

EFFECT OF *TOTAL* MODELING UNCERTAINTY ON THE ACCURACY OF NUMERICAL SIMULATIONS

by

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Assessment of the Reliability of Mathematical Models**
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OVERVIEW OF PRESENTATION

- ❑ ***Total Modeling Uncertainty - Definition***
- ❑ **A General Approach Suitable for All Simulations**
- ❑ **Specialization - Linear Structural Dynamics**
- ❑ **Quantification of *Total Modeling Uncertainty* - Examples**
 - Frequency uncertainty
 - Mode shape uncertainty
 - Mass and stiffness uncertainty
 - Uncertainty expressed in terms of generalized “modal” metrics
- ❑ **Accuracy of Numerical Simulations - Examples**
 - Frequency response of a truss beam
 - Airblast response of a reinforced concrete wall

TOTAL MODELING UNCERTAINTY

- ❑ **Truth Assumption**

That which we observe in the physical world

- ❑ **Simulation Uncertainty**

Statistical representation of the difference between simulated and observed phenomena, e.g. a covariance matrix of a discrete time series or frequency response function (FRF).

- ❑ **Total Modeling Uncertainty**

Statistical representation of the difference between simulated and observed phenomena *expressed in terms of generic, dimensionless metrics that may be averaged over a family of generically similar phenomena*, e.g. ratio of measured to predicted natural frequency; normalized analysis-test eigenvector products.

SOURCES OF VARIABILITY / UNCERTAINTY

- ❑ **Test Article or Product Variability**
 - Tolerances
 - Material properties
 - Construction

- ❑ **Experimental Variability**
 - Variability in experimental setup
 - Measurement variability
 - Variability in data processing

- ❑ **Modeling Variability**
 - Variability in modeling assumptions - parameterization
 - Parametric variability
 - Computational variability

- ❑ ***Total Modeling Uncertainty - All of the Above***

GENERAL APPROACH

❑ Observed Phenomena

$$Y(t) = \begin{Bmatrix} y_i(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{Bmatrix} = \begin{bmatrix} y_1(t_1), & y_1(t_2), & \cdots & y_1(t_m) \\ y_2(t_1), & y_2(t_2), & \cdots & y_2(t_m) \\ \vdots & \vdots & & \vdots \\ y_n(t_1), & y_n(t_2), & \cdots & y_n(t_m) \end{bmatrix}$$

❑ Corresponding Numerical Simulation

$${}^0Y(t) = \begin{Bmatrix} {}^0y_i(t) \\ {}^0y_2(t) \\ \vdots \\ {}^0y_n(t) \end{Bmatrix} = \begin{bmatrix} {}^0y_1(t_1), & {}^0y_1(t_2), & \cdots & {}^0y_1(t_m) \\ {}^0y_2(t_1), & {}^0y_2(t_2), & \cdots & {}^0y_2(t_m) \\ \vdots & \vdots & & \vdots \\ {}^0y_n(t_1), & {}^0y_n(t_2), & \cdots & {}^0y_n(t_m) \end{bmatrix}$$

❑ Singular Value Decomposition

$$Y(t) = U\Sigma V^T = \phi D \eta$$

$${}^0Y(t) = {}^0U {}^0\Sigma {}^0V^T = {}^0\phi {}^0D {}^0\eta$$

PRINCIPAL COMPONENTS ANALYSIS

□ Singular Values and Singular (Eigen) Vectors

- D_{ii} = Singular value (from the diagonal matrix, D)
- ϕ_i = Left (column) eigenvector
- η_i = Right (row) eigenvector

□ Orthonormal properties of Eigenvectors

$$\begin{aligned}\phi_i^T \phi_j &= \delta_{ij} & \eta_i \eta_j^T &= \delta_{ij} \\ {}^0\phi_i^T {}^0\phi_j &= \delta_{ij} & {}^0\eta_i {}^0\eta_j^T &= \delta_{ij}\end{aligned}$$

□ Principal Components : *Energy*-ordered “Modes”

$$YY^T = (\phi D \eta)(\eta^T D \phi^T) = \phi D^2 \phi^T$$

$$\sum_i^n \sum_j^m y_i^2(t_j) = \text{Tr}(YY^T) = \text{Tr}(\phi D^2 \phi^T) = \sum_i^n D_{ii}^2$$

PC METRICS FOR QUANTIFYING TOTAL MODELING UNCERTAINTY

- Express ϕ_i 's and η_i 's as linear combinations of ${}^0\phi_i$'s and ${}^0\eta_i$'s

$$\phi \cong ({}^0\phi)\psi : {}^0\phi^T\phi = \psi$$

$$\eta \cong ({}^0\nu)^T\eta : {}^0\eta\eta^T = \nu$$

- Evaluate differences between observation and simulation

$$\Delta\phi = \phi - {}^0\phi \cong {}^0\phi(\psi - I) = {}^0\phi\Delta\psi$$

$$\Delta D = D - {}^0D; \quad \Delta D {}^0D^{-1} = \Delta\tilde{D}$$

$$\Delta\eta = \eta - {}^0\eta \cong (\nu - I) {}^0\eta = \Delta\nu {}^0\eta$$

- Vectorize $\Delta\psi, \Delta\tilde{D}, \Delta\nu$

$$\Delta\tilde{r} = \begin{Bmatrix} \text{vec}(\Delta\psi) \\ \text{vec}(\Delta\tilde{D}) \\ \text{vec}(\Delta\nu) \end{Bmatrix}$$

EVALUATION OF TOTAL MODELING UNCERTAINTY

□ Using Replicate Test Data for a Particular Test Article

$$S_{\tilde{r}\tilde{r}} = E[(\Delta\tilde{r} - \mu_{\Delta\tilde{r}})(\Delta\tilde{r} - \mu_{\Delta\tilde{r}})^T] = \frac{1}{N-1} \sum_{i=1}^N (\Delta\tilde{r} - \mu_{\Delta\tilde{r}})(\Delta\tilde{r} - \mu_{\Delta\tilde{r}})^T$$

$$\mu_{\Delta\tilde{r}} = E[\Delta\tilde{r}] = \frac{1}{N} \sum_{i=1}^N \Delta\tilde{r} \text{ (Systematic Error)}$$

□ Using Test Data from Generically Similar Test Articles

$$S_{\tilde{r}\tilde{r}} = E[(\Delta\tilde{r}\Delta\tilde{r}^T)] = \frac{1}{N} \sum_{i=1}^N \Delta\tilde{r}\Delta\tilde{r}^T$$

$\mu_{\Delta\tilde{r}}$ is assumed to be zero

FAST RUNNING SIMULATION

□ Recall Singular Value Decomposition

$$Y(t) = \phi D \eta$$

$$y_i(t_j) = \sum_{k=1}^p \phi_{ik} D_{kk} \eta_k(t_j) = \sum_{k=1}^p \phi_{ik} D_{kk} \eta_{kj}$$

□ Recall Difference Analysis

$$\phi = {}^0\phi + \Delta\phi \cong {}^0\phi(I + \Delta\psi)$$

$$D = {}^0D + \Delta D = {}^0D(I + \Delta\tilde{D})$$

$$\eta = {}^0\eta + \Delta\eta \cong {}^0\eta(I + \Delta\nu)$$

□ Fast Running Simulation

$$\hat{Y}_{ij} = {}^0\phi(I + \Delta\psi) {}^0D(I + \Delta\tilde{D}) {}^0\eta(I + \Delta\nu)$$

EVALUATING THE ACCURACY OF NUMERICAL SIMULATIONS

□ Linear Covariance Propagation

$$S_{YY} \cong S_{\hat{Y}\hat{Y}} = T_{\hat{Y}\tilde{r}} S_{\tilde{r}\tilde{r}} T_{\hat{Y}\tilde{r}}^T$$

$$T_{\hat{Y}\tilde{r}} = \partial \hat{Y} / \partial \tilde{r}$$

□ Interval Propagation - Vertex Method

$$\Delta Y = \left\{ \max_{i,j} [Y(c_i), Y(e_j)] - \min_{i,j} [Y(c_i), Y(e_j)] \right\}$$

c_i = coordinates of vertices of rectangular hyperspace

e_j = coordinates of extrema within rectangular hyperspace

□ Monte Carlo Simulation

- Diagonalize $S_{\tilde{r}\tilde{r}} : \Delta \tilde{r} = \Theta \Delta s$
- Execute PC Model for randomly selected values of Δs

SPECIALIZATION TO LINEAR STRUCTURAL DYNAMICS

□ Classical Normal Mode Analysis

$$({}^0K - {}^0\lambda_j {}^0M) {}^0\phi_j = 0$$

${}^0\lambda_j$ = Analytical eigenvalues

${}^0\phi_j$ = Analytical Eigenvectors

0M = Analytical mass matrix

0k = Analytical stiffness matrix

□ Modal Mass and Stiffness Matrices

$${}^0m = {}^0\phi^T {}^0M {}^0\phi = I$$

$${}^0k = {}^0\phi^T {}^0K {}^0\phi = {}^0\lambda$$

□ Assumed “True” Modal Mass and Stiffness Matrices

$$m = {}^0m + \Delta m = I + \Delta m$$

$$k = {}^0k + \Delta k = {}^0\lambda + \Delta k$$

MODAL METRICS FOR QUANTIFYING MODELING UNCERTAINTY

- Normalization of Test Modes, ϕ

$$\phi_j^T {}^0M \phi_j = 1$$

- Cross-orthogonality of Analysis and Test Modes

$${}^0\phi_j^T {}^0M \phi_j = \phi_j^T {}^0M {}^0\phi \psi = \psi$$

- Difference Between Analysis and Test Modes

$$\Delta\lambda = \lambda - {}^0\lambda$$

$$\Delta\phi = \phi - {}^0\phi = {}^0\phi(\psi - I) = {}^0\phi\Delta\psi$$

- Modal Metrics for Quantifying Total Modeling Uncertainty

$$\Delta m = -(\Delta\psi + \Delta\psi^T)$$

$$\Delta\tilde{k} = {}^0\lambda^{-1/2}(\Delta\lambda - {}^0\lambda\Delta\psi - \Delta\psi^T {}^0\lambda) {}^0\lambda^{-1/2}$$

$$\Delta\tilde{r} = \begin{Bmatrix} \text{vec}(\Delta m) \\ \text{vec}(\Delta\tilde{k}) \\ \text{vec}(\Delta\zeta) \end{Bmatrix}$$

COMPARISON OF FAST-RUNNING SIMULATIONS

□ General Approach

$$y_i(t_j) = \sum_{k=1}^p \phi_{ik} D_{kk} \eta_k(t_j)$$

where ϕ , D and η are the *principal component metrics* corresponding to the *energy-ordered modes* of a SVD.

□ Linear Structural Dynamics Approach

$$y_i(t_j) = \sum_{k=1}^p \phi_{ik} \Gamma_{kk} \eta_k(t_j)$$

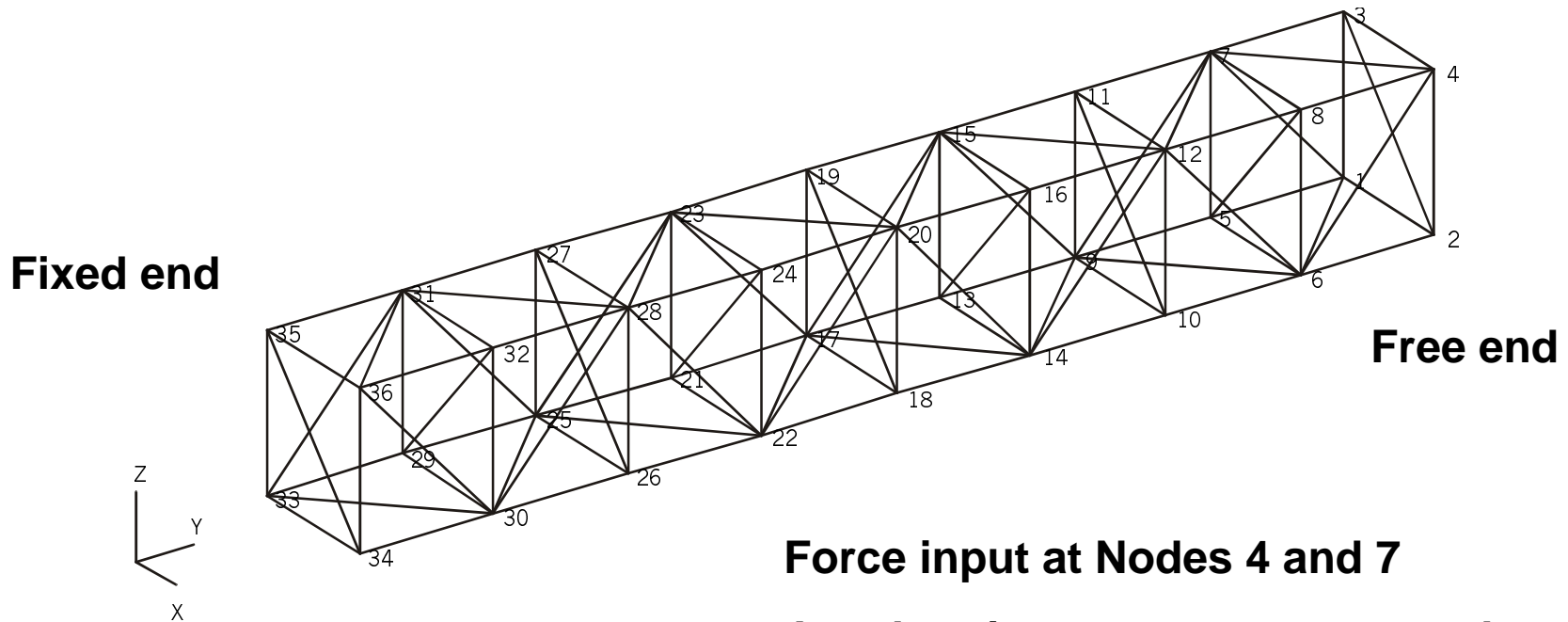
where ϕ , Γ and η are the *classical modal metrics* from a real symmetric eigenvalue/eigenvector analysis.

ϕ_k = Classical normal mode

Γ_{kk} = Modal participation factor

$\eta_k(t)$ = Modal response time history

NASA / LaRC EIGHT BAY TRUSS



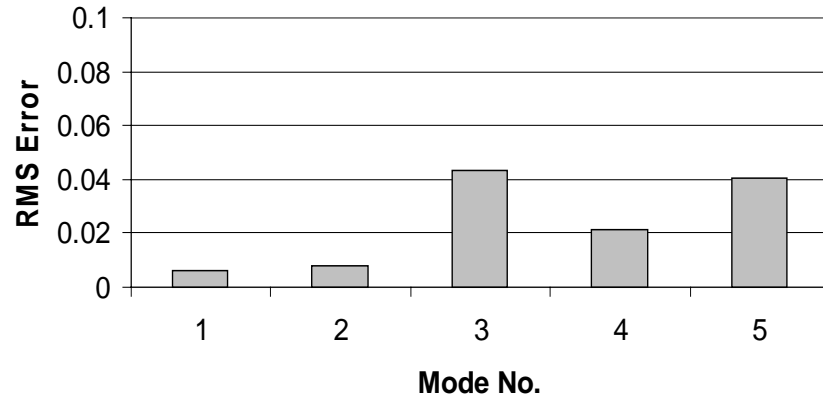
Force input at Nodes 4 and 7

**Acceleration response measured at
Nodes 1 through 32**

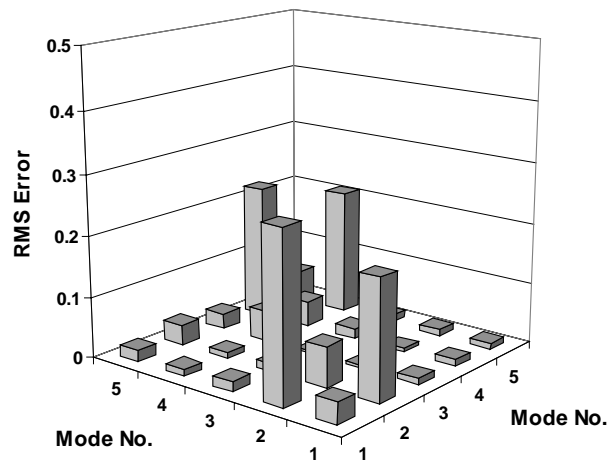
EXAMPLE OF TOTAL MODELING UNCERTAINTY CLASSICAL MODAL ANALYSIS

Variations of an 8-bay truss beam

Normalized Frequency

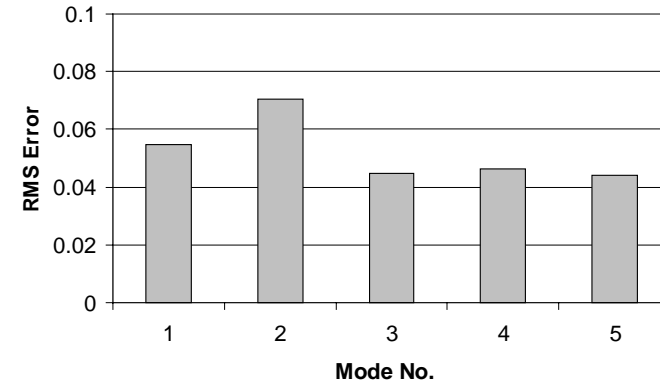


Eigenvector Cross-orthogonality

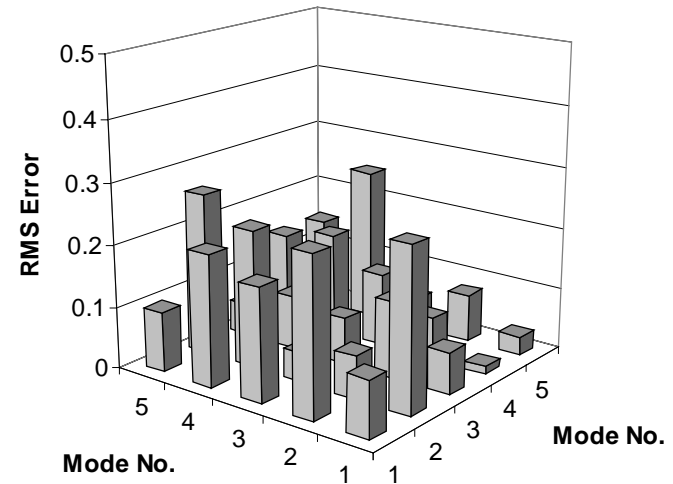


Generic family of space structures

Normalized Frequency

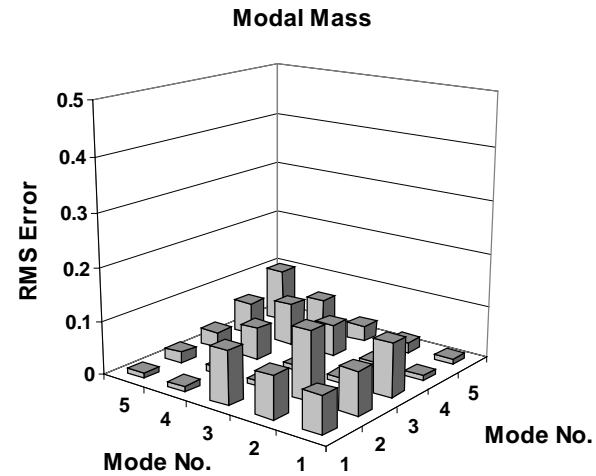


Eigenvector Cross-orthogonality

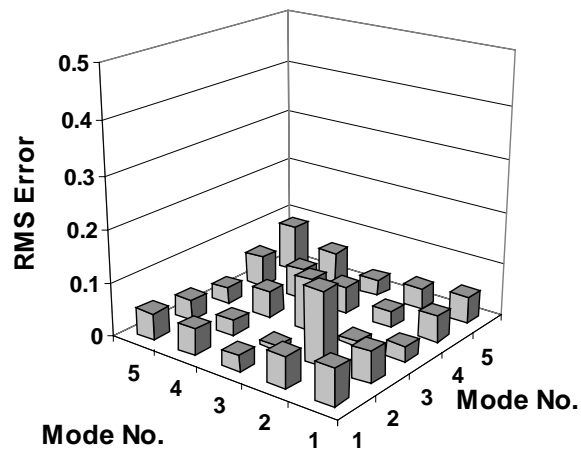


TOTAL MODELING UNCERTAINTY IN TERMS OF MODAL MASS AND STIFFNESS VARIABILITY

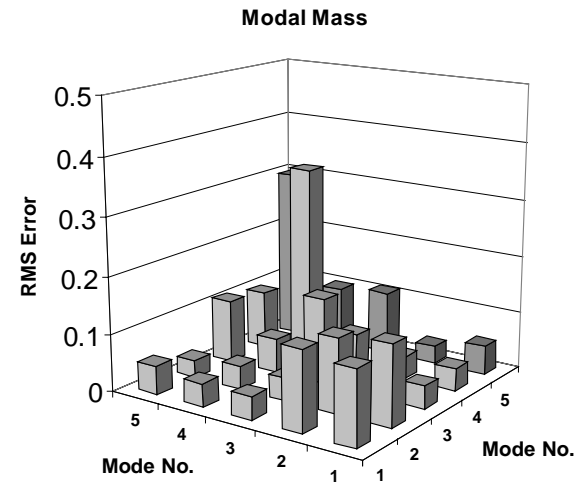
Variations of an 8-bay truss beam



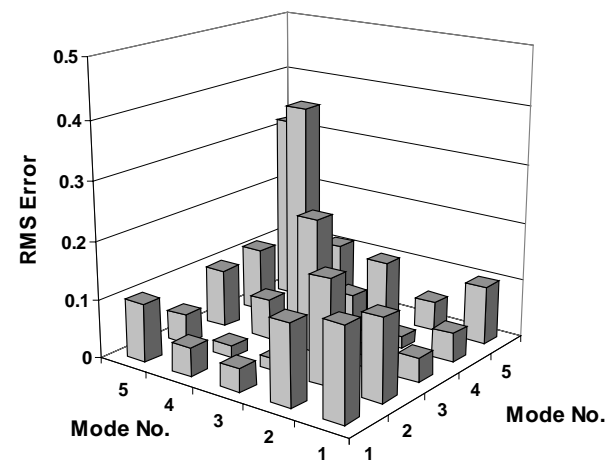
Normalized Modal Stiffness



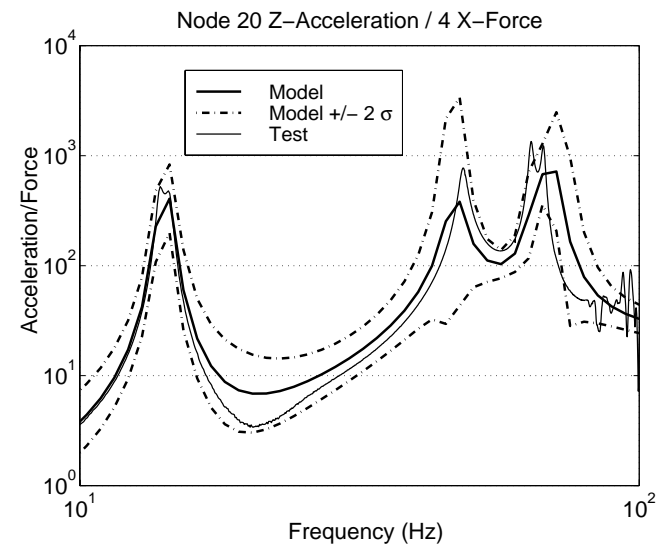
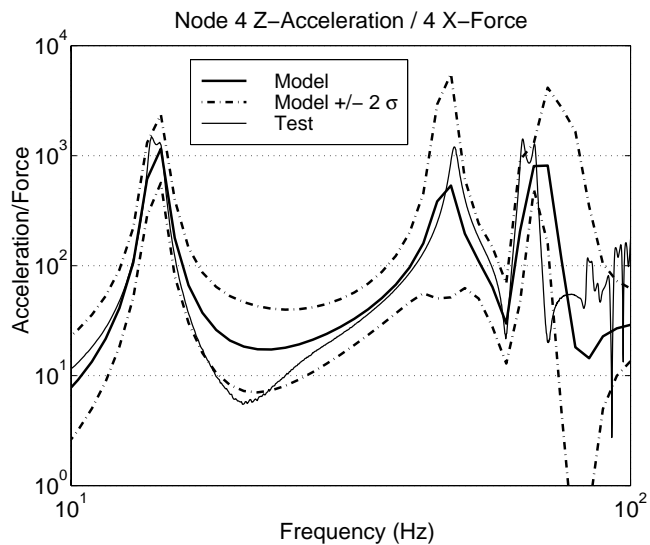
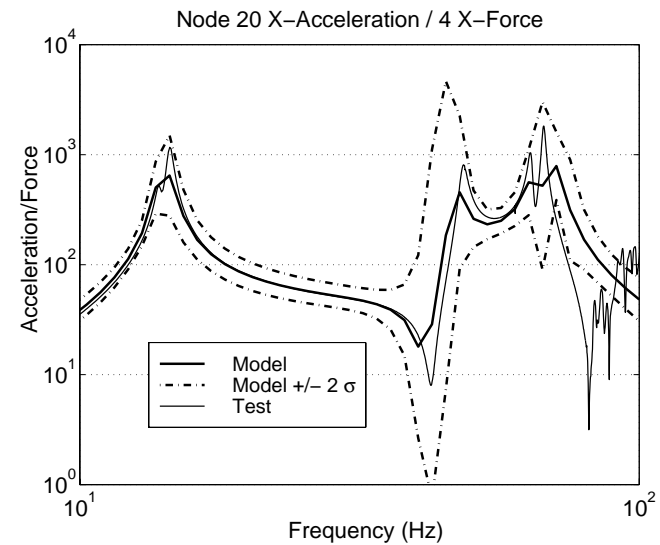
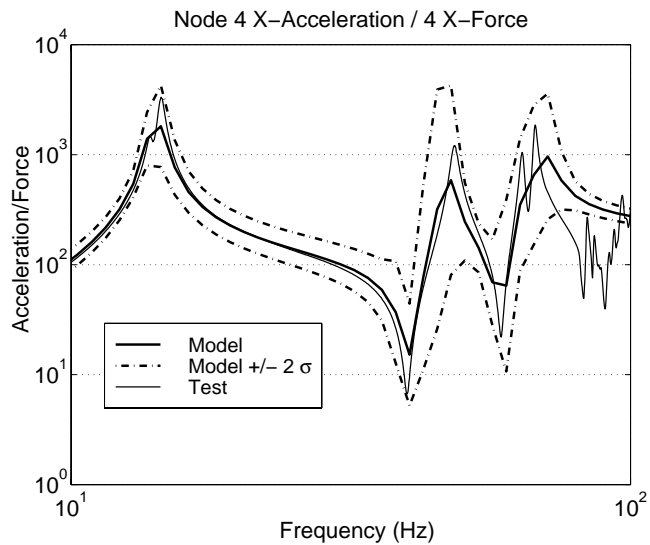
Generic family of space structures



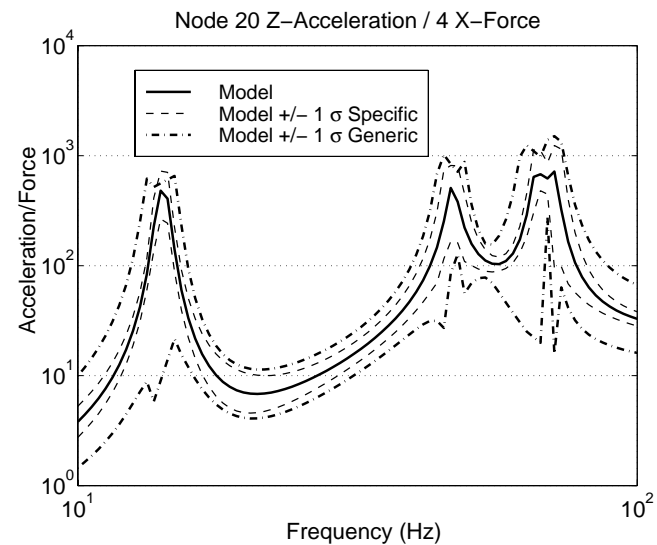
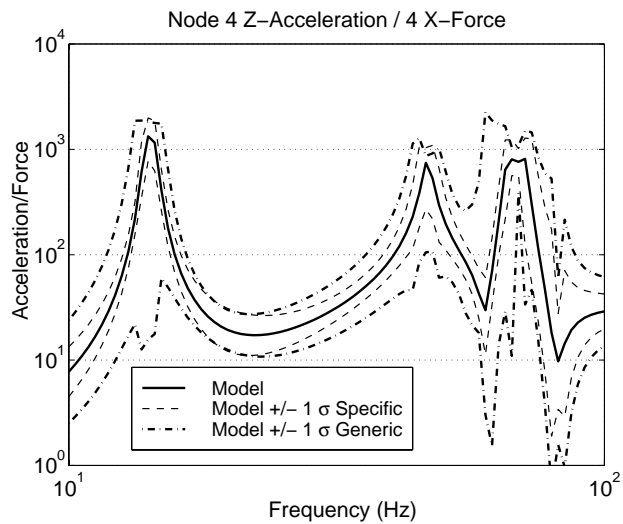
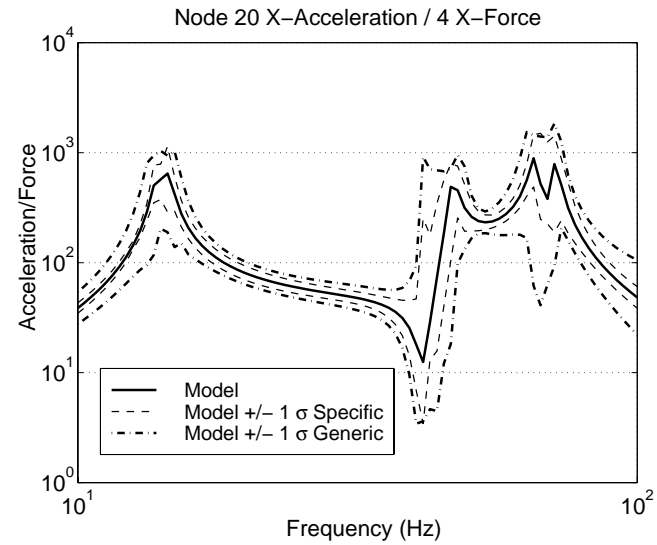
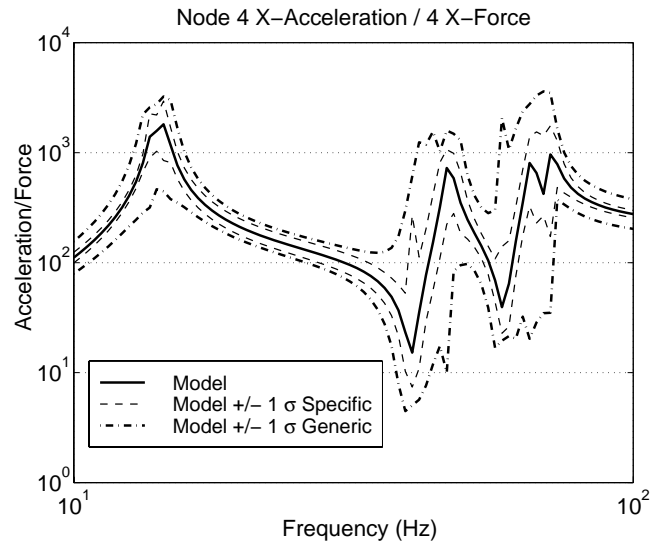
Normalized Modal Stiffness



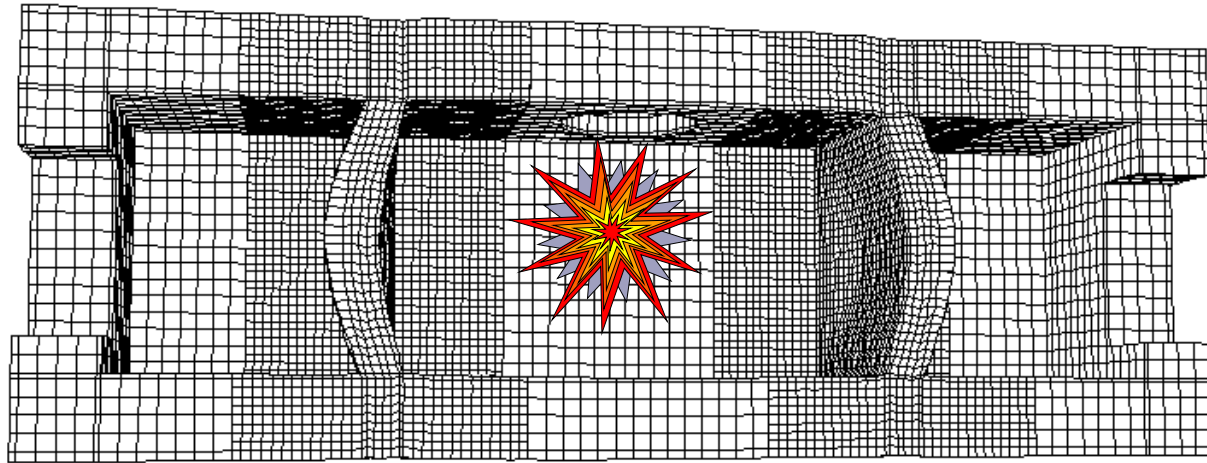
FRF SIMULATION ACCURACY BASED ON STRUCTURE-SPECIFIC MODELING UNCERTAINTY



COMPARISON OF SIMULATION ACCURACY FOR SPECIFIC AND GENERIC MODELING UNCERTAINTY



BLAST RESPONSE OF A BURIED R/C STRUCTURE



Finite Element Model

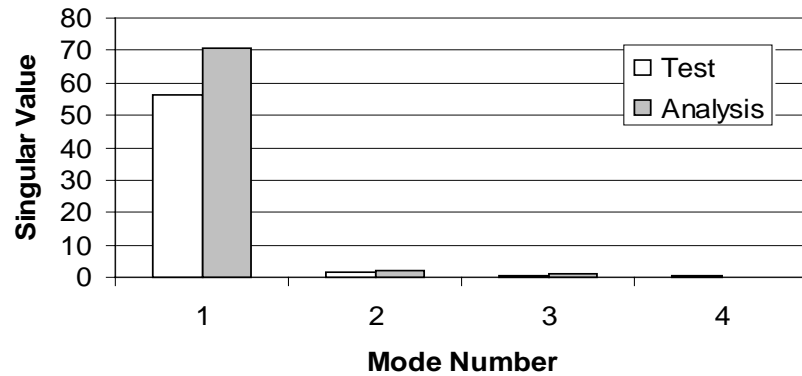
- ❑ ~ 80,000 continuum elements
(concrete and soil)
- ❑ ~ 20,000 rebar elements

Test Measurements

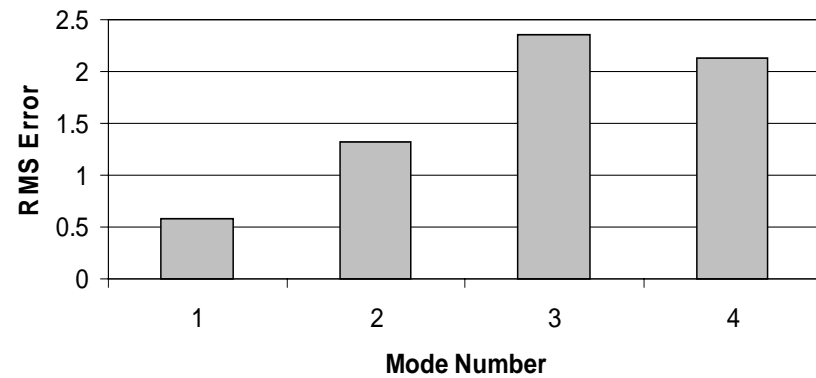
- ❑ 12 Pressure gages
(10 on interior walls)
- ❑ 12 Accelerometers on interior walls
- ❑ 5 Locations each wall

EXAMPLE OF TOTAL MODELING UNCERTAINTY PRINCIPAL COMPONENTS ANALYSIS

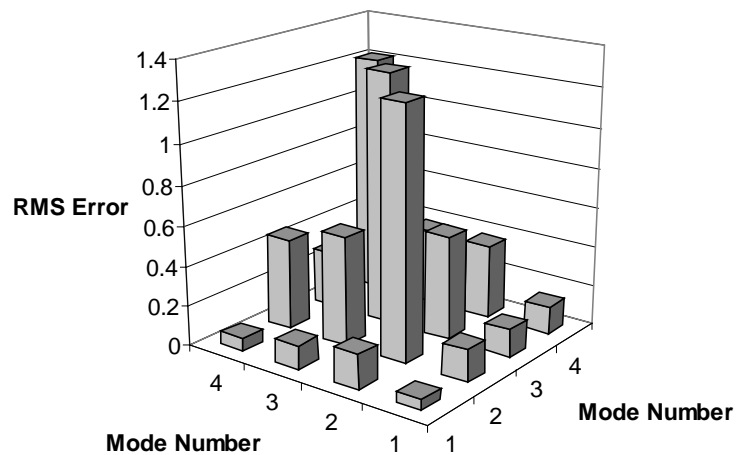
Typical Singular Values



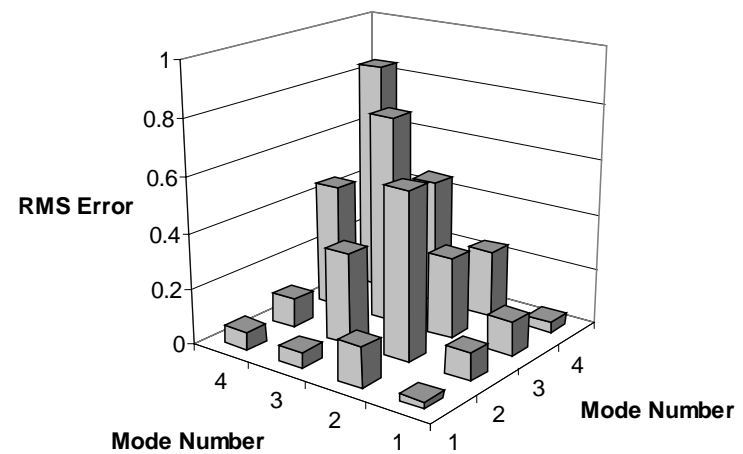
Normalized Singular Values



Left Eigenvector Cross-Orthogonality

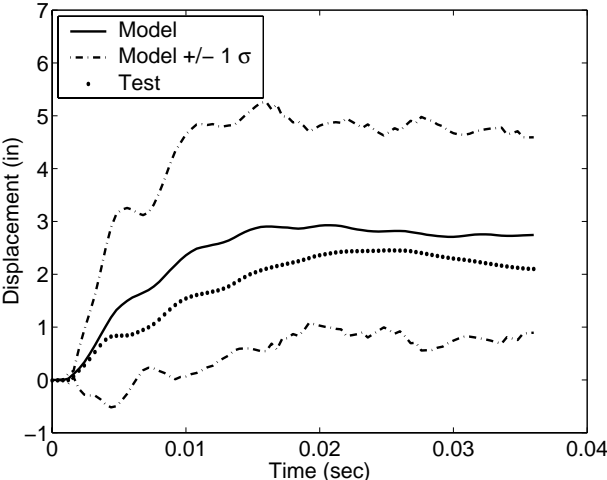


Right Eigenvector Cross-Orthogonality

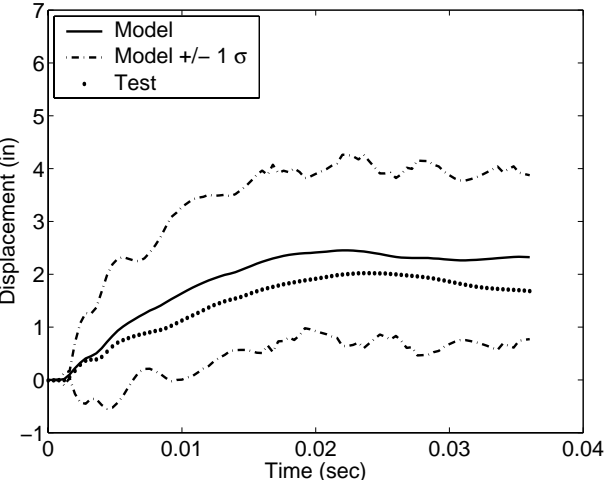


DISPLACEMENT SIMULATION UNCERTAINTY FOR R/C WALL PRE-UPDATE MODEL OF STRUCTURE 1

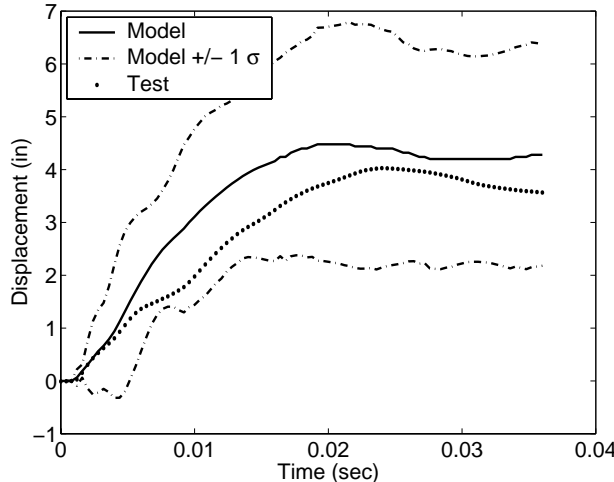
Left Horizontal Quarter Point



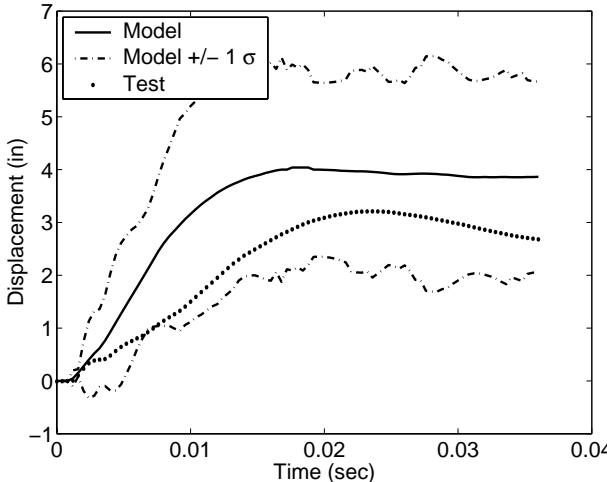
Upper Vertical Quarter Point



Center Point

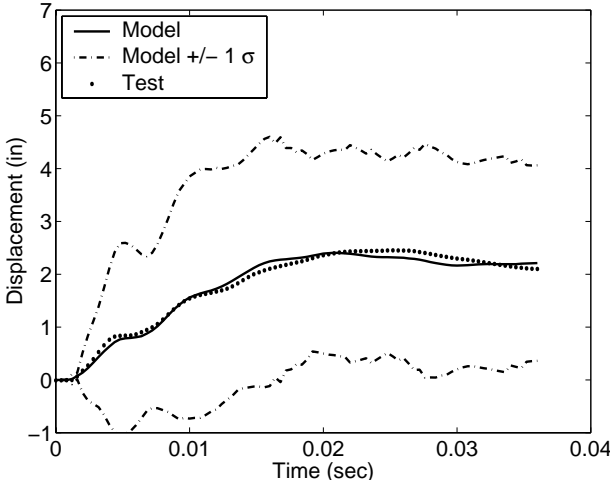


Lower Vertical Quarter Point

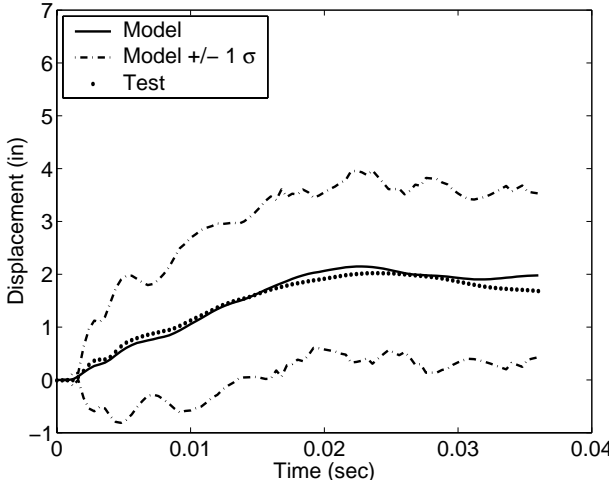


DISPLACEMENT SIMULATION UNCERTAINTY FOR R/C WALL POST-UPDATE MODEL OF STRUCTURE 1

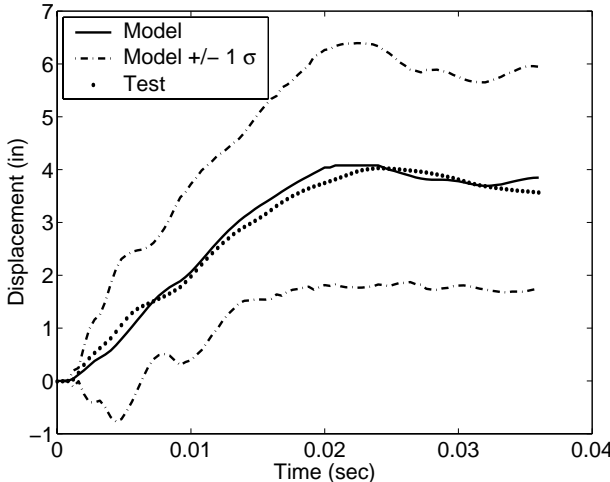
Left Horizontal Quarter Point



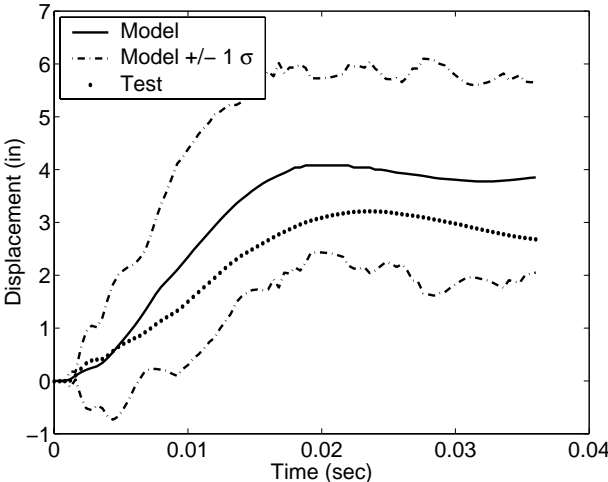
Upper Vertical Quarter Point



Center Point

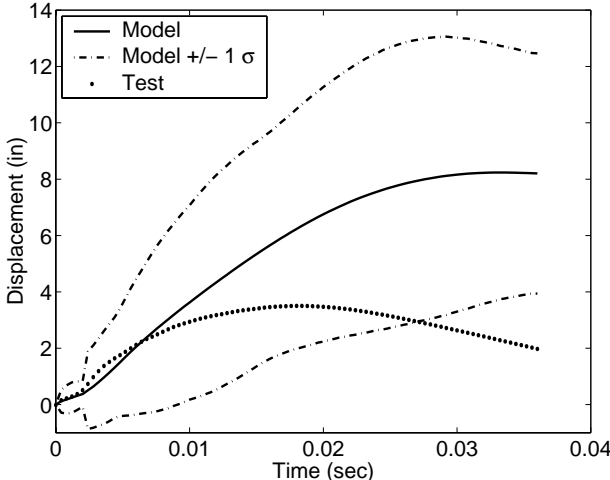


Lower Vertical Quarter Point

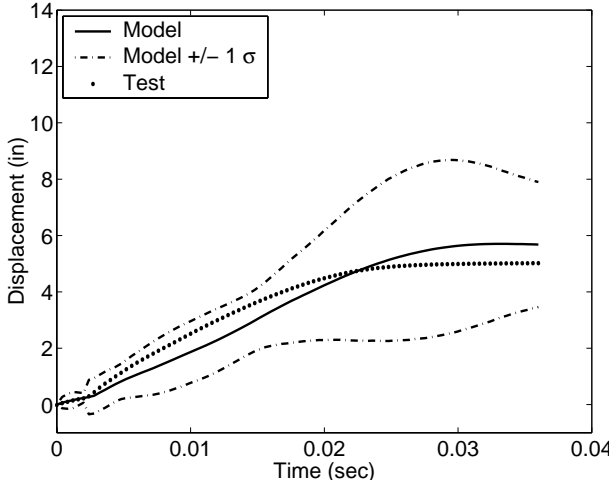


DISPLACEMENT SIMULATION UNCERTAINTY FOR R/C WALL PRE-UPDATE MODEL OF STRUCTURE 2

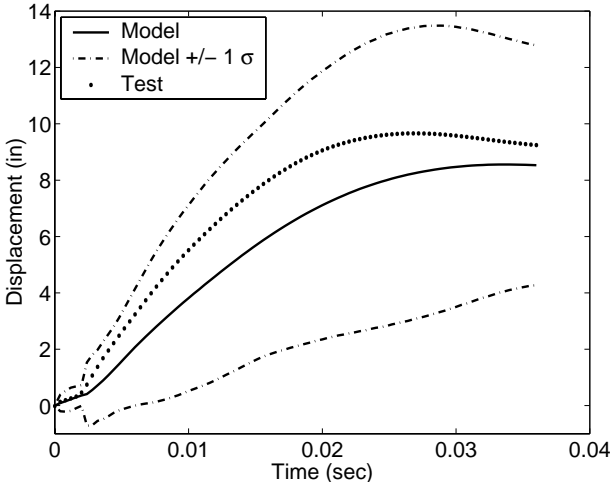
Gage 1



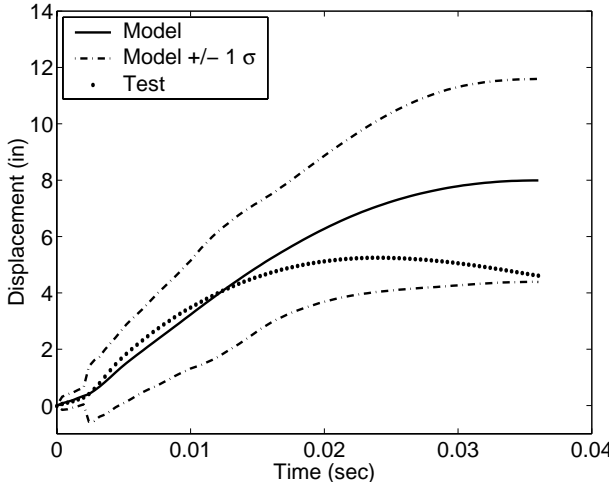
Gage 2



Gage 3



Gage 4



SUMMARY

❑ **Why *total* modeling uncertainty?**

The predictive accuracy of numerical simulations can be reliably estimated only to the extent that *total* modeling uncertainty is quantified.

❑ **How does one obtain the necessary data?**

By decomposing corresponding sets of simulated and measured response into their principal components, and statistically processing the differences between corresponding PC (modal) metrics.

❑ **How is the predictive accuracy of numerical simulations derived from total modeling uncertainty?**

- By first order covariance propagation,
- By interval propagation, and/or
- By Monte Carlo simulation

CONCLUSIONS

- ❑ **Numerical simulations are increasingly relied upon for making critical decisions.**
- ❑ **The reliability of these simulations depends on uncertainty inherent in the underlying mathematical models.**
- ❑ **A true representation of this uncertainty must include all possible sources of uncertainty, i.e. *total* modeling uncertainty.**
- ❑ **Decision making under uncertainty demands that *total* modeling uncertainty be quantified realistically.**