

**Classification and Reformulation:  
a Direct Way to Tackle Multi-Item Lot-Sizing Problems  
by MIP**

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# Outline

- Decomposition of Multi-Item Lot-Sizing
- Four Instances
- Single Item Classification
- Single Item: Reformulation, Separation, Optimization
- Further Classification
- The Instances: Classification, Reformulation, Solution
- Further Results
- Open Questions

## A Typical Multi-Item Lot-Sizing Problem

$NI$  is number of items

$NT$  is number of periods

$d_t^i$  is demand for  $i$  in  $t$

$p_t^i$  is unit production cost of  $i$  in  $t$

$h_t^i$  is unit storage cost per unit of  $i$  in  $t$

$f_t^i$  is fixed set-up cost if  $i$  produced in  $t$

$C_t^i$  is maximum production of  $i$  in  $t$

Single Machine - Capacity Constraints in Each Period,

Variables:

$x_t^i$  is production of  $i$  in  $t$

$s_t^i$  is stock of  $i$  at end of  $t$

$y_t^i = 1$  if set up to produce  $i$  in  $t$ ,  $y_t^i = 0$  otherwise

## Basic Formulation and Structure

$$\begin{aligned}
 \min \quad & \sum_{i,t} (p_t^i x_t^i + h_t^i s_t^i + f_t^i y_t^i) \\
 & s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t \\
 & x_t^i \leq C_t^i y_t^i \quad \forall i, t \\
 & \sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t \quad \forall t \\
 & x, s \geq 0, y \in \{0, 1\}
 \end{aligned}$$

What is the structure of the feasible region?

$$X = \left( \bigcap_{i=1}^{NI} Y^i \right) \cap \left( \bigcap_{t=1}^{NT} Z^t \right)$$

$Y^i$  is the single item lot-sizing region

$$\begin{aligned}
 Y^i = \{ & (x, s, y) \in R_+^{NT} \times R_+^{NT} \times \{0, 1\}^{NT} : \\
 & s_{t-1} + x_t = d_t + s_t, x_t \leq C_t y_t \quad \forall t \}
 \end{aligned}$$

$Z^t$  is the single period resource constraint region

$$\begin{aligned}
 Z^t = \{ & (x_t, y_t) \in R_+^{NI} \times \{0, 1\}^{NI} : \\
 & \sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t, x_t^i \leq C_t^i y_t^i \quad \forall i \}.
 \end{aligned}$$

Strategy: Find or approximate  $\text{conv}(Y^i)$  and  $\text{conv}(Z^t)$ .

## Four Problems

### Problem 1. Bottling lines

Four families of products. 30 day planning horizon. The line is available 16 hours per day, and only one family can be produced per day. There are storage, set-up and start-up costs.

$$\begin{aligned} \min \sum_{i,t} (p_t^i x_t^i + h s_t^i + f_t^i y_t^i + g_t^i z_t^i) \\ s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t \\ x_t^i \leq C_t^i y_t^i \quad \forall i, t \\ \sum_i y_t^i \leq 1 \quad \forall t \\ z_t^i \geq y_t^i - y_{t-1}^i \quad \forall i, t \\ z_t^i \leq y_t^i \quad \forall i, t \\ x, s \geq 0, y, z \in \{0, 1\} \end{aligned}$$

$z_t^i$  is a start-up variable. It takes value 1, if item  $i$  is set up in  $t$ , but not in  $t - 1$ .

## Problem 2. Sequence-dependent Changeover Costs

Produce at full capacity, so demands can be normalized with  $d^i \in \{0, 1\}$ . 10 items and a 35 period planning horizon. There are storage and sequence-dependent changeover costs.

$$\begin{aligned} \min \quad & \sum_{i,t} h^i s_t^i + \sum_{i,j,t} q^{ij} \chi_t^{i,j} \\ & s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t \\ & x_t^i \leq y_t^i \quad \forall i, t \\ & \sum_i y_t^i = 1 \quad \forall t \\ & \chi_t^{ij} \geq y_{t-1}^i + y_t^j - 1 \quad \forall i, j, t \\ & x, y \in \{0, 1\}, s, \chi \geq 0 \end{aligned}$$

### Problem 3. Pringles

Production line - 30 products (6 flavours), 60 periods. Each item is produced at full capacity, and only one item is produced per period. Backlogging is allowed.

$$\begin{aligned} \min \quad & \sum_{i,t} (b_t^i r_t^i + h_t^i s_t^i) \\ s_{t-1}^i - r_{t-1}^i + C^i y_t^i &= d_t^i + s_t^i - r_t^i \quad \forall i, t \\ \sum_i y_t^i &\leq 1 \quad \forall t \\ s, r &\geq 0, y \in \{0, 1\} \end{aligned}$$

+ constraints on the sequencing of flavours.

## Problem4: worms

$$\min \sum_{it} \alpha_t^i \text{above}_t^i + 10000 \sum_{it} \text{late}_t^i + \sum_{it} 0.25 \text{below}_t^i$$

$$\sum_i b^i y_t^{ik} + \sum_i a^i x_t^{ik} \leq 2L_t/NK$$

$$(S_0^i) + \text{inv}_{t-1}^i + \sum_k x_t^{ik} = d_t^i + \text{inv}_t^i$$

$$\text{inv}_t^i = SS^i - \text{below}_t^i + \text{above}_t^i$$

$$x_t^{ik} \leq C_t^{ik} y_t^{ik}$$

$$x_t^{ik} \geq \text{MINBATCH} y_t^{ik}$$

$$\text{below}_t^i \leq SS^i$$

$$x, \text{inv}, \text{below} \geq 0, y \in \{0, 1\}$$

## Classification: Single Item ( $d_{tl} \equiv \sum_{j=t}^l d_j$ )

### Lot-Sizing (LS)

$$\min \sum_t (k_t' + p_t' x_t + f_t y_t)$$

$$s_{t-1} + x_t = d_t + s_t \quad \forall t$$

$$x_t \leq C_t y_t \quad \forall t$$

$$x, s \geq 0, y \in \{0, 1\}$$

### Wagner-Whitin (WW) $p_{t-1}' + h_{t-1}' \geq h_t' \forall t.$

Produce as late as possible.

$$\min \sum_t (h_t s_t + f_t y_t)$$

$$s_{t-1} + \sum_{j=t}^l C_j y_j \geq d_{tl} \quad \forall t, l, t \leq l$$

$$s \geq 0, y \in \{0, 1\}$$

### Discrete Lot-Sizing with Initial Stock (DLSI)

$x_t = C_t y_t$  for  $t = 1, \dots, n.$

$$\min h_0 s_0 + \sum_t f_t y_t$$

$$s_0 + \sum_{j=1}^l C_j y_j \geq d_{1l} \quad \forall l$$

$$s_0 \geq 0, y \in \{0, 1\}$$

**Discrete Lot-Sizing (DLS)**  $s_0 = 0$

$$\sum_t f_t y_t$$
$$\sum_{j=1}^l C_j y_j \geq d_{1l} \quad \forall l$$
$$y \in \{0, 1\}$$

**Production Classification: LS, or WW, or DLSI, or DLS**

### **Capacity Classification**

Varying Capacity (**C**):  $C_t$  varying

Constant Capacity (**CC**):  $C_t = C$  for all  $t$

Uncapacitated (**U**):  $C_t \geq d_{tn}$  for all  $t$

### **Complexity**

DLS-C is NP-hard, so all four varying capacity problems are hard.

LS-CC is easy, so all eight CC and U problems are easy.

## Further Single Item Variants

The third field *VAR* concerns extensions/variants to one of the twelve problems  $\{PROB\} - \{CAP\}$  considered so far.

- Backlogging (B).
- Start-Up Costs (SC).
- Start-Up Times (ST).
- Minimum Production Levels (LB).
- Sales (SL).  $d_t$  is now an upper bound on the sales in period  $t$ .
- Safety Stocks (SS). There is a lower bound  $s_t \geq \underline{S}_t$ .

Three fields for a single item lot-sizing problem

$\{LS, WW, DLSI, DLS\} - \{C, CC, U\} - \{B, SC, ST(C), SL, LB(C), SS\}^*$

## Multi-Item Classification: Machines

### Machines

$NK$  is the number of machines

$MI$  indicates that machines are identical, having the same production rate and capacity ( $C_t^{ik} = C_t^i$  for all  $k$ ), but possibly differing costs

$MD$  indicates that the machines are different.

$LT$  indicates that there are lead times.

## Multi-Item Classification: Resource Constraints $Z^t$

*Small time buckets* (Short time periods)  $SB[1, 2]$  small bucket model with at most one or at most two set-ups per period.

$SB[1]$  is often referred to as a model with *mode* constraints.

*Big time buckets*. (Longer time periods)

$BB[.]$  denotes a big bucket model with at least one joint capacity constraint imposing a limit  $L_t^k$

$SET$  indicates that there are also set-up times.

$ST$  indicates that there are start-up times.

$SQT$  indicates that there are sequence dependent changeover times  $qt^{ijk}$ .

$SQC$  indicates that there are sequence dependent changeover costs  $qc^{ijk}$ .

Thus the machine classification looks like

$$\{NK = [.] , [IM, VM, LT] : [SB(1, 2), BB(SET, ST, SQT), SQC]\}.$$

## Resource Constraints $Z^t$ : Modelling

Single Set-up  $SB1$ :

$$Z^t = \{y \in \{0, 1\}^{NI} : \sum_i y_t^i \leq 1\}$$

Two Set-ups  $SB2$ : Typical Sequence (2, 3)(3, 4)(4, 4)(4, 1)

Last item in period  $t$  is first item in period  $t + 1$

*BB – SET*

$$Z^t = \{(x, y) \in R_+^{NI} \times \{0, 1\}^{NI} : \sum_i (a^i x^i + b^i y_t^i) \leq B_t, x_t \leq C_t y_t \forall t\}$$

*BB – ST*

$$Z^t = \{(x, y) \in R_+^{NI} \times \{0, 1\}^{NI} + \times \{0, 1\}^{NI} : \sum_i (a^i x_t^i + b^i y_t^i) \leq B_t, \\ x_t^i \leq C_t^i y_t^i, z_t^i \leq y_t^i, z_t^i \geq y_t^i - y_{t-1}^i \text{ etc. } \forall i\}$$

*BB – SQT*

$$Z^t = \{(x, y) \in R_+^{NI} \times \{0, 1\}^{NI} + \times \{0, 1\}^{NI} : \sum_i (a^i x_t^i + \sum_j \tau^{ij} \chi_t^{ij}) \leq B_t, \\ x_t^i \leq C_t^i y_t, \sum_i \chi_t^{ij} = y_t^j, \text{ etc. } \forall i, j\}$$

## Multi-Item Classification: Multi-Level Production

$NL$  denotes the number of levels, with  $\rho_t^{ijk}$  the number of units of item  $i$  needed to produce one item of  $j$  on machine  $k$  in period  $t$  for each item  $j \in S(i)$ , the set of successors of  $i$ .

$G$  denotes a general product structure

$A$  denotes assembly structure

$S$  denotes in series assembly structure, i.e. linear.

The production structure classification is simple

$$\{NL = [.\], [G, A, S]\}.$$

## Reformulations $NI = 1, NT = n$

	<i>LS</i>	<i>WW</i>	<i>DLSI</i>	<i>DLS</i>
<b>FORMULATION</b>				
<i>U</i>	<i>SP</i> $O(n) \times O(n^2)$ <i>FL</i> $O(n^2) \times O(n^2)$ [19, 13]	<i>WW</i> $O(n^2) \times O(n)$ [33]	–	–
<i>CC</i>	$O(n^3) \times O(n^3)$ [44]	$O(n^2) \times O(n^2)$ [33]	$O(n) \times O(n)$ [26, 33]	$O(n) \times O(n)$ <i>Folklore</i>
<b>SEPARATION</b>				
<i>U</i>	$(l, S)$ $O(n \log n)$ [4]	<i>WW</i> $O(n)$	–	–
<i>CC</i>	<i>klSI</i> * <i>HEUR</i> [32]	<i>klSI(WW)</i> $O(n^2 \log n)$ [33]	<i>Mixing</i> $O(n \log n)$ [17, 26, 33]	<i>Above</i>  <i>Folklore</i>
<b>OPTIMIZATION</b>				
<i>U</i>	$O(n \log n)$ [1, 14, 46]	$O(n)$ [1, 14, 46]	–	–
<i>CC</i>	$O(n^3)$ [16, 42]	$O(n^2 \log n)$ [44]	$O(n)$	$O(n)$

Table 1: Model *PROB* – {*CC*, *U*}

## With Backlogging

	<i>LS</i>	<i>WW</i>	<i>DLSI</i>	<i>DLS</i>
<b>FORMULATION</b>				
<i>U</i>	$SP(B) O(n) \times O(n^2)$ $FL(B) O(n^2) \times O(n^2)$ [3, 31]	$O(n^2) \times O(n)$ [33]	– –	–
<i>CC</i>	$O(n^3) \times O(n^3)$	$O(n^3) \times O(n^2)$ [45]	$O(n^2) \times O(n^2)^*$ [26, 45]	$O(n) \times O(n)$ [26]
<b>SEPARATION</b>				
<i>U</i>	$Ext(l, S)^*$ [31]	<i>Cycles</i> <i>MCF, HSep(VV)</i> [33]		
<i>CC</i>	$HEUR(Con)$	<i>FC, RC, GMix</i> [28, 21, 26]	<i>GMix</i> [26]	<i>MIR</i> [26]
<b>OPTIMIZATION</b>				
<i>U</i>	$O(n \log n)$ [1, 14, 46]	$O(n)$ [1, 14, 46]	–	–
<i>CC</i>	$O(n^3)$	–	$O(n \log n)$	$O(n \log n)$

Table 2: Model *PROB* – {*CC, U*} – *B* with Backlogging

## With Start-Ups

	<i>LS</i>	<i>WW</i>	–	<i>DLS</i>
<b>FORMULATION</b>				
<i>U</i>	<i>SP(ST)</i> $O(n^2) \times O(n^2)$ <i>FL(ST)</i> $O(n^3) \times O(n^2)$ [48], [43]	$O(n^2) \times O(n)$ [33]	–	–
<i>CC</i>				$O(n^2) \times O(n^2)$ [41] ( <i>WW</i> ) $O(n^2) \times O(n)$ [38]
<b>SEPARATION</b>				
<i>U</i>	( <i>l, R, S</i> ) <i>Sep</i> $O(n^3)$ [48], [43]	<i>Above</i>		
<i>CC</i>	<i>left/right, submod</i> <i>SEP</i> ( $O(\text{easy?})$ ) [9]			<i>hole/bucket</i> <i>SEP?</i> [40]
<b>OPTIMIZATION</b>				
<i>U</i>	$O(n \log n)$ [46, 1, 14]	$O(n)$ [46, 1, 14]	–	–
<i>CC</i>	?	?		$O(n^2)$ <i>WW</i> $O(n \log n)vH$

Table 3: Model *PROB* – {*CC, U*} – *SC* with Start-Ups

## Problem 1. Reformulation

Classification NI=4, NK=1, NT=30, NL=1: **WW-CC-SC/SB1**

Two possible relaxations for which a tight reformulation is known.

### WW-U-SC and WW-CC

Reformulation **WW-U-SC**

$$s_{t-1} \geq \sum_{j=t}^l d_j (1 - y_t - z_{t+1} - \dots - z_j)$$

Reformulation **WW-CC**

$$s_{k-1} \geq C \sum_{t \in [k, n]} f_t^k \delta_t^k + C \mu^k \quad \forall k$$

$$\sum_{u=k}^t y_u \geq \sum_{\tau \in \{0\} \cup [k, n]} \left[ \frac{d_{k\tau}}{C} - f_\tau^k \right] \delta_\tau^k - \mu^k \quad \forall k, t, k \leq t$$

$$\sum_{t \in \{0\} \cup [k, n]} \delta_t^k = 1 \quad \forall k$$

$$\mu^k \geq 0, \delta_t^k \geq 0, \text{ for } t \in \{0\} \cup [k, n], \quad \forall k$$

$$0 \leq y_t \leq 1 \text{ for } t = 1, \dots, NT$$

(here  $f_0^k = 0$  and  $f_\tau^k = \frac{d_{k\tau}}{C} - \lfloor \frac{d_{k\tau}}{C} \rfloor$ ).

## Problem 1. Results of Reformulation

Instance cl-1a consists of the original formulation.

Instance cl-1b is with the addition of the inequalities for  $WW - U - SC$ .

Instance cl-1c has in addition the reformulation of  $WW - CC$  for each item.

instance	m	n	int	LP	XLP	IP	secs	nodes
cl-1a	511	720	120	1509.1	3549.6	4414.2	5000*	$3.8 \times 10^5$
cl-1b	2354	720	120	3800.6	4305.1	4404.5	383	3826
cl-1c	4454	2824	120	4309.9	4310.5	4404.5	82	175

Table 4: Results for Problem 1

\* indicates that the run was terminated before optimality was proved. For formulation cl1a the best lower bound on termination was 4251.2 leaving a gap of 3.7%.

## Problem 2. Reformulation

**Observation 1** Reformulation of changeover variables

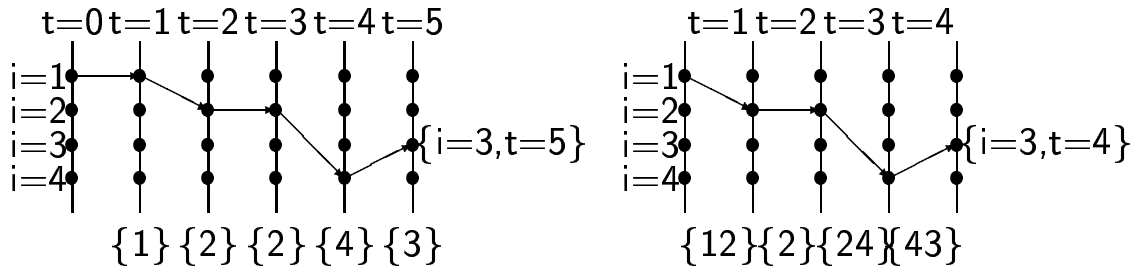


Figure 1: Small Buckets: One and Two Set-ups

Consider the polyhedron representing the flow of a single unit from item to item over time:

$$\begin{aligned} \sum_i \chi_t^{ij} &= y_t^j \quad \forall j, t \\ \sum_j \chi_t^{ij} &= y_{t-1}^i \quad \forall i, t \\ \sum_i y_0^i &= 1 \\ \chi_t^{ij} &\geq 0 \quad \forall i, j, t \end{aligned}$$

shown in Figure 1, where  $y_t^i$  is the flow through node  $(i, t)$  and  $\chi_t^{ij}$  is the flow from node  $(i, t - 1)$  to node  $(j, t)$ . This formulation representing a flow of one unit (or a path) is as tight as possible.

**Observation 2: Inclusion of start-up variables.**

$$z_t^j = \sum_{i:i \neq j} \chi_t^{ij} \text{ gives a start-up variable.}$$

**Observation 3: Single item subproblems: DLS-CC-SC**

Use the reformulation of Van Eyl (WW).

**Proposition 1.** *Suppose that  $d_{t_1} = \dots = d_{t_p} = 1$  with  $t < t_1 < \dots < t_p = l$  and  $d_\tau = 0$  in intervening periods. The inequality*

$$s_{t-1}^i + \sum_{u=t}^{t+p-1} y_u^i + \sum_{u=t+1}^{t+p-1} (d_{ul} - (t+p-u))z_u + \sum_{u=t+p}^l d_{ul}z_u \geq p$$

*is valid.*

## Problem 2: Results

Instance cl2-NTa is the initial formulation

Instance cl2-NTb is the formulation with reformulation from Observations 1 and 2.

Instance cl2-NTc also includes the reformulation of  $DLS - CC - SC(WW)$ .

instance	m	n	int	LP	XLP	IP	secs	nodes
cl2-35a	3797	4110	350	27.2	34.7	2056	1800*	51500*
cl2-35b	2062	5130	690	180.9	531.6	1599	1800*	8000*
cl2-35c	2599	5130	690	1361.5	1361.5	1387	9	17
cl2-60c	4817	8880	1190	1453.6	1454.0	1560	17579	8117

Table 5: Results for Problem 2

Note that cl2-35a and cl2-35b are unsolved after 1800 seconds. The best lower bounds obtained are 240.9 and 804.3 respectively.

## Problem 3: Reformulation

Classification: NI=30,NK=1,NT=60,NL=1, DLS-CC-B

Reformulation: Suffices to observe that the flow conservation can be written as

$$s_t^i \geq C^i \sum_{u=1}^t y_u^i - d_{1t}^i \text{ for all } i, t,$$

and then adding the MIR inequalities

$$s_t^i \geq C^i (1 - f_t^i) \left( \sum_{u=1}^t y_u^i - \left\lfloor \frac{d_{1t}^i}{C^i} \right\rfloor \right) \text{ for } i = 1, \dots, NI, t = 1, \dots, NT$$

where  $f_t^i = \frac{d_{1t}^i}{C^i} - \left\lfloor \frac{d_{1t}^i}{C^i} \right\rfloor$  gives the convex hull even with  $NI > 1$ .

### Problem 3: Results

Instance	#col	#row	#non-0	LP	LP time	Best LB	Best UB	IP time
DLSB-p	5160	1621	12668	1627056	1	1759969	2188995	900*
DLSB-o	5520	2353	16723	2507612	3	2711504	3274036	900*
DLSB-pr	5160	3181	51974	1858181	7	1858181	1858181	7
DLSB-or	5520	3913	65863	3059263	15	3069539	3069539	56

Table 6: *DLSB* instances before and after reformulation

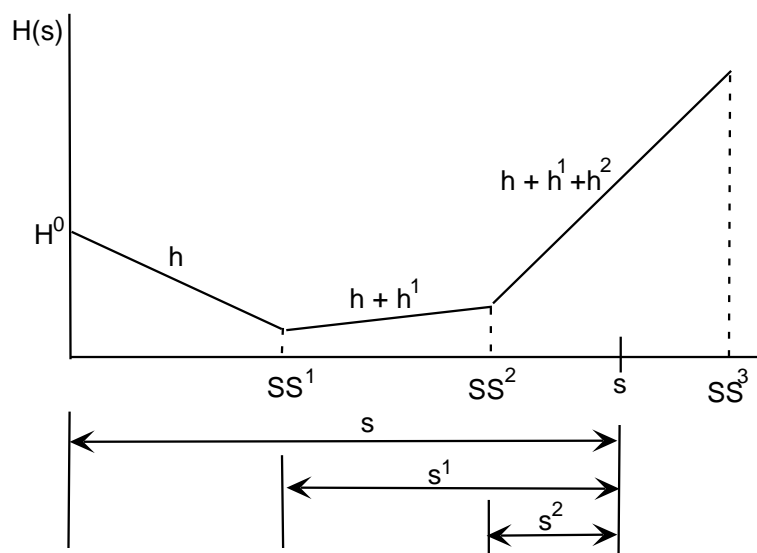
## Problem 4: Wagner-Whitin with Safety Stocks

$$s_{t-1} + x_t = d_t + s_t \quad \forall t$$

$$x_t \leq C y_t \quad \forall t$$

$$\sigma_t \geq s_t - SS \quad \forall t$$

$$x_t, s_t \geq 0, y_t \in \{0, 1\} \quad \forall t$$



$$s_{k-1} + C \sum_{u=k}^t y_u \geq d_{kt} \quad \forall k, t$$

$$\sigma_{k-1} + C \sum_{u=k}^t y_u \geq d_{kt} - SS \quad \forall k, t$$

$$s_t, \sigma_t \geq 0, y_t \in \{0, 1\} \quad \forall t$$

Again mixing suffices.

## Problem 4: Initial Stocks/Lower Bounds

$s_{k-1} \geq \underline{S} > 0$ . In place of the inequality

$$s_{k-1} + \sum_u d_{ul} y_u \geq d_{kl},$$

use the values  $d_{ku} - \underline{S}$  to calculate new demands  $\tilde{d}_u$  for  $u = k, \dots, n$ , and use this to calculate new demands  $\tilde{d}_{ul}$  for  $u = k, \dots, l$ .

Example.

$(d_2, d_3, d_4) = (2, 3, 6)$  and  $s_1 \geq 4$ .

Basic Wagner-Whitin inequality

$$s_1 + 11y_2 + 9y_3 + 6y_4 \geq 11.$$

Using the lower bound with  $s' = s_1 - 4$ ,  $(d_2 - 4, d_{23} - 4, d_{24} - 4)^+ = (0, 1, 7)$ .

So  $(\tilde{d}_2, \tilde{d}_3, \tilde{d}_4) = (0, 1, 6)$ .

The new inequality is

$$s_1 + 7y_2 + 7y_3 + 6y_4 \geq 7 + 4, \text{ or}$$

$$s_1 + 7y_2 + 7y_3 + 6y_4 \geq 11.$$

## Problem 4: Results

Instance	#rows	#cols	#int	LP	LP time	Best LB	Best UB	IP time
worms1	14500	15685	3586	5572357	5585163	5586203	5587910	900*
w1d	32975	14660	3155	5579906	5587453	5587910	5587910	435

Table 7: *worms* instances before and after reformulation

## Mixed Models: WW-U-{B,SC}

Consider the problem

$$\begin{aligned}
 \min \quad & \sum_t h_t s_t + \sum_t b_t r_t + \sum_t f_t y_t + \sum_t g_t z_t \\
 & s_{t-1} - r_{t-1} + x_t = d_t + s_t - r_t \quad \forall t \\
 & x_t \leq M y_t \quad \forall t \\
 & z_t - w_{t-1} = y_t - y_{t-1} \quad \forall t \\
 & z_t \leq y_t \quad \forall t \\
 & x, s, r \in \mathbb{R}_+^n, y, z \in \{0, 1\}^n
 \end{aligned}$$

The extended formulation  $Q^{WW-U-B-SC}$ :

$$\begin{aligned}
 & \alpha_t + y_t + \beta_t = 1 \quad \forall t \text{ with } d_t > 0 \\
 & \beta_{t+1} + z_{t+1} \geq \beta_t \quad \forall t \\
 & s_{t-1} \geq \sum_{u=t}^k d_u (\alpha_u - \sum_{j=t}^{u-1} w_j) \quad \forall t, k \text{ with } t \leq k \\
 & r_t \geq \sum_{u=k}^t d_u (\beta_u - \sum_{j=u+1}^t z_j) \quad \forall k, t \text{ with } k \leq t \\
 & z_t - w_{t-1} = y_t - y_{t-1} \quad \forall t \\
 & z_t \leq y_t \quad \forall t \\
 & x, s, r \in \mathbb{R}_+^n, y, z \in [0, 1]^n
 \end{aligned}$$

## LS-U-{B,Sales,Piecewise Concave Production Costs}

The problem can be formulated as

$$\begin{aligned}
 \min \quad & \sum_{k,t} p_t^k x_t^k + \sum_{k,t} f_t^k y_t^k + \sum_{k,t} h_t^k s_t^k + \sum_{k,t} b_t^k r_t^k - \sum_{l,t} e_t^l v_t^l \\
 & s_{t-1} - r_{t-1} + \sum_k x_t^k = d_t + \sum_l v_t^l + s_t - r_t \quad \forall t \\
 & v_t^l \leq V_t^l \quad \forall l, t \\
 & x_t^k \leq M y_t^k \quad \forall k, t \\
 & s, r, x, v \geq 0, \quad y \in \{0, 1\}^n
 \end{aligned}$$

where we assume that  $0 \leq f_t^1 \leq f_t^2 \leq \dots$  for all  $t$ , and  $e_t^1 \leq e_t^2 \dots$  for all  $t$ .

Note that if  $f_t^k = 0$ ,  $x_t^k$  can be viewed as the amount bought in from outside in period  $t$ .

## An extended formulation

Letting  $\bar{V}_l^t = \sum_{\lambda} V_t^{\lambda}$  for all  $l, t$ , we observe that, because  $e_t^1 \leq e_t^2 \dots$  for all  $t$ ,  $\sum_{\lambda} v_t^{\lambda} = \bar{V}_t^l$  for some  $l$  and all  $t$ .

New variables in our extended formulation.

$\alpha_{ut}^{kl} = 1$  if production takes place in period  $u$  using production type  $k$  to satisfy a demand of  $d_t + \bar{V}_t^l$  in period  $t$ .

$z_{ut}^k = 1$  if production takes place in period  $u$  using production type  $k$  to satisfy the demand in  $t$ .

## The Formulation

$$\min \sum_{k,t} p_t^k x_t^k + \sum_{k,t} f_t^k y_t^k + \sum_{k,t} h_t^k s_t^k + \sum_{k,t} b_t^k r_t^k - \sum_{l,t} e_t^l v_t^l$$

$$z_{ut}^k = \sum_l \alpha_{ut}^{kl} \text{ for } 1 \leq u, t \leq n, \forall k$$

$$\sum_k \sum_u z_{ut}^k = 1 \text{ for } 1 \leq t \leq n,$$

$$y_t^k \geq z_{tt}^k \text{ for } 1 \leq t \leq n,$$

$$z_{ut}^k \geq z_{u,t+1}^k \text{ for } 1 \leq u \leq t \leq n, \forall k$$

$$z_{ut}^k \geq z_{u,t-1}^k \text{ for } 1 \leq t \leq u \leq n, \forall k$$

$$v_t^l = \sum_k \sum_u \sum_{\lambda: \lambda \geq l} V_t^\lambda \alpha_{ut}^{kl} \text{ for } 1 \leq t \leq n, \forall l$$

$$x_u^k = \sum_l \sum_t (d_t + \bar{V}_t^l) \alpha_{ut}^{kl} \text{ for } 1 \leq u \leq n, \forall k$$

$$s_{t-1} = \sum_{k,l} \sum_{u,\tau: u < t \leq \tau} (d_\tau + \bar{V}_\tau^l) \alpha_{u\tau}^{kl} \text{ for } 1 \leq t \leq n$$

$$r_t = \sum_{k,l} \sum_{u,\tau: \tau \leq t < u} (d_\tau + \bar{V}_\tau^l) \alpha_{u\tau}^{kl} \text{ for } 1 \leq t \leq n$$

$$\alpha_{ut}^{kl} \geq 0, z_{ut}^k \geq 0, 0 \leq y_t \leq 1 \forall t$$

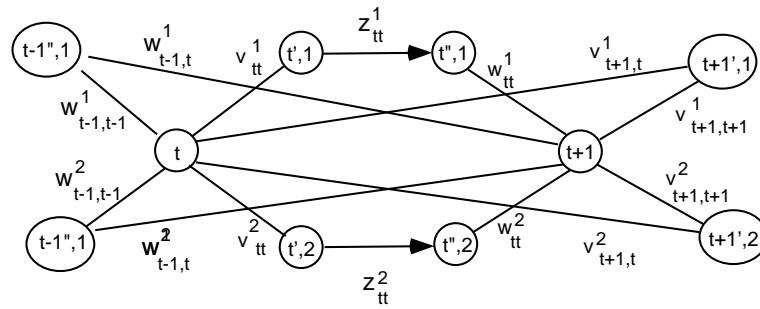


Figure 2: Formulation as a fixed charge network flow

## Open Questions

### Compact Formulations?

$WW - CC - B$  recent result  $O(n^3) \times O(n^2)$

$WW - CC - ST$

$DLS - CC - \{B, ST\}$

### Approximate Formulations

What to do if  $n = NT = 60/90$ ? Even  $O(n^2)$  becomes problematic.

### Resource Constraint Submodels

$BB - SET - ST$

How to generate cuts easily?

Extend results of Miller et al.

# Supply Chain Models?

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