The Price-Directed Approach to Approximate Dynamic Programming:

Application to Inventory Routing

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Price-Directed Control

Using an operating policy that exploits optimal dual prices from a mathematical programming relaxation of the underlying control problem.

Approximate Dynamic Programming

The Price-Directed Approach

Formulate a Markov Decision Process (MDP)

$\Downarrow$

Construct a functional approximation to value function

$\Downarrow$

Math programming relaxation

$\Downarrow$

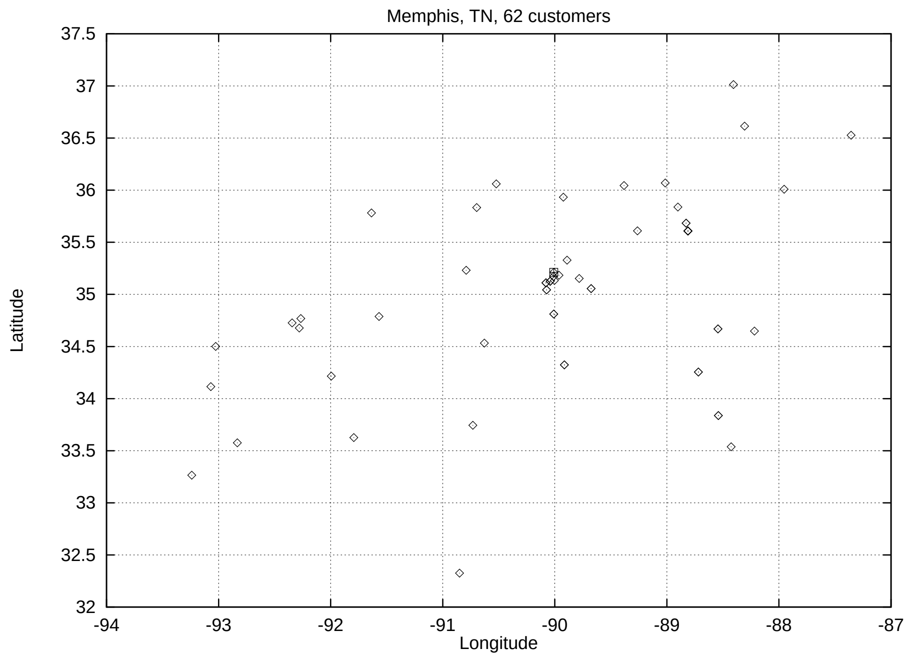
Price-directed operating policy

$\Downarrow$

Evaluate computational/economic performance

Praxair

Show picture of truck.



http://tli.isye.gatech.edu/research/casestudy/cs.cfm

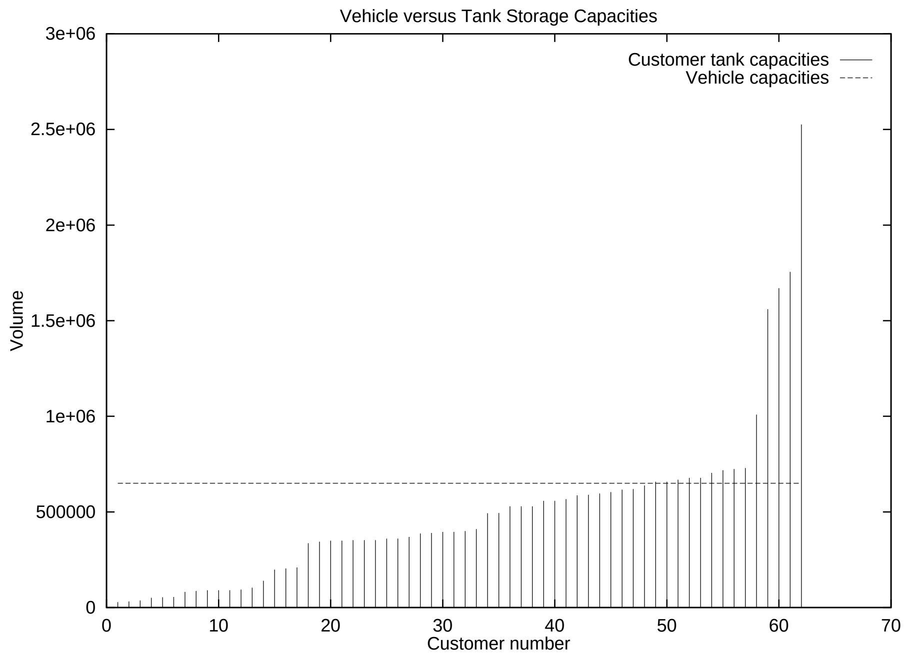
Problem Statement

- Dispatch trucks through time.
- For each truck dispatched, decide
  - which subset of customers to visit
  - how much to deliver to each customer,

  given current customer inventories.
- Replenishment costs = transportation + fixed drop-off costs.

Minimize Long-run time average replenishment costs

subject to No stockouts


Vehicle versus Tank Storage Capacities
Customer tank capacities
Vehicle capacities
Volume
Customer number
3e+06
2.5e+06
2e+06
1.5e+06
1e+06
500000
0
0
10
20
30
40
50
60
70


Maximum days until next delivery
Maximum days until next delivery
Customer number
140
133
126
119
112
105
98
91
84
77
70
63
56
49
42
35
28
21
14
7
0
0
10
20
30
40
50
60
70

Data

$\mathcal{I}$ = set of all customers

λ_i = Demand rate for customer i

$\overline{S}_i$ = Max storage level for customer i

C_I = Cost to replenish $I \subseteq \mathcal{I}$

Monotonicity: $C_{I_1} \leq C_{I_2} \quad \forall I_1 \subseteq I_2$

$\overline{D}$ = Truck storage capacity

Assumptions

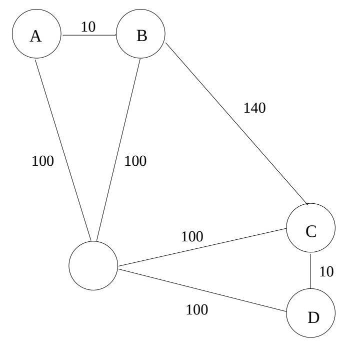
- Continuous inventory state space
- Continuous-time
- Constant (deterministic) demand rates
- Vehicle always available
- No time windows
- Instantaneous replenishment

Numerical Example: Data

From Bell, Dalberto, and Fisher, et al. (1983).

Item i	$\overline{S}_i$	λ_i
A	5000 gallons	1000 gallons per day
B	3000	3000
C	2000	2000
D	4000	1500

$\overline{D} = 5000$



Numerical Example: Policies

- **Policy 1:**
  Every day make 2 trips

Deliver 1000 to A and 3000 to B	\$210
Deliver 2000 to C and 1500 to D	\$210
Average cost	\$420 per day

- **Policy 2:**

Day 1	Deliver 3000 to B and 2000 to C	\$340
Day 2	Deliver 2000 to A and 3000 to B	\$210
	Deliver 2000 to C and 3000 to D	\$210
	Average cost	\$380 per day

Questions

Is Policy 2 optimal?

How to find Policy 2?

What should you do *now*?

Most expensive customer? *(cost per unit)*

Literature

- Bell,Dalberto, and Fisher, et. al. (1983). Air Products.
- Dror, Ball (1987).
- Campbell, Savelsbergh, Clarke, and Kleywegt (1998). Review paper, inventory routing.
- Kleywegt, Nori, Savelsbergh (1999).
- Federgruen, Zheng (1992). Joint replenishment.
- Herer, Roundy (1997). One-warehouse, multiretailer.
- Adelman (2001). Stochastic demand, holding/stockout costs.

Decision Variables

$s_{i,n}$ = inventory at customer i immediately prior to dispatch n $\quad \forall i \in \mathcal{I}, n$

$Z_{I,n} = 1$ if dispatch n is to subset I, 0 otherwise $\quad \forall I \subseteq \mathcal{I}, n$
(I_n = the nth subset)

$d_{i,n}$ = delivery quantity to customer i on dispatch n $\quad \forall i, n$

T_n = time duration between dispatch n and $n+1$ $\quad \forall n$

$$\Big(\vec{s}_n, (I_n, \vec{d}_n)\Big) \xrightarrow{T_n} \vec{s}_{n+1}$$

Dynamic System Formulation

$$\text{Minimize} \quad \limsup_{N\to\infty} \quad \frac{\sum_{n=1}^{N}\sum_{I\subseteq\mathcal{I}} C_I Z_{I,n}}{\sum_{n=1}^{N} T_n}$$

$$\begin{aligned}
s_{i,n+1} &= s_{i,n} + d_{i,n} - \lambda_i T_n && \forall i \in \mathcal{I}, n \\
d_{i,n} &\le (\overline{S}_i - s_{i,n}) \sum_{\{I\subseteq\mathcal{I}: i\in I\}} Z_{I,n} && \forall i \in \mathcal{I}, n \\
\sum_{i\in\mathcal{I}} d_{i,n} &\le \overline{D} && \forall n \\
\sum_{I\subseteq\mathcal{I}} Z_{I,n} &= 1 && \forall n \\
Z_{I,n} &\in \{0,1\} && \forall I \subseteq \mathcal{I}, n \\
d, s, T &\ge 0.
\end{aligned}$$

Semi-Markov Decision Process: Dynamics

$$(\text{state}, \text{action}) \xrightarrow{\text{time}} (\text{state}, \text{action}) \xrightarrow{\text{time}} \ldots$$

$$\left(\vec{s}_1, (I_1, \vec{d}_1)\right) \xrightarrow{\tau_1} \left(\vec{s}_2, (I_2, \vec{d}_2)\right) \xrightarrow{\tau_2} \ldots$$

SMDP: Primitives

- State-space:

$$\mathcal{S} \equiv \{\text{feasible inventory levels } \vec{s}\}$$

- Action-space:

$$\mathcal{D}_{\vec{s}} \equiv \{\text{feasible itineraries } (I, \vec{d})\} \qquad \forall \vec{s}$$

- Cost of $(\vec{s}, (I, \vec{d})) = C_I$
- Time until next stockout:

$$\tau(\vec{s}, (I, \vec{d})) = \min_{i \in \mathcal{I}} \left(\frac{s_i + d_i}{\lambda_i} \right) \qquad \forall (\vec{s}, (I, \vec{d})) \in \mathcal{S} \times \mathcal{D}_{\vec{s}}$$

- Next state:

$$s_i \leftarrow s_i + d_i - \lambda_i \tau(\vec{s}, (I, \vec{d})) \qquad \forall i \in \mathcal{I}.$$

SMDP: Optimality Equations

Gain: g

Bias function: $h(\vec{s}) \quad \forall \vec{s} \in \mathcal{S}$

Optimality Equations:

$$h(\vec{s}) \quad = \quad \inf_{(I,\vec{d}) \in \mathcal{D}_{\vec{s}}} \left\{ C_I - g\tau(\vec{s},(I,\vec{d})) + h(\vec{s}+\vec{d}-\vec{\lambda}\tau(\vec{s},(I,\vec{d}))) \right\} \quad \forall \vec{s} \in \mathcal{S}$$

SMDP: Functional Approximation

Gain: $g \approx \sum_{i \in \mathcal{I}} \lambda_i V_i$

Bias function: $h(\vec{s}) \approx \theta - \sum_{i \in \mathcal{I}} s_i V_i \quad \forall \vec{s} \in \mathcal{S}$

Policy: In state $\vec{s} \in \mathcal{S}$ solve

$$\begin{aligned} & \inf_{(I,\vec{d}) \in \mathcal{D}_{\vec{s}}} \left\{ C_I - g\tau(\vec{s},(I,\vec{d})) + h(\vec{s}+\vec{d}-\vec{\lambda}\tau(\vec{s},(I,\vec{d}))) - h(\vec{s}) \right\} \\ &= \inf_{(I,\vec{d}) \in \mathcal{D}_{\vec{s}}} \left\{ C_I - \vec{\lambda} \bullet \vec{V}\tau(\vec{s},(I,\vec{d})) - \left(\vec{s}+\vec{d}-\vec{\lambda}\tau(\vec{s},(I,\vec{d}))\right) \bullet \vec{V} + \vec{s} \bullet \vec{V} \right\} \\ &= \inf_{(I,\vec{d}) \in \mathcal{D}_{\vec{s}}} \left\{ C_I - \vec{V} \bullet \vec{d} \right\} \end{aligned}$$

$$\boxed{\max_{(I,\vec{d}) \in \mathcal{D}_{\vec{s}}} \left\{ \vec{V} \bullet \vec{d} - C_I \right\}}$$

Price-Directed Dispatching

- Let V_i = value given to the dispatcher (by management) for replenishing one unit of item i, $\forall i$.
- Suppose the dispatcher is considering $(I, \vec{d})$ where
  - $I \subseteq \mathcal{I}$ is a subset of customers, and
  - $\vec{d} = \langle d_1, d_2, \ldots \rangle$ are delivery quantities to each customer.

$$\sum_{i \in I} V_i d_i - C_I < 0 \quad \text{Not Favorable}$$

$$\sum_{i \in I} V_i d_i - C_I \geq 0 \quad \text{Favorable}$$

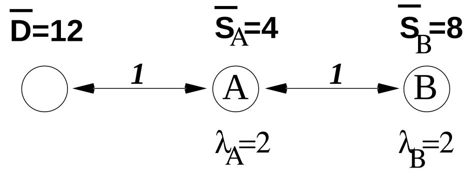
Operating Policy

Dispatch one truck at a time.
Whenever a customer stocks out solve

$$\begin{aligned} \underset{(I,\vec{d})}{\text{Maximize}} \quad & \sum_{i \in I} V_i d_i - C_I \\ & \sum_{i \in I} d_i \leq \overline{D} \\ & 0 \leq d_i \leq \overline{S}_i - s_i \quad \forall i \in I \end{aligned}$$

where the current inventory levels are $\vec{s} = \langle s_1, s_2, \ldots \rangle$, $I \subseteq \mathcal{I}$, and $\exists j \in I$ with $s_j = 0$.

Example



$C_A = 2$, $C_B = 4$, $C_{AB} = 4$

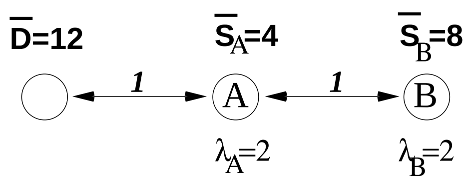
- $V_A = 1/2$, $V_B = 1/4$
- $(s_A = 0,\ s_B = 1)$ Is $A4B7$ favorable?

$$4V_A + 7V_B - C_{AB} = 4(1/2) + 7(1/4) - 4 = -.25 < 0 \qquad \text{NO}$$

- $(s_A = 0,\ s_B = 0)$ Is $A4B8$ favorable?

$$4V_A + 8V_B - C_{AB} = 4(1/2) + 8(1/4) - 4 = 0 \qquad \text{YES}$$

Example: Minimum Delivery Quantities



- $(s_A = 0, s_B > 0)$ **At A, deliver to B?**

$$\begin{aligned} d_B V_B - (C_{AB} - C_A) &\geq 0 \\ d_B(1/4) - 2 &\geq 0 \\ \mathbf{d_B} &\geq \mathbf{8} \end{aligned}$$

- $(s_A > 0, s_B = 0)$ **At B, deliver to A?**

$$\begin{aligned} d_A V_A - (C_{AB} - C_B) &\geq 0 \\ d_A(1/2) - 0 &\geq 0 \\ \mathbf{d_A} &\geq \mathbf{0} \end{aligned}$$

Minimum Delivery Quantities

- $(s_i > 0)$ **At I, deliver to $I \cup \{i\}$?**

$$\boxed{d_i \geq \frac{C_{I\cup\{i\}} - C_I}{V_i}}$$

- Exclude i if $s_i > 0$ and

$$\overline{S}_i - s_i < \min_{\{I : i \notin I\}} \left(\frac{C_{I\cup\{i\}} - C_I}{V_i} \right)$$

Research Questions

1. Where does this operating policy come from?
2. How should management compute the V_i's?
3. What properties should they satisfy?
4. How well do they perform?

Management's Problem

How to set the V_i's?

- Net-value $> 0 \Longrightarrow$ too pessimistic
- Net-value $< 0 \Longrightarrow$ too optimistic (and unfair to the dispatcher)
- Ideally, Net-value $= 0 \Longrightarrow V_i$ is marginal cost

Who should have larger V's?

Customers who, to avoid stockouts, must generate higher costs per unit delivered.

- Smaller storage capacities
- More geographically isolated, further from depot

Dual: Linear semi-infinite program

Sup g

$$h(\vec{s}) - h(\vec{s} + \vec{d} - \vec{\lambda}\tau(\vec{s},(I,\vec{d}))) \leq C_I - g\tau(\vec{s},(I,\vec{d})) \quad \forall \vec{s} \in \mathcal{S}, (I,\vec{d}) \in \mathcal{D}_{\vec{s}},$$

$\Downarrow$ (Functional approximation)

$$\textbf{(D)} \quad \text{Maximize} \quad \sum_{i \in \mathcal{I}} \lambda_i V_i$$

$$\sum_{i \in I} d_i V_i \leq C_I \quad \forall (I,\vec{d}) \in \mathcal{D}_{\vec{0}}.$$

THEOREM *Lower bound on any attainable cost rate.*

Primal: Decision variables

Z_I = Long-run time average rate at which dispatches use subset I $\forall I \subseteq \mathcal{I}$

$d_{i,I}$ = Long-run average delivery quantity to customer i on dispatches to subset I $\quad \forall I \subseteq \mathcal{I}, i \in I$

Define these as limits of underlying control process quantities.

Primal: Non-linear program

$$
\begin{aligned}
\textbf{(NLP)} \quad \text{Minimize} \quad & \sum_{I \subseteq \mathcal{I}} C_I Z_I \\
\sum_{I \subseteq \mathcal{I}} d_{i,I} Z_I &= \lambda_i \quad \forall i \in \mathcal{I} \\
\sum_{i \in I} d_{i,I} &\leq \overline{D} \quad \forall I \subseteq \mathcal{I} \\
d_{i,I} &\leq \overline{S}_i \quad \forall I \subseteq \mathcal{I}, i \in I \\
Z, d &\geq 0.
\end{aligned}
$$

THEOREM *Lower bound on optimal cost rate.*

Duality

THEOREM *There exists a pair of optimal solutions V^*, (Z^*, d^*) to* **(D)** *and* **(NLP)***, respectively, for which*

$$\text{(no duality gap)} \quad \sum_{i \in \mathcal{I}} \lambda_i V_i^* = \sum_{I \subseteq \mathcal{I}} C_I Z_I^*$$

and

$$\text{(complementary slackness)} \quad Z_I^* \left(\sum_{i \in I} d_{i,I}^* V_i^* - C_I \right) = 0 \quad \forall (I, \vec{d}) \in \mathcal{D}_{\vec{0}}.$$

Numerical Example: Solution

Primal solution:

$$\begin{aligned} Z^*_{B3000,C2000} &= 1/2 \\ Z^*_{A2000,B3000} &= 1/2 \\ Z^*_{C2000,D3000} &= 1/2 \end{aligned}$$

Dual solution:

$$V_A^* = .005,\ V_B^* = .06\overline{6},\ V_C^* = .07,\ V_D^* = .02\overline{3}$$

Complementary slackness:

$$\begin{aligned} 3000V_B^* + 2000V_C^* &= 340 \\ 2000V_A^* + 3000V_B^* &= 210 \\ 2000V_C^* + 3000V_D^* &= 210, \end{aligned}$$

Economic Properties

Property 1 (Bounded, non-negative)

$$\frac{\min\limits_{I \subseteq \mathcal{I} \setminus \{i\}} \left\{C_{I \cup \{i\}} - C_I\right\}}{\min\{\overline{D}, \overline{S}_i\}} \leq V_i^* \leq \frac{C_{\{i\}}}{\min\{\overline{D}, \overline{S}_i\}} \qquad \forall i \in \mathcal{I}$$

Economic Properties

PROPERTY 2 (Comparability)
If $i_1, i_2 \in \mathcal{I}$ *satisfy*

$$\begin{aligned} \overline{S}_{i_1} &\leq \overline{S}_{i_2} \\ C_{I\cup\{i_1\}} &\geq C_{I\cup\{i_2\}} \qquad \forall I \subseteq \mathcal{I} \setminus \{i_1, i_2\}, \end{aligned}$$

then $V^*_{i_1} \geq V^*_{i_2}$.

PROPERTY 3 (Completeness)
Given an optimal V^* *to* **(D)**, *any alternative optimal solution to* **(NLP)** *satisfies complementary slackness with respect to this* V^*.

Economic Properties

PROPERTY 4 (Homogeneity)
For $\alpha, \beta, \gamma > 0$, *let*

$$\begin{aligned} \lambda_i &\to \alpha\lambda_i \quad \forall i \\ \overline{S}_i &\to \beta\overline{S}_i \quad \forall i \\ \overline{D} &\to \beta\overline{D} \\ C_I &\to \gamma C_I \quad \forall I. \end{aligned}$$

Then

$$\begin{aligned} V^*_i &\to (\gamma/\beta)V^*_i \quad \forall i \\ d^*_{i,I} &\to \beta d^*_{i,I} \quad \forall I, i \\ Z^*_I &\to (\alpha/\beta)Z^*_I \quad \forall I. \end{aligned}$$

Solution: Column generation

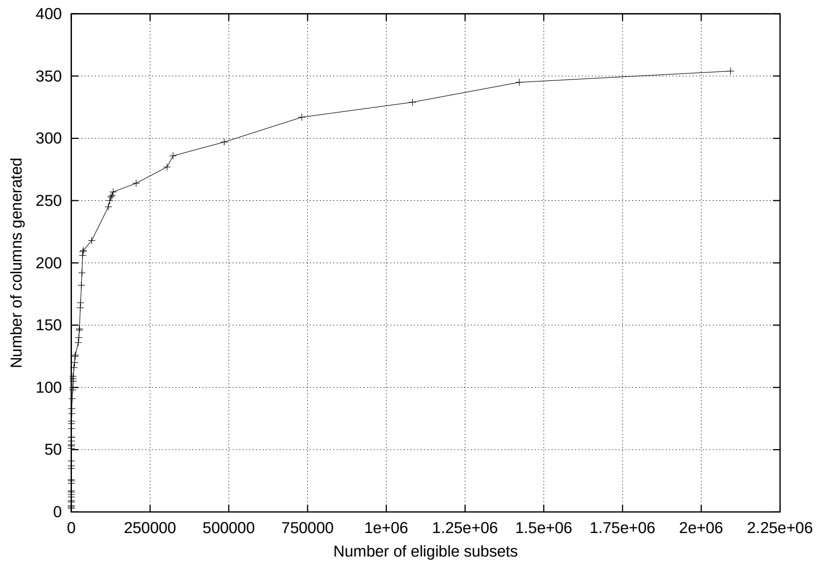
- Find an $(I, \vec{d})$ such that
$$\sum_{i \in I} V_i d_i - C_I > 0.$$

- **Proposition:** $Z_I^* = 0$ for all $I \subseteq \mathcal{I}$ such that
$$\sum_{i \in I} \overline{S}_i - \max_{j \in I} \overline{S}_j \quad \geq \quad \overline{D}.$$

Subset Elimination

No. customers	No. total subsets	No. eligible subsets
25	33.5 Million	1,110
35	34.4 Billion	14,215
50	1.1×10^{15}	136,405
62	4.6×10^{18}	2,092,842

Columns Generated



Price-Directed Operating Policy

Use optimal V_i^* from math program.

When in state $\vec{s}$, solve

$$\max_{(I,\vec{d})\in\mathcal{D}_{\vec{s}}} \left\{ \sum_{i\in\mathcal{I}} V_i^* d_i - C_I \right\}$$

Certificate of Optimality

Theorem *Any feasible sequence* $(I_n, \vec{d}_n)$ *for which*

$$\vec{V}^* \bullet \vec{d}_n - C_{I_n} = 0 \qquad \forall n = 1, \ldots$$

is optimal.

Proof

$$0 = \limsup_{N\to\infty} \frac{\sum_{n=1}^{N} \left(C_{I_n} - \sum_{i\in\mathcal{I}} V_i^* d_{i,n} \right)}{\sum_{n=1}^{N} T_n} = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} C_{I_n}}{\sum_{n=1}^{N} T_n} - \sum_{i\in\mathcal{I}} V_i^* \lambda_i$$

Optimality of Direct Shipment

Proposition *If* $\overline{S}_i \geq \overline{D}$ $\forall i \in \mathcal{I}$, *then direct replenishment is optimal.*

Simulation Results: Real-world data

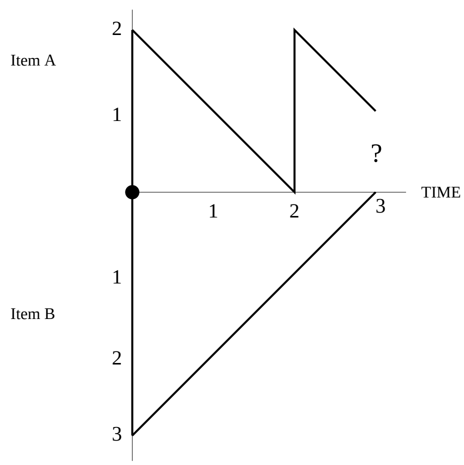
Instance	Relative gap
memphis25s.irp	1.009
memphis25c.irp	1.270
memphis35.irp	1.010
memphis49c.irp	1.015
memphis50c.irp	1.034
memphis50s.irp	1.009
memphis62.irp	1.029

(error $< .0012$)

Relaxation: Example

$\overline{S}_A = 2, \overline{S}_B = 3, \overline{D} = 5$ $\lambda_A = \lambda_B = 1$ $C_A = 50, C_B = 50, C_{AB} = 85$

Optimal solution: $Z^*_{\text{A2}} = 1/6, Z^*_{\text{A2B3}} = 1/3$



How to generate new primal constraints?

Enhance the functional approximation.

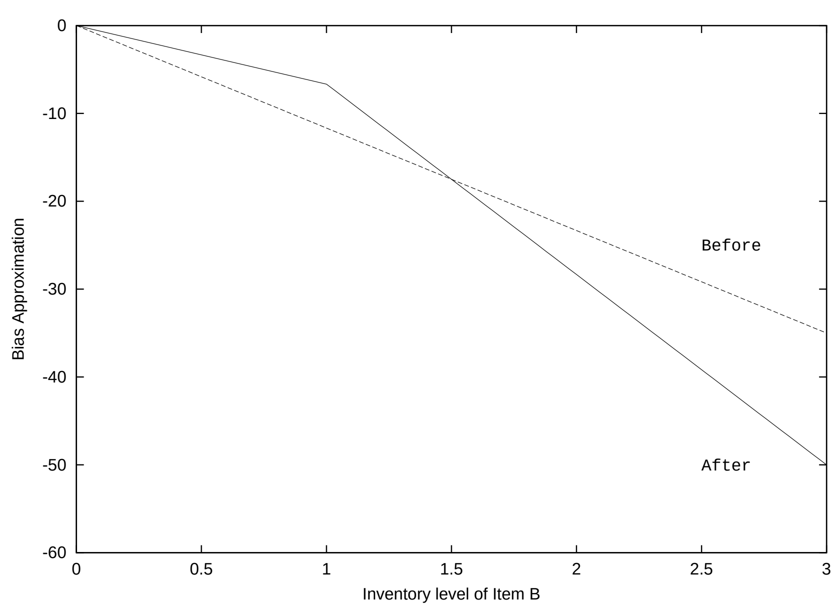
Gain: $g \approx \sum_{i \in \mathcal{I}} \lambda_i V_i$

Bias function: $h(\vec{s}) \approx \theta - \sum_{i \in \mathcal{I}} s_i V_i - \sum_{i \in \mathcal{I}} f_i(s_i) \quad \forall \vec{s} \in \mathcal{S}$

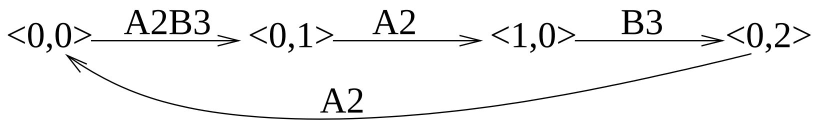
where $f_i(\cdot)$ is a piecewise-linear function $\forall i$.

(We consider more general piecewise-linear functions.)

Example: Piecewise-Linear Approximation



Example: Optimal cyclic schedule



Lower bound on cost increases from 36.667 to 39.167.

Optimal prices yield certificate of optimality.

Lower bound improvement

Instance	Original lower bound	New lower bound	Relative
1	2915.42	3111.16	1.067
2	4888.88	4972.95	1.017
3	8880.89	9605.20	1.082
4	5565.12	5892.06	1.059

General Research Questions

- Other contexts and problem features?
- Stronger formulations?
  Generate cuts through more powerful functional approximations
- General classes of solvable math programs?
- Algorithms

Approximate Dynamic Programming

Powell, Kleywegt, van Roy & de Farias, Bertsimas, Tsitsiklis & Bertsekas

Literature: *Approximate DP*

- Bellman and Dreyfus (1959).
  Functional fit to a polynomial value function.
- Schweitzer and Seidman (1985).
  Formulated generic linear program.
- Bertsekas and Tsitsiklis (1996).
  Neuro-dynamic programming.
- Powell and Carvalho (1998).
  Logistics queueing network. Adaptive DP.
- Adelman, Nemhauser, et al. (1999,1999).
  Cable manufacturing, remnant inventory.
- de Farias and van Roy (2001,2001).
  Queueing, constraint sampling.
- Adelman (2002,2002).
  Deterministic and stochastic inventory/routing.