
7	0.5946 , 0.7444	-0.220036	starH	0.2986 , 0.7643	-1.83519	lemon	7
8	0.7425 , 0.7511	-2.50583	lemon	0.3036 , 0.7600	-1.70052	starH	8
9	0.8377 , 0.9070	-4.95059	starH	0.3151 , 0.2370	3.83861	lemon	9
10	0.9463 , 0.2216	1.55737	starH	0.7776 , 0.9027	1.63854	starH	10
11	x , x	x	x	0.8147 , 0.4362	0.2912	starH	11
12	x , x	x	x	0.9027 , 0.1218	-3.819	starE	12

Table 2: Umbilics for Test B

	SURFACE 1			SURFACE 2			
	u coord.,v coord.	mean curv.	type	u coord.,v coord.	mean curv.	type	
1	0.0925 , 0.2316	-1.77872	starH	0.0936 , 0.2222	1.78356	starH	1
2	0.1536 , 0.8464	3.64467	starE	0.1548 , 0.8010	-3.6117	starE	2
3	0.3143 , 0.6857	1.12016	lemon	0.3106 , 0.6608	-1.1499	lemon	3
4	0.7327 , 0.2672	-0.709637	starE	0.7306 , 0.2521	0.702023	starE	4
5	0.7684 , 0.9075	-1.77872	starH	0.7700 , 0.8712	1.69311	starH	5

Table 3: Assignment problem for Test A

Surface 1	1	2	3	4	5	6	7	8	9	10	Phantoms	
Surface 2	6	12	9	1	3	11	5	4	2	10	8	7

Table 4: Assignment problem for Test B

Surface 1		1	2	3	4	5
Surface 2	First choice	1	2	3	4	5
Surface 2	Second choice	5	2	3	4	1

Table 5: Maximum distance between aligned surfaces

Test A	0.00571279155461
Test B	0.00338807941712
Test C	1.6401720477e-10

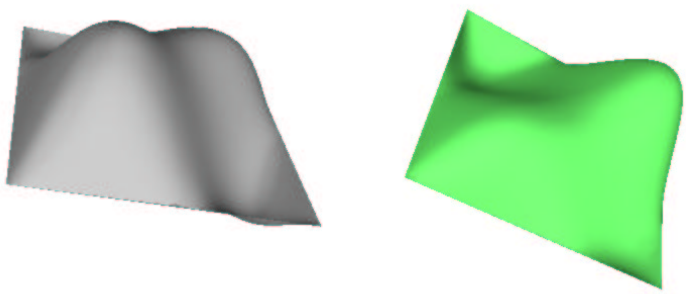

Figure 1: Surface 1 (green) and surface 2 (gray) from TEST A. Surface 2 is a small perturbation of surface 1.

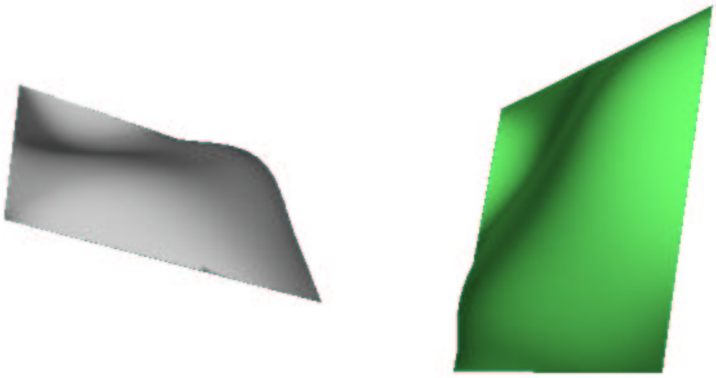

Figure 2: Surface 1 (green) and surface 2 (gray) from TEST B. Surface 2 is a small perturbation of surface 1.

Figure 3: Surface 1 (green) and surface 2 (gray) from TEST C. Surface 2 is an unperturbed reflection of surface 1.

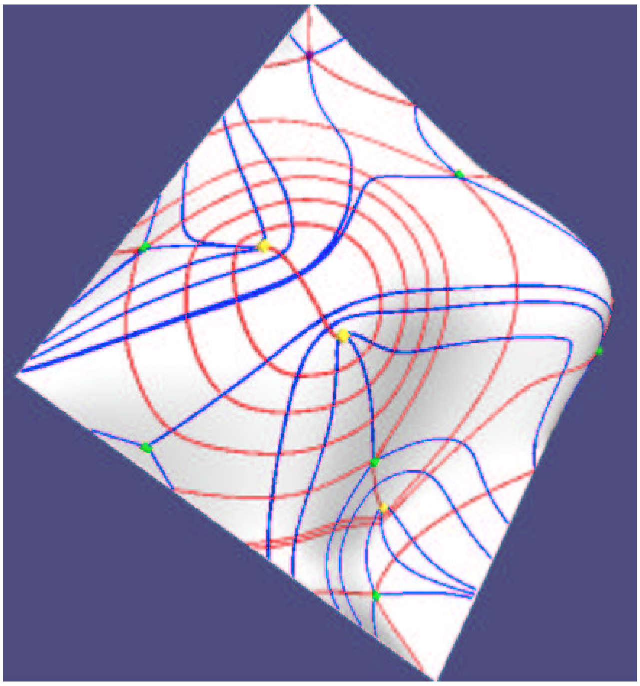

Figure 4: Surface 1 from TEST A. Red and blue lines are lines of maximum and minimum curvature, resp.

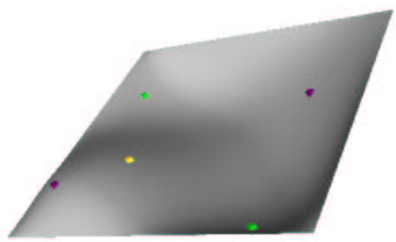

Figure 5: Surface 1 from TEST B.

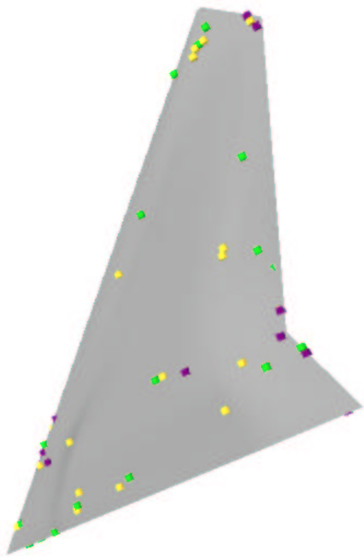

Figure 6: Surface 1 from TEST C.

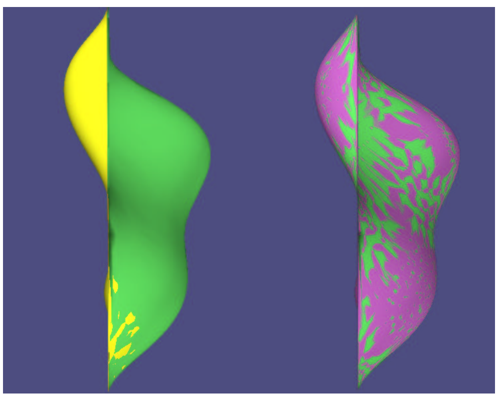

Figure 7: Our matching vs 'original' matching.