Numerical Methods for Sensitivity Computations

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IMA WORKSHOP

Decision Making under Uncertainty:
Assessment of the Reliability of Mathematical Models

18 September, 1999

Minneapolis, MN

Group Effort

- Virginia Tech
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  - J. Atwell, C. Camphouse, K. Hulsing, L. Stanley, D. Stewart
- University of Virginia
  - P. Parrish, H. Wadley
- Princeton University
  - Yannis Kevrekedis

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- BEAM Technologies
  - G. Berkooz
    - CFETools ™
- Commonwealth Scientific Corp.
  - D. Baldwin
- Oak Ridge National Lab
  - W. Butler

Motivation and Outline

MATHEMATICAL MODELS SHOULD BE "GOOD ENOUGH" FOR ...

DECISION MAKING & DESIGN
CONTROL & OPTIMIZATION

OUTLINE OF PRESENTATION

- **Present a project to motivate current interest**
  - **CONTROL OF THIN FILM GROWTH**
- **Use elementary examples to illustrate the ideas**
- **Present numerical results for a complex problems**
  - **HYDROGEN INJECTION SCRAMJET INJECTION DESIGN AND OPTIMIZATION**
  - **CHEMICALLY REACTING FLOWS**

Why Sensitivities?

- USEFUL IN OPTIMIZATION BASED DESIGN
- SENSITIVITIES HAVE MANY OTHER USES
  - PRIORITIZE DESIGN & CONTROL VARIABLES
  - EVALUATE DESIGNS & CONTROL LAWS
  - NON- OPTIMIZATION BASED DESIGN
  - FAST SOLVERS
  - ANALYZE UNCERTAINTIES
  - PREDICT “FAILURE” (FLOW SEPARATION, ETC.)
- SOME OBSERVATIONS
  - DO NUMERICS CAREFULLY
  - “ORDER” MATTERS

MAY REQUIRE COMPLEX MATHEMATICAL THEORIES

DIFFERENTIATION OF SET-VALUED FUNCTIONS

Key Points and Goal of the Talk

POINT #1: Infinite dimensional “theory’’ can be helpful in solving real engineering problems.

KEY QUESTION: How can information about the infinite dimensional problem be maximized?

POINT #2: The order of development in the design of an algorithm can have great impact on the construction of efficient computational methods.

KEY QUESTION: When should approximation take place?

GOAL OF THE TALK: To make a case that points #1 and #2 are valid for some real problems

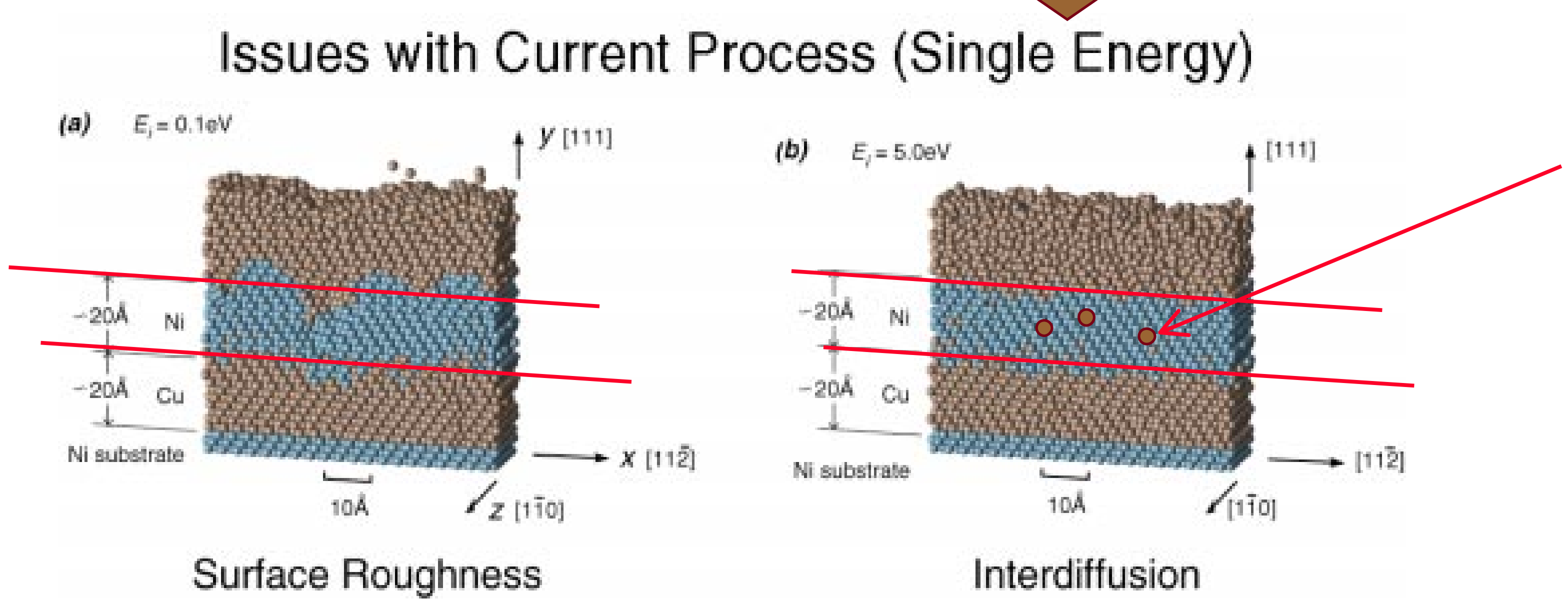
Control of Thin Film Growth

"VARIABLE ENERGY ION SOURCE"

OR

E_i = .1 eV

E_i = 5.0 eV



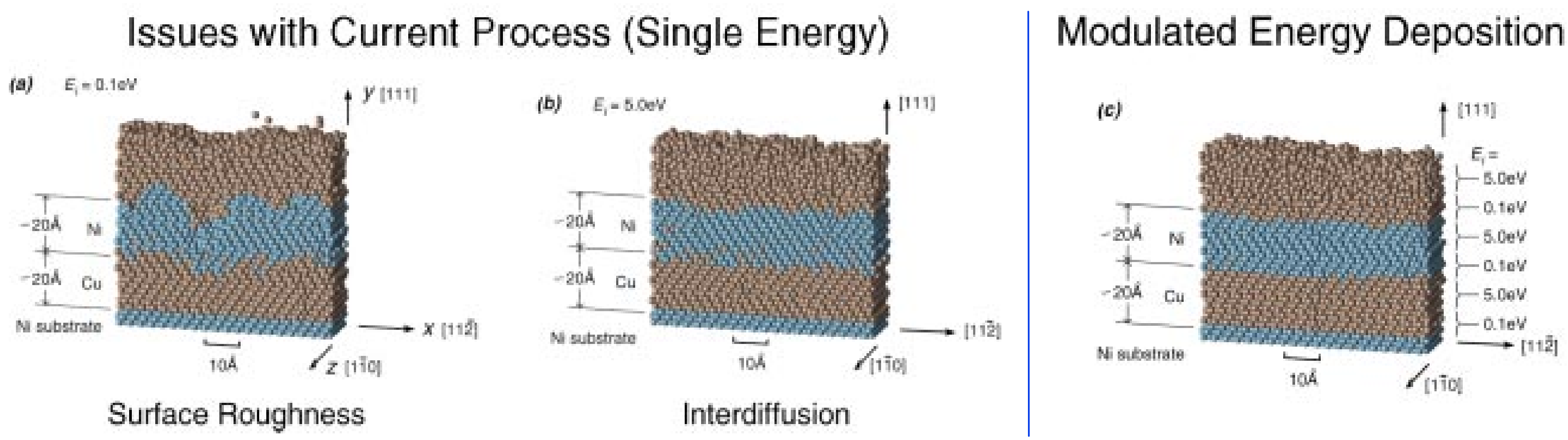
Control of Thin Film Growth

Cambridge Hydrodynamics, SC Solutions, Colorado, Oak Ridge National Lab

Atomistic Model-Based Design of GMR Processes. Virginia (PI: H. Wadley)

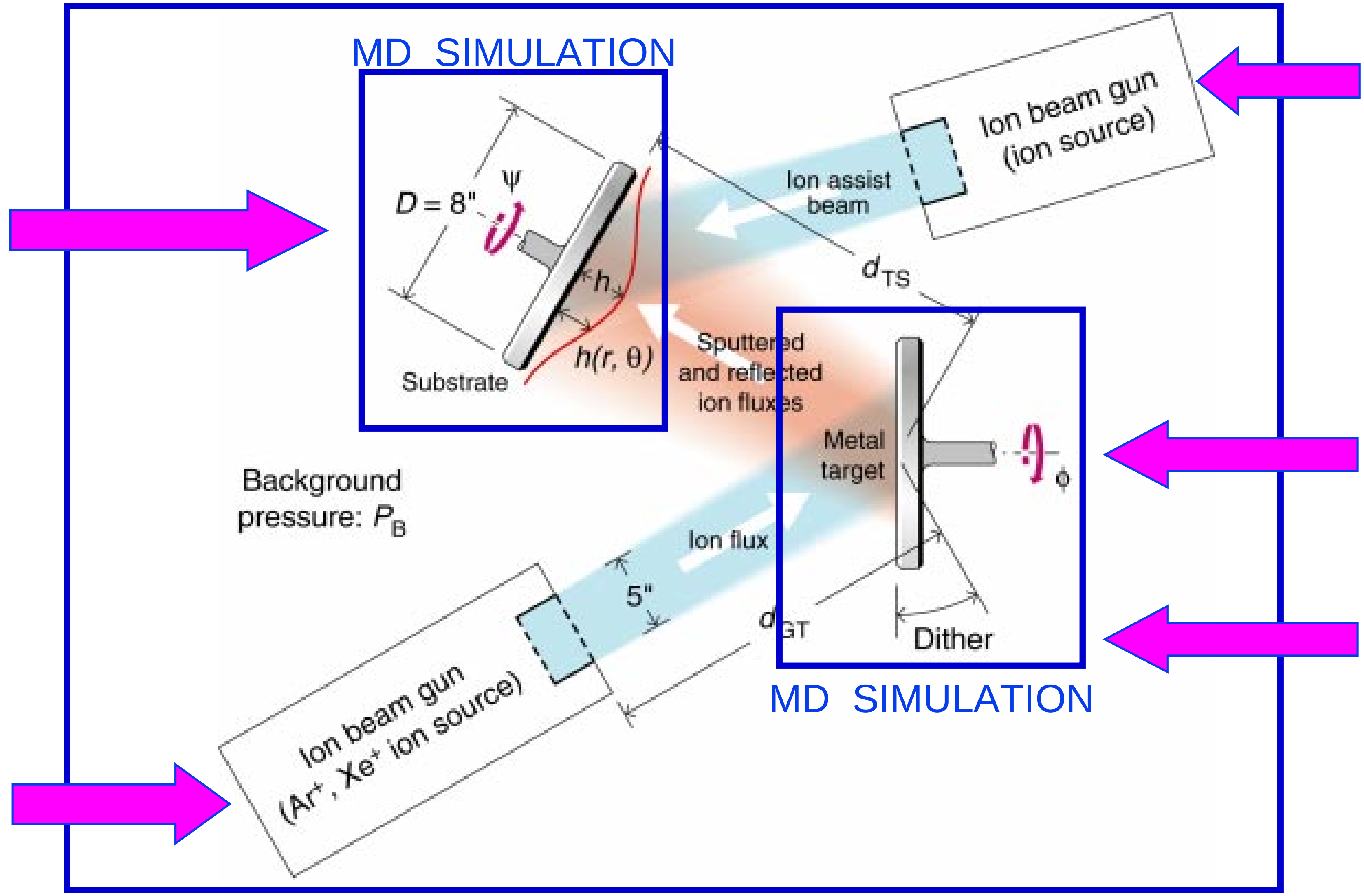
Optimized ion beam processing through Modulated Energy Deposition

- **Low energy for initial monolayers**
- **Moderate energy for intermediate layers**
- **High energy to flatten film surface**



Successful proof-of-concept experiments using Modulated Energy Deposition approach (Honeywell)

Ion Beam Deposition Process



? Models ?

Phenomenological models (Ortiz, Repetteo, Si, Zangwill, … 1990s)

$$\frac{\partial}{\partial t} h(t,x,y) = \nu \nabla \cdot [(\kappa^{-2} |\nabla h(t,x,y)|^2 - 1) \nabla h(t,x,y)] - D\nabla^4 h(t,x,y) + F(t,x,y,\mathbf{q})$$

Molecular Dynamic Models (Alder, Wainwright, … 1950s)

$$r^N = [\vec{r}_1, \quad \vec{r}_2, \quad \ldots \quad \vec{r}_N]^T$$

Position of N - atoms

$$\mathcal{U}(r^N) = \sum_{\substack{i,j \\ i<j}} u(\|\vec{r}_i - \vec{r}_j\|, \varepsilon, \sigma, p, \mathbf{q}), \qquad f_i = -\frac{\partial}{\partial \vec{r}_i} \mathcal{U}(r^N)$$

$$m \frac{d^2}{dt^2} r^N(t) = f(t, r^N(t), \varepsilon, \sigma, p, \mathbf{q})$$

Optimization/Control Problem

OPTIMIZATION/CONTROL PROBLEM: Find $q=q(t)$ to minimize

$$\mathbf{F}(h(\cdot,\mathbf{q}),\mathbf{q}) = \int_0^T \frac{1}{2}\int_\Omega \left| h(t,x,y,\mathbf{q}) - d(t,x,y)\right|^2 \mathrm{dtdxdy}$$

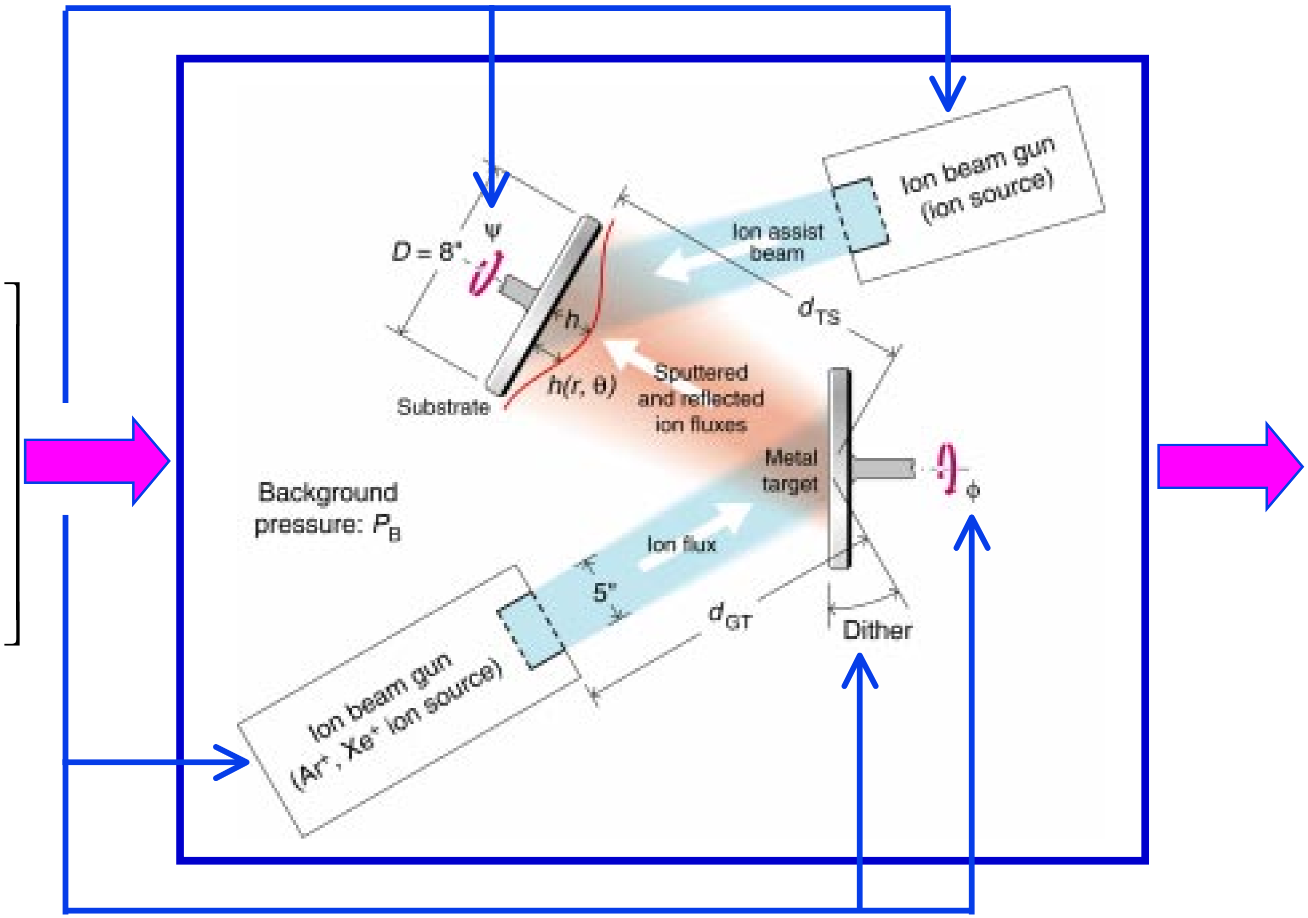
$$\mathbf{J}(\mathbf{q}) = \mathbf{F}(h(\cdot,\mathbf{q}),\mathbf{q})$$

SENSITIVITY

$$\nabla_q \mathbf{J}(\mathbf{q}) = \int_0^T \int_\Omega \left| h(t,x,y,\mathbf{q}) - d(t,x,y)\right| \boxed{\frac{\partial}{\partial q} h(t,x,y,\mathbf{q})}\, dtdxdy$$

Ion Beam Deposition Process

$$\mathbf{q} = \begin{bmatrix} \psi \\ \phi \\ \mu \\ \varphi \\ d \\ \vdots \end{bmatrix}$$



$$h(r,\theta)$$

How sensitive is $h(r,\theta,\psi,\phi,\mu,\varphi,d)$ **to** ψ **? -** $\partial_{\psi} h(r,\theta,\psi,\phi,\mu,\varphi,d)$

1D Model Problem

LET 1 < q < ∞ and consider the boundary value problem

$$(\Sigma) \quad \frac{d^2}{dx^2}w(x)+\frac{1}{8}\left[\frac{d}{dx}w(x)\right]^3=0,\ 0<x<q, \qquad w(0)=0, \qquad w(q)=4$$

Given data $\hat{w}(x)$, $0<x<1$ the goal is to match $\hat{w}(x)$ by solving the following

OPTIMAL DESIGN PROBLEM: Find the parameter 1 < q_0, to minimize the cost function

$$(COST) \qquad \mathbf{J}(q)=\mathbf{F}(w(\cdot,q),q)=\tfrac{1}{2}\int_0^1 |\, w(x,q)-\hat{w}(x)\,|^2\, dx$$

Model Problem

$$\frac{d}{dq}\left[\mathbf{J}(q)\right]=\int_0^1\left\langle w(x,q)-\hat{w}(x),\frac{\partial}{\partial q}(w(x,q))\right\rangle dx$$

SENSITIVITY

$$s(x)=s(x,q)=\frac{\partial}{\partial q}w(x,q)\quad 0<x<q$$

The sensitivity equation for $s(x,q)=\partial_q w(x,q)$ in the "physical" domain $\Omega(q)=(0,q)$ is given by

$(\delta\Sigma)$

$$\frac{d^2}{dx^2}s(x)+\frac{3}{8}\left[\frac{d}{dx}w(x,q)\right]^2\frac{d}{dx}s(x)=0,\quad 0<x<q,$$

$$s(0)=0,\quad s(q)=-\tfrac{d}{dx}w(x)\Big|_{x=q}$$

Can be made "rigorous" by the method of mappings.
MORE ABOUT THIS NEAR THE END

Typical Cost Function

$$\mathbf{J}(q) = \mathbf{F}(w(\cdot, q), q) = \tfrac{1}{2}\int_0^1 | w(x, q) - \hat{w}(x) |^2 \, dx$$

WHERE $w(x, q)$ **USUALLY SATISFIES A DIFFERENTIAL EQUATION AND** q **IS A PARAMETER (OR VECTOR OF PARAMETERS)**

THE CHAIN RULE PRODUCES

CONTINUOUS SENSITIVITY

$$\tfrac{d}{dq}\mathbf{J}(q) = \tfrac{d}{dq}\mathbf{F}(w(\cdot, q), q) = \int_0^1 [w(x, q) - \hat{w}(x)] \cdot [\tfrac{\partial}{\partial q} w(x, q)] dx$$

OR (Reality) USING NUMERICAL SOLUTIONS

DISCRETE SENSITIVITY

$$\tfrac{d}{dq}[\mathbf{J}^{\mathbf{h}}(q)] = \tfrac{d}{dq}\mathbf{F}(w^{\mathbf{h}}(\cdot, q), q) = \int_0^1 \langle w^{\mathbf{h}}(x, q) - \hat{w}(x), \tfrac{\partial}{\partial q}[w^{\mathbf{h}}(x, q)] \rangle dx$$

Computing Gradients

TYPICAL APPROACHES TO COMPUTE $\left.\frac{d}{dq}\left[\mathbf{J}^{h}(q)\right]\right|_{q=q_0}$

(I) BY FINITE DIFFERENCES

$$\frac{d}{dq}\left[\mathbf{J}^{h}(q_0)\right] \cong \left[\frac{\mathbf{J}^{h}(q_0+\Delta q) - \mathbf{J}^{h}(q_0)}{\Delta q}\right]$$

(II) BY DISCRETE SENSITIVITIES

$$\tfrac{d}{dq}[\mathbf{J}^{h}(q_0)] = \tfrac{d}{dq}\mathbf{F}(w^{h}(\cdot,q_0),q_0) = \int_0^1 \langle w^{h}(x,q_0) - \hat{w}(x), \tfrac{\partial}{\partial q}[w^{h}(x,q_0)]\rangle dx$$

Computing Gradients

FINITE DIFFERENCES

- REQUIRES 2 NON-LINEAR SOLVES
- IF SHAPE IS A DESIGN VARIABLE, FD REQUIRES 2 MESH GENERATIONS

DISCRETE SENSITIVITIES

- REQUIRES THE EXISTENCE OF THE DISCRETE SENSITIVITY
- IF SHAPE IS A DESIGN VARIABLE, THE DISCRETE SENSITIVITY LEADS TO MESH DERIVATIVES COMPUTATIONS

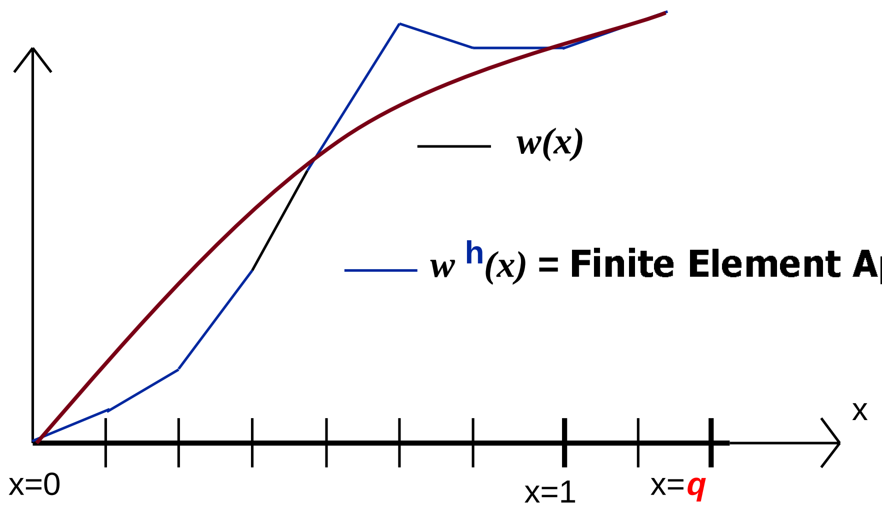
WHAT IS THE "CONTINUOUS / HYBRID" SENSITIVITY EQUATION METHOD? --- SEM

$$\frac{d}{dq}[\mathbf{J}^{\mathbf{h}}(q_0)] \approx \int_0^1 \langle w^{\mathbf{h}}(x,q_0) - \hat{w}(x), [\tfrac{\partial}{\partial q} w \ (x,q_0)]^{\mathbf{h},\,\mathbf{k}} \rangle dx$$

APPROXIMATE

A Sensitivity Equation Method

FOR q > 1 AND $h=q/(N+1)$ CONSIDER (FORMAL)



NUMERICAL APPROXIMATION

$$(\Sigma)^{h} \quad \frac{d^2}{dx^2} w^{h}(x) + \frac{1}{8}\left[\frac{d}{dx} w^{h}(x)\right]^3 = 0,\ 0 < x < q, \qquad w^{h}(0) = 0, \qquad w^{h}(q) = 4$$

DISCRETE STATE EQUATION

A Sensitivity Equation Method

$$(\Sigma)^h \quad \frac{d^2}{dx^2} w^h(x) + \frac{1}{8}\left[\frac{d}{dx} w^h(x)\right]^3 = 0,\ 0 < x < q, \qquad w^h(0) = 0, \quad w^h(q) = 4$$

$$(\delta\Sigma)^h \quad \frac{d^2}{dx^2} s(x) + \frac{3}{8}\left[\frac{d}{dx} w^h(x,q)\right]^2 \frac{d}{dx} s(x) = 0, \quad 0 < x < q,$$

$$s(0) = 0, \quad s(q) = -\frac{d}{dx} w^h(x)\Big|_{x=q}$$

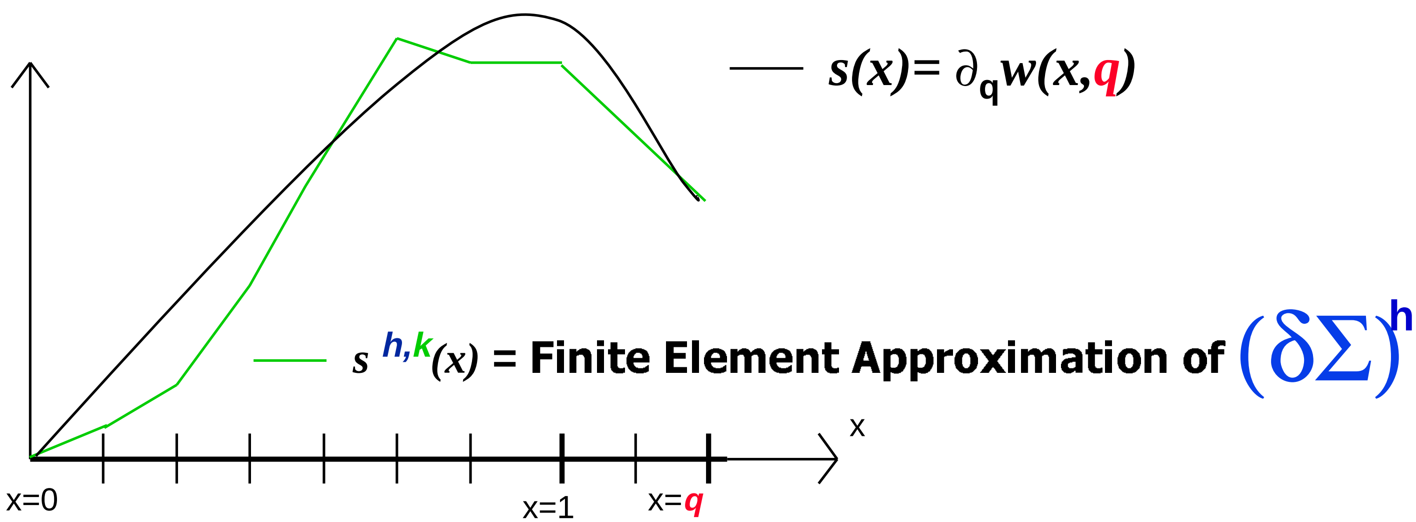
- **IMPORTANT OBSERVATIONS**
  - **The sensitivity equations are linear**
  - **The sensitivity equation "solver" can be constructed independently of the forward solver -- *SENSE™***
  - **When done correctly "mesh gradients" are not required**

A Sensitivity Equation Method

$$(\delta\Sigma)^h \qquad \frac{d^2}{dx^2}s(x)+\frac{3}{8}\left[\frac{d}{dx}w^h(x,q)\right]^2\frac{d}{dx}s(x)=0, \quad 0<x<q,$$

$$s(0)=0, \quad s(q)=-\frac{d}{dx}w^h(x)\Big|_{x=q}$$

FOR $q > 1$ **AND** $k = q/(M+1)$ **CONSIDER (FORMAL)**



2nd NUMERICAL APPROXIMATION → $s^{h,k}(x,q)=\left[\frac{\partial}{\partial q}w(x,q)\right]^{h,k}$

Convergence Issues

$$\frac{d}{dq}[\mathbf{J}^{\mathbf{h}}(q)] \approx \int_0^1 \langle w^{\mathbf{h}}(x,q) - \hat{w}(x), [\frac{\partial}{\partial q} w(x,q)]^{h,k} \rangle dx \overset{def}{=} \mathbf{G}(x,q,\mathbf{h},k)$$

THEOREM. The finite element scheme is asymptotically consistent.

$$\lim_{\substack{h \to 0 \\ k \to 0}} \left| \frac{d}{dq}[\mathbf{J}^{\mathbf{h}}(q)] - \mathbf{G}(x,q,\mathbf{h},k) \right| = 0$$

IDEA: When the error $\left| \frac{d}{dq}[\mathbf{J}^{\mathbf{h}}(q)] - \mathbf{G}(x,q,\mathbf{h},k) \right|$ **is small, then**

a trust region method should (might?) converge.

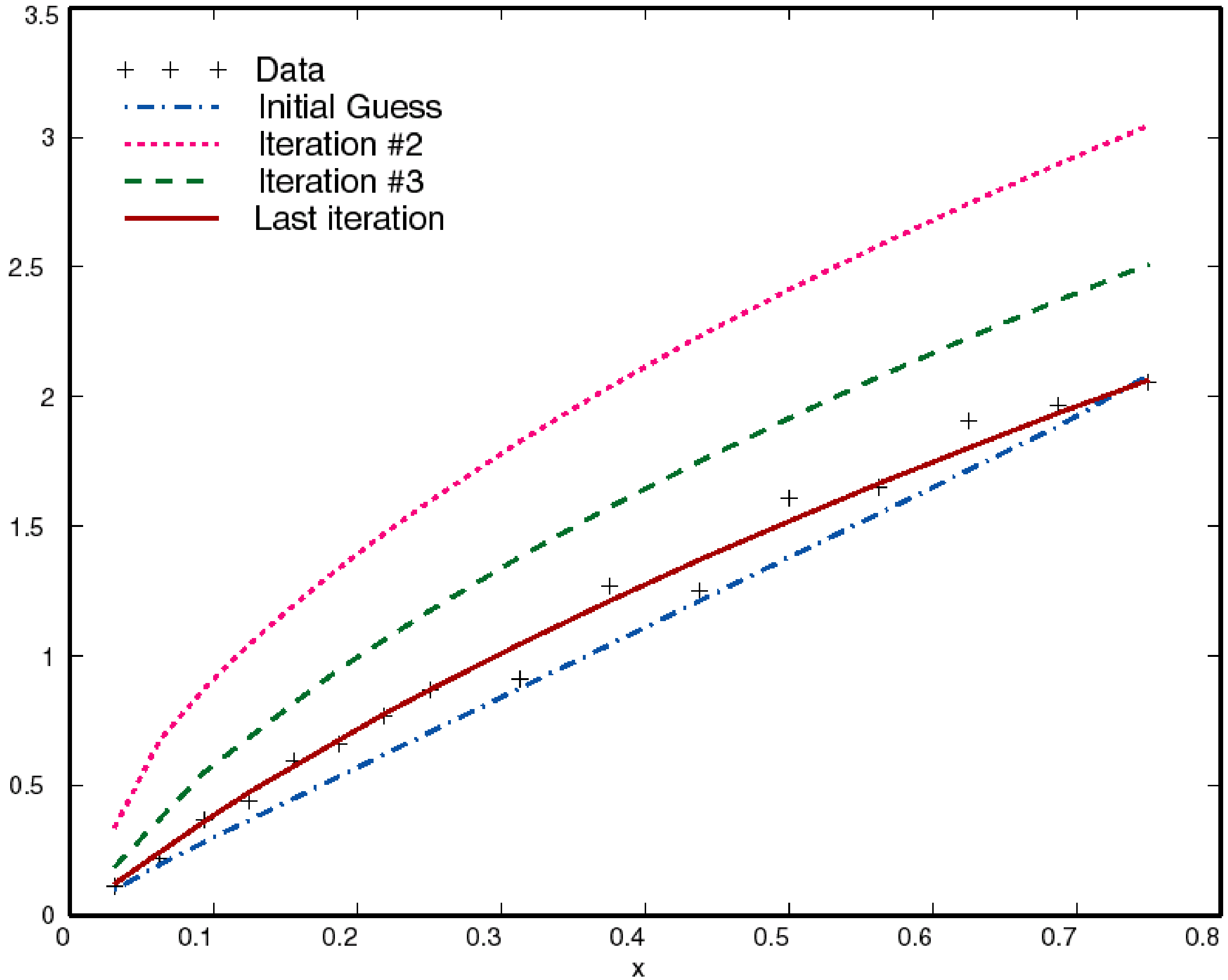
R. G. Carter, "On the Global Convergence of Trust-Region Algorithms Using Inexact Gradient Information", SIAM J. Num. Anal., Vol 28 (1991), 251-265.

J. T. Borggaard, "The Sensitivity Equation Method for Optimal Design", Ph.D. Thesis, Virginia Tech, Blacksburg, VA, 1995.

J. T. Borggaard and J. A. Burns, "A PDE Sensitivity Equation Method for Optimal Aerodynamic Design", Journal of Computational Physics, Vol.136 (1997), 366-384.

Convergence Issues

N=16, M=32



Convergence Issues

THE CASE k = h is often used, but may not be “good enough”

$$\left|\frac{d}{dq}[\mathbf{J}^{\mathbf{h}}(q)] - \mathbf{G}(x, q, \mathbf{h}, \mathbf{h})\right| \Rightarrow$$ **NOT CONVERGENT**

N=M=16		Tol = 0.00001	Tol = 0.0001	Total: 378.82		
Iter	q	Grad. Norm	Step	Time (secs)	Cost Time	Grad. Time
0	1.2000	4.3998E+00	-3.6427E-02	0.1231		
1	1.1636	3.1583E+03	1.4051E-03	31.3210	31.2697	0.0478
2	1.1650	3.0910E+03	-1.4339E-03	36.2310	36.1798	0.0480
3	1.1635	5.8909E+02	7.8372E-03	46.1160	46.0075	0.1043
4	1.1714	5.0139E+03	-8.8462E-04	45.3550	45.3006	0.0511
5	1.1705	2.9396E+03	-1.5052E-03	43.6720	43.6208	0.0470
6	1.1690	1.7238E+04	2.5880E-04	42.5810	42.5301	0.0468
7	1.1693	2.5888E+03	1.7342E-03	46.3470	46.2965	0.0472
8	1.1710	4.6995E+04	-9.4732E-05	44.1900	44.1396	0.0468
9	1.1709	1.5743E+02	0.0000E+00	42.8790	42.8265	0.0485

Timing Issues

THE CASE $k = 2h$ offers flexibility and $\left|\frac{d}{dq}[J^{h}(q)] - G(x, q, h, 2h)\right|$

convergence. But, what about timings?

N=16, M=32		Tol = 0.00001	Tol = 0.0001	Total: 39.81		
Iter	q	Grad. Norm	Step	Time (secs)	Cost Time	Grad. Time
0	1.2000	4.8489E+00	-3.2414E-02	0.1968		
1	1.1676	2.0720E+01	4.0347E-01	34.9270	34.8053	0.0911
2	1.5711	4.4544E+00	3.7808E-01	1.2613	1.1234	0.1075
3	1.9491	6.9846E-02	-1.2442E-02	0.9941	0.8714	0.0925
4	1.9367	1.5779E-02	2.7472E-03	0.4190	0.2907	0.0938
5	1.9394	3.3558E-03	-5.8723E-04	0.4095	0.2845	0.0943
6	1.9389	7.3235E-04	1.2801E-04	0.4525	0.3135	0.1083
7	1.9390	1.5892E-04	-2.7785E-05	0.8602	0.6327	0.1968
8	1.9390	3.0703E-05	0.0000E+00	0.2914	0.1451	0.1083

Approximately 96 .6% of cpu time spent in function evaluations
Approximately 02 .4% of cpu time spent in gradient evaluations

Timing Issues

N=32,M=64		Tol = 0.00001	Tol = 0.0001	Total: 96.91		
Iter	q	Grad. Norm	Step	Time (secs)	Cost Time	Grad Time
0	1.2000	4.9082E+00	-3.2679E-02	0.3508		
1	1.1673	1.8722E+01	5.0219E-01	82.9600	82.6775	0.1826
2	1.6695	2.6636E+00	2.9462E-01	3.5167	3.2308	0.1826
3	1.9641	1.7279E-01	-3.3035E-02	1.9769	1.6607	0.1964
4	1.9311	4.4619E-02	8.0364E-03	1.5274	1.1746	0.2383
5	1.9391	1.0217E-02	-1.8672E-03	1.0228	0.7004	0.2174
6	1.9373	2.4206E-03	4.4087E-04	1.1680	0.8792	0.1847
7	1.9377	5.6533E-04	-1.0305E-04	1.7608	1.3065	0.3556
8	1.9376	9.9844E-05	-1.8196E-05	1.6172	1.2212	0.2802
9	1.9376	1.7636E-05	0.0000E+00	1.0120	0.7050	0.2070

Approximately 96 .9% of cpu time spent in function evaluations
Approximately 02 .1% of cpu time spent in gradient evaluations

NOW FOR A “SERIOUS” PROBLEM

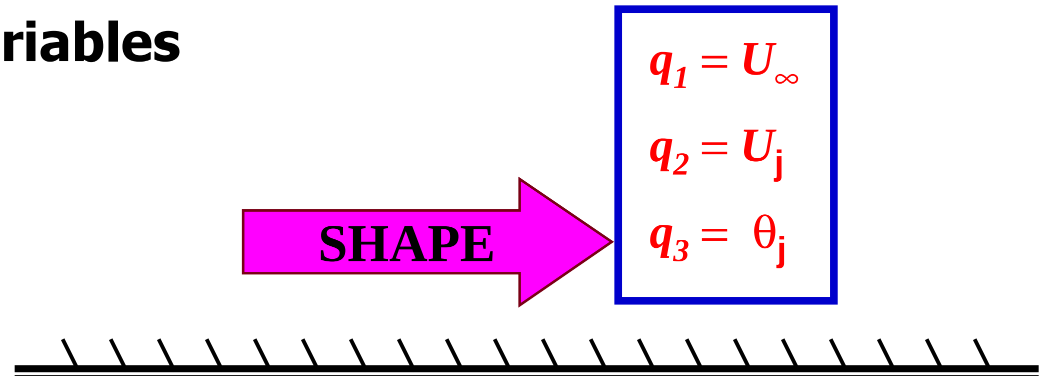
Injection Scramjet Design

- **Objective:**
  - Prioritization of Design / Control Variables
- **Free-stream & Design Variables**
  - **Free-stream: N_2 / O_2 mixture**

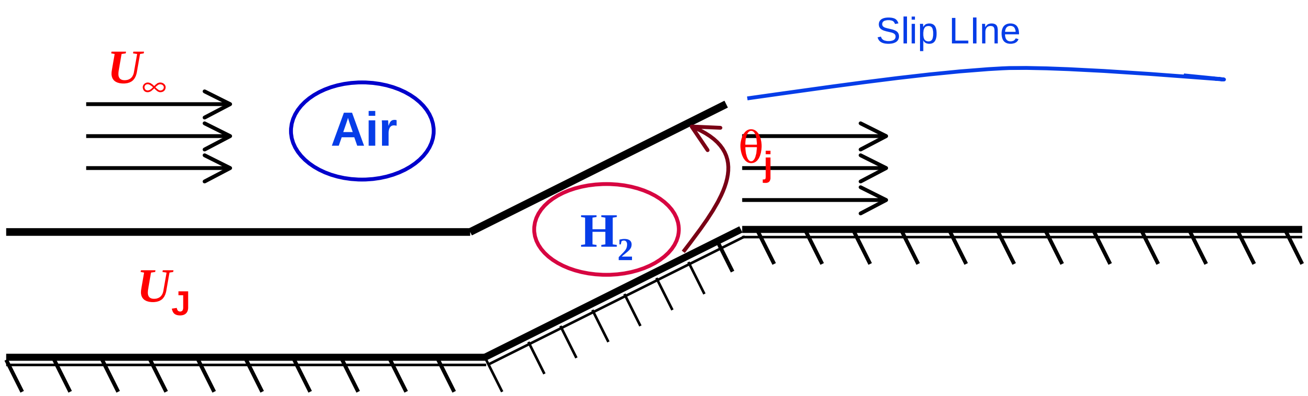
    M = 3, T = 800 K
  - **Injectant: H_2**

    M = 1.7, T = 291 K
  - **Momentum ratio = 1.7**

Design/Control Variables



Virginia Tech
(Gene Cliff)
&
AeroSoft, Inc.
(Andy Godfrey)



Finite Rate Chemistry 5 - Species

$$\vec{W}=\vec{W}(x,y)=[\vec{\rho}(x,y),\bar{\rho}u(x,y),\bar{\rho}v(x,y),\bar{\rho}e(x,y)]^T$$

WHERE

$$\vec{\rho}(x,y)=[\rho_{N2}(x,y),\rho_{O2}(x,y),\rho_{H2}(x,y),\rho_{H2O}(x,y),\rho_{NO}(x,y)]^T$$

$$(\Sigma) \quad \frac{\partial}{\partial x}F(\vec{W}(x,y))+\frac{\partial}{\partial y}G(\vec{W}(x,y))=0, \ \text{for } (x,y)\in\Omega(q)$$

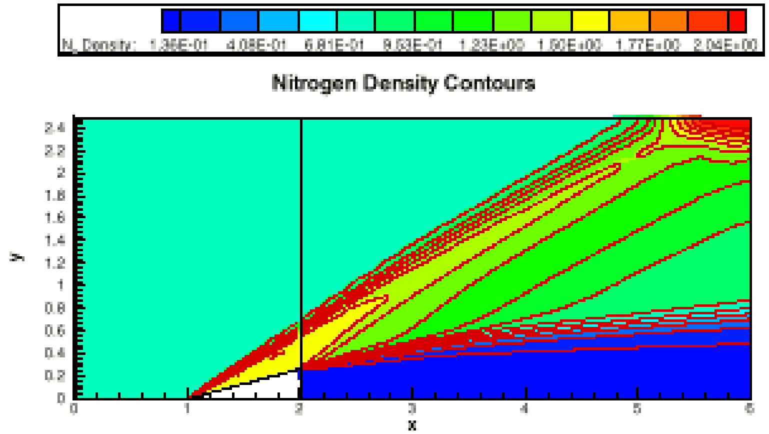
THE H_2O MASS-FRACTION DENSITY HAS THE FORM

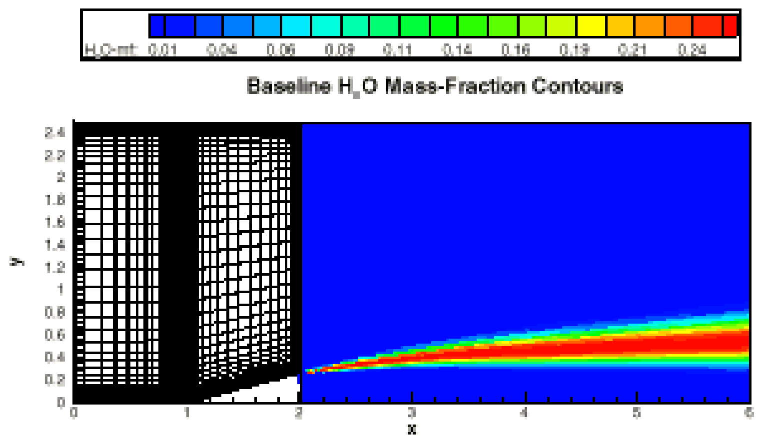
$$f_{H2O}(x,y,U_\infty,U_j,\theta_j)=f_{H2O}(x,y,q_1,q_2,q_3)$$

WANT → $\frac{\partial}{\partial q_i} f_{H2O}(x,y,q_1,q_2,q_3)$

N_2 and H_2O Contours

- Wedge Angle: 15 deg
- Shock Angle: 32 deg
- Flow Solver
  - *GASP™*
  - Marching
    - 2nd Order Upwind
    - 3rd Order
  - Converges 70 planes
  - 3 OM in 60-70 Iters/plane
  - Grid Sizes:
    - Zone 1: 41 x 57 x 2
    - Zone 2: 31 x 81 x 2

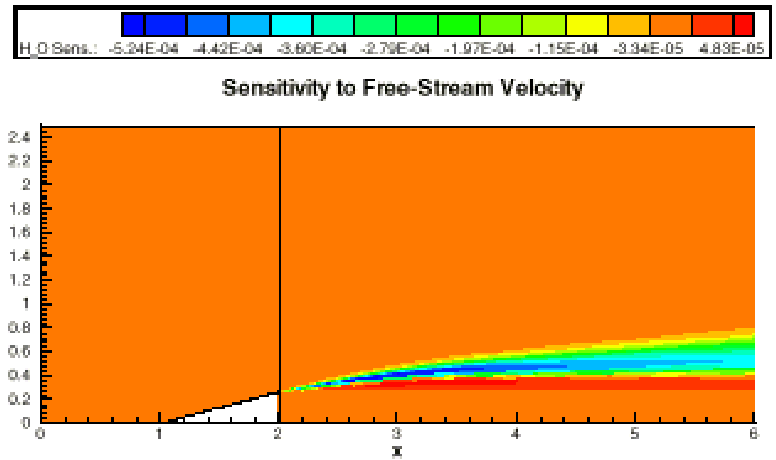




H_2O Mass-Fraction Sensitivities

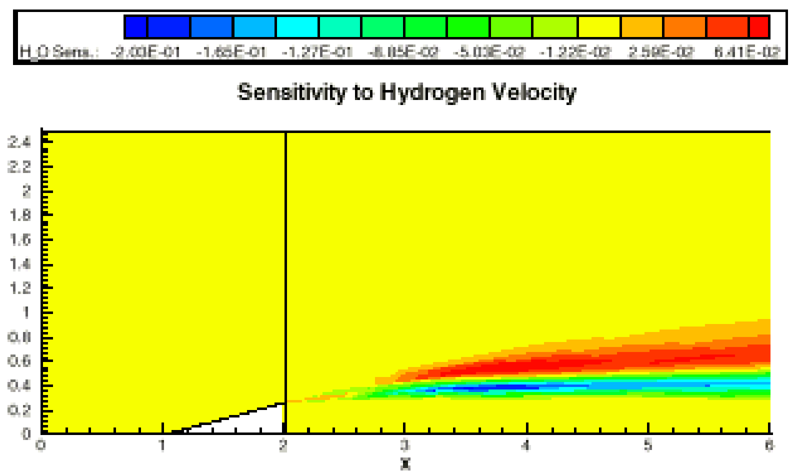
◆ **Sensitivity to $q_1 = U_\infty$**

- **Converges 14 OM in 4 Iterations**
- **H_2O mass fraction decreases**



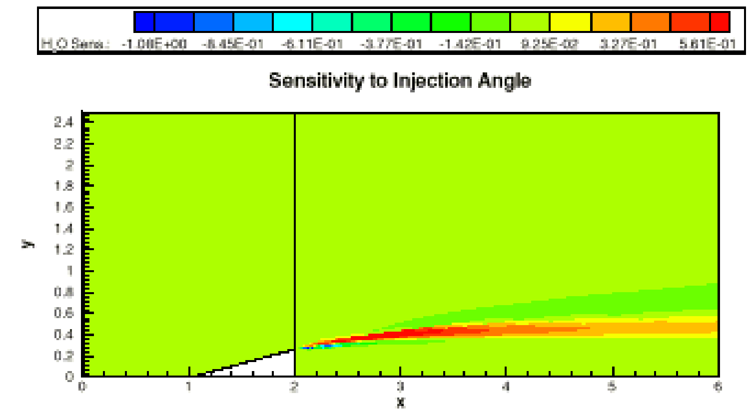
◆ **Sensitivity to $q_2 = U_j$**

- **Converges 13 OM in 5 iterations**
- **Slip line shifts up**

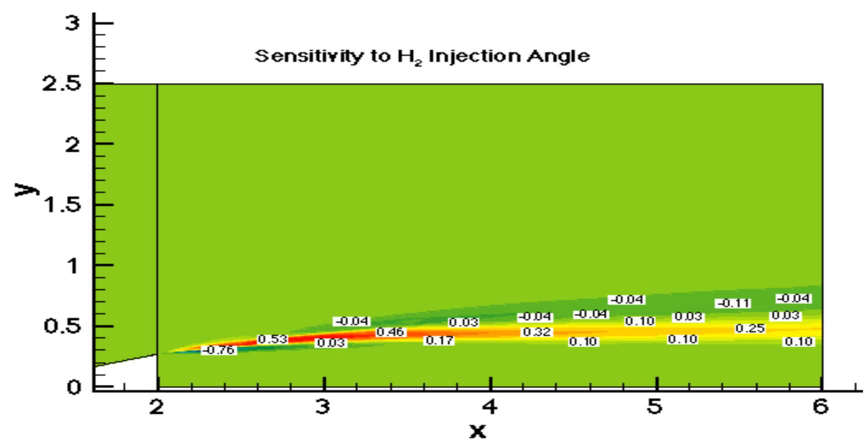


H_2O Mass Fraction Sensitivity

- Sensitivity to q_3 = θ_j
  - Converges 15 OM in 4 iterations



- Slip line shifts down



Mathematics Impacts “Practically”

- UNDERSTANDING THE PROPER MATHEMATICAL FRAMEWORK CAN BE EXPLOITED TO PRODUCE BETTER COMPUTATIONAL TOOLS

- A JET EXAMPLE WITH 20 DESIGN VARIABLES
  - PREVIOUS ENGINEERING METHODOLOGY REQUIRED 8400 CPU HRS ~ 1 YEAR
  - USING A HYBRID SEM DEVELOPED AT CODAC AS IMPLEMENTED IN *SENSE™* REDUCED THE DESIGN CYCLE TIME FROM ...

8400 CPU HRS ~ 1 YEAR TO 480 CPU HRS ~ 3 WEEKS

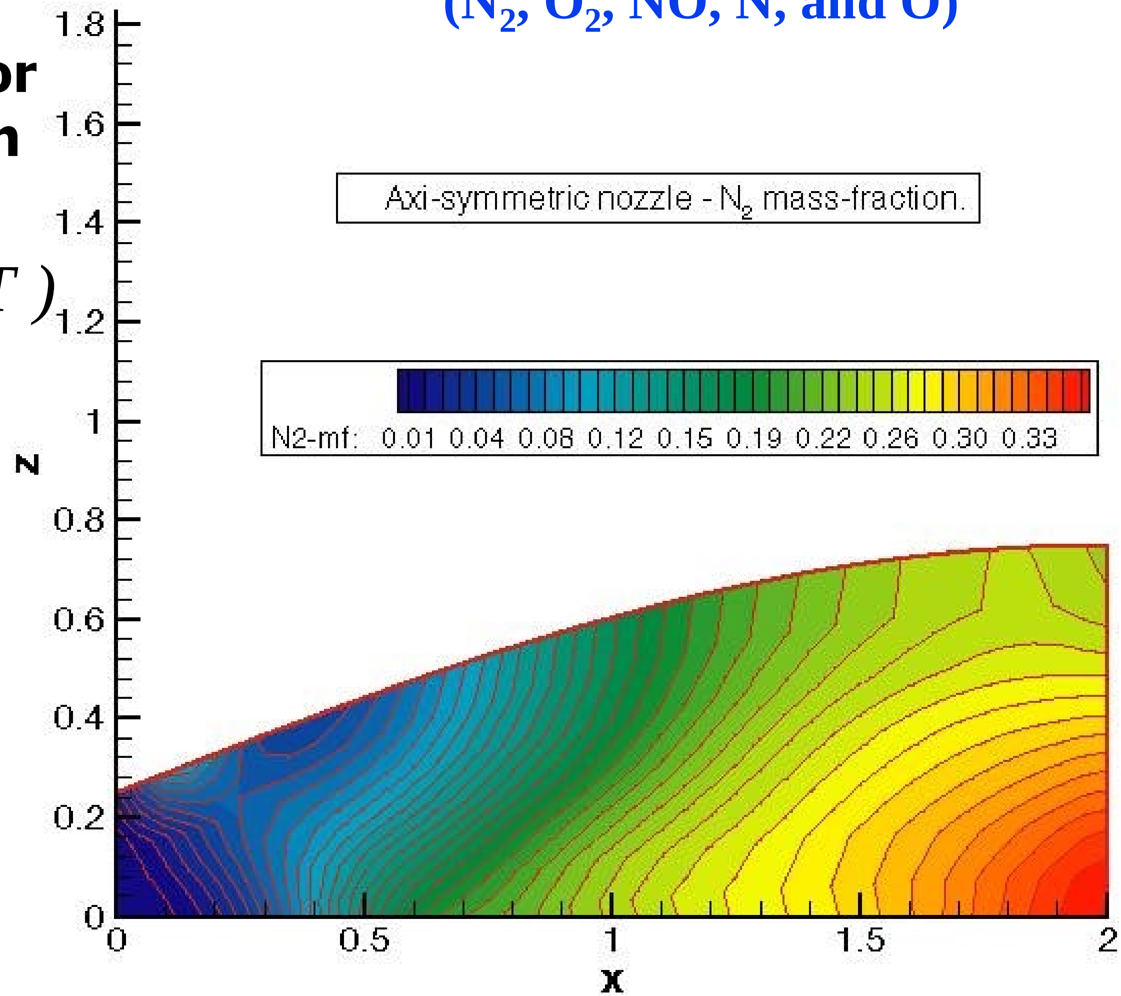
Another *SENSE*™ Application

- **Sensitivity to Modeling Coefficients**
- **Arrhenius Equation for the Forward Reaction Rate**

$$k_f = C_f T^{\eta} \exp(-\varepsilon_a / kT)$$

- **Sensitivity with Respect to η**
  - **For the Diffusion Reaction of N_2**
    $N_2 + C \rightarrow 2N + C$
    - Mach = 1.37
    - T = 9000 K
    - p = 101317 N/M^2
    - 5 species
    - 17 reactions

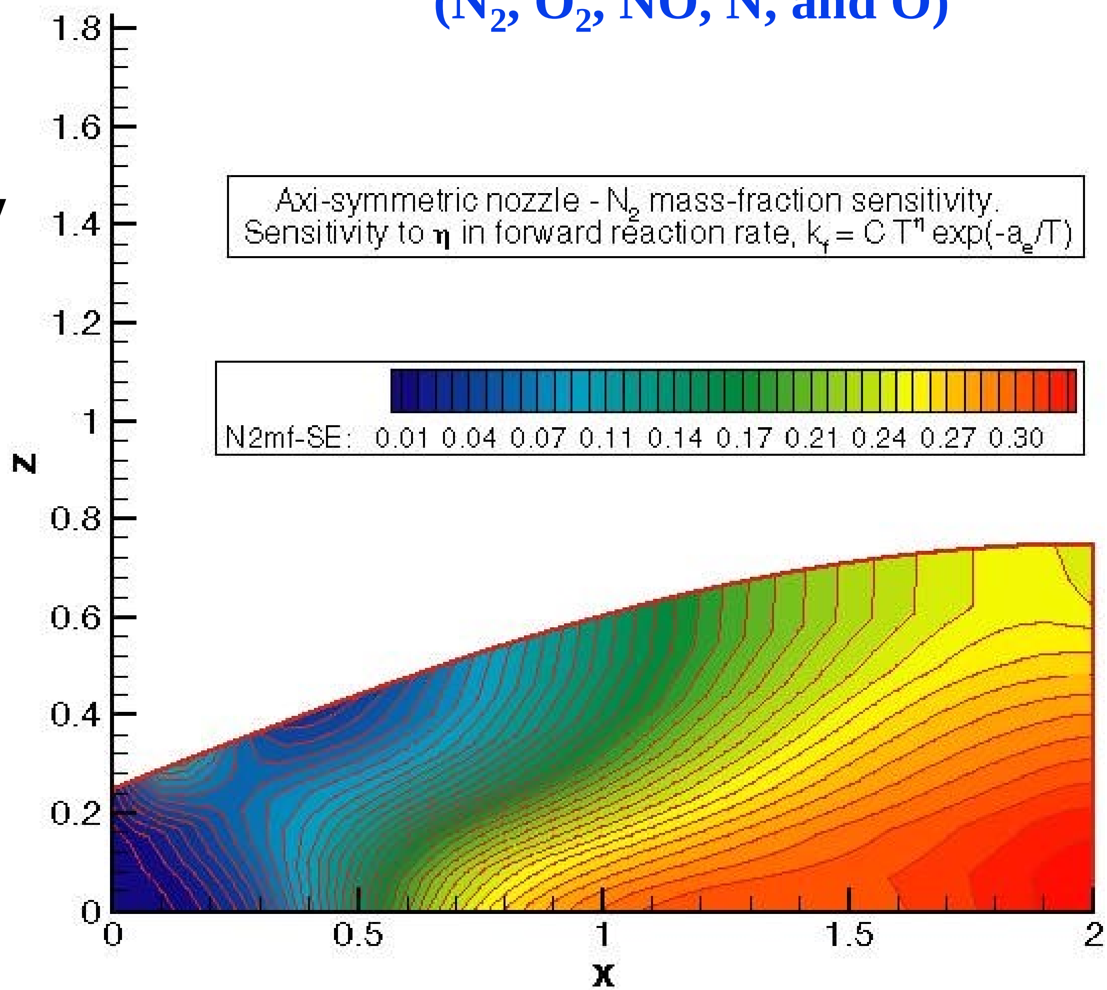
(N_2, O_2, NO, N, and O)



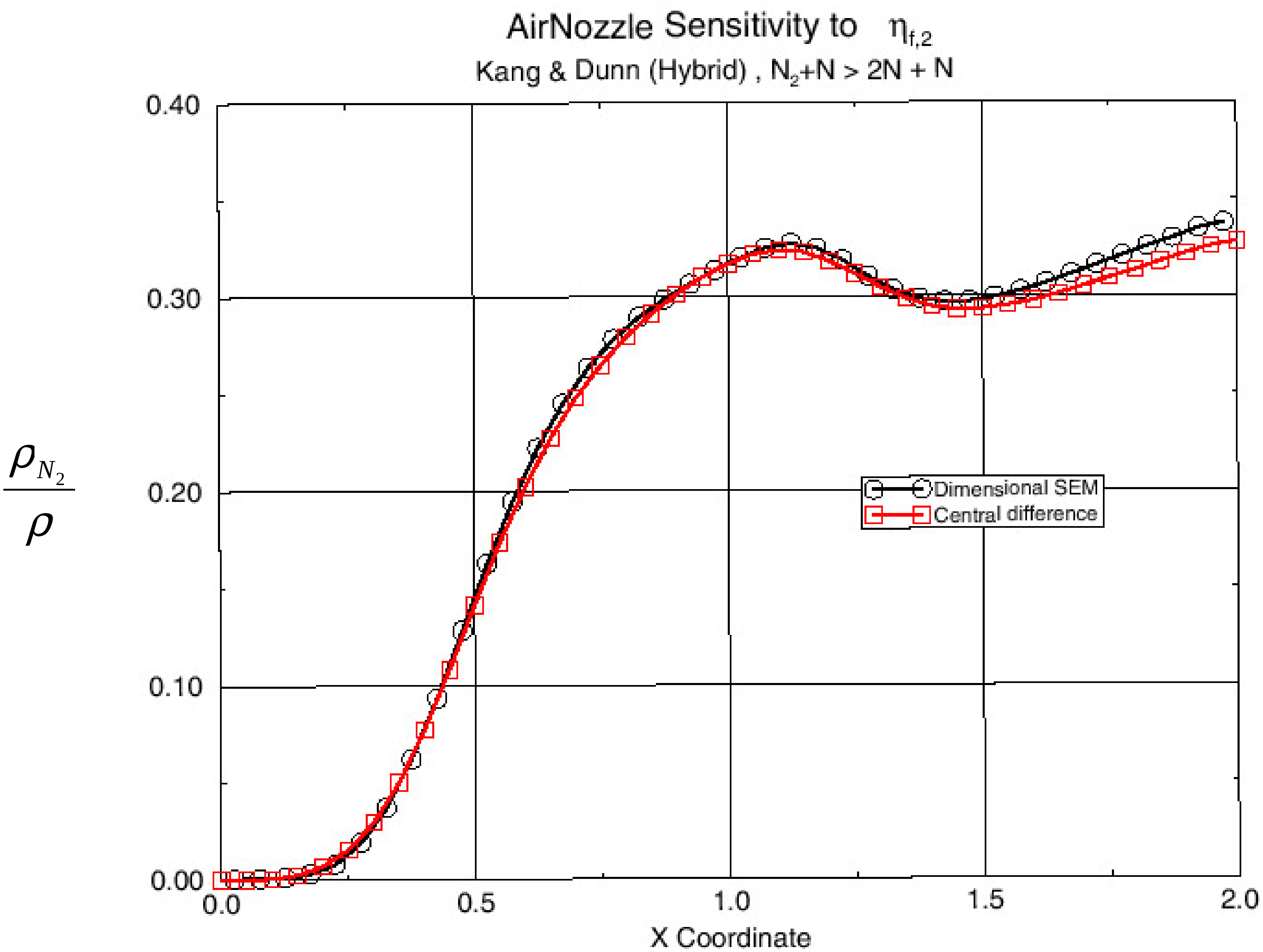
Another *SENSE*™ Application

- *GASP™* - Spaced-Marched Flow Solve
  - 41.7 sec on a Cray J90
- *SENSE™* - Sensitivity Solve
  - 2.89 sec on a Cray J90
- Compared to Central Difference in η
- Results are "Typical"

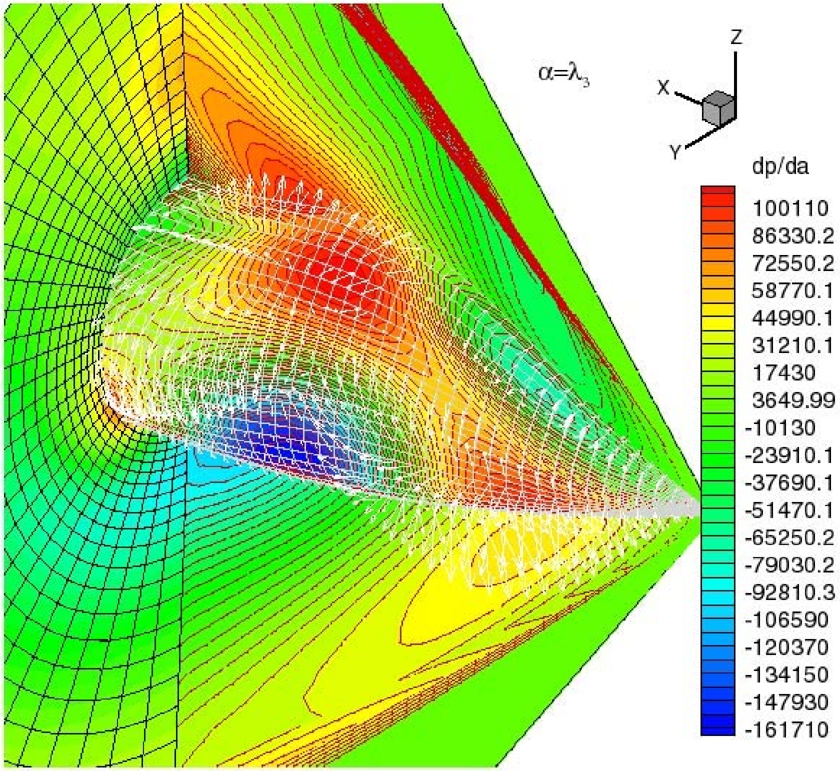
(N_2, O_2, NO, N, and O)



Another *SENSE*™ Application



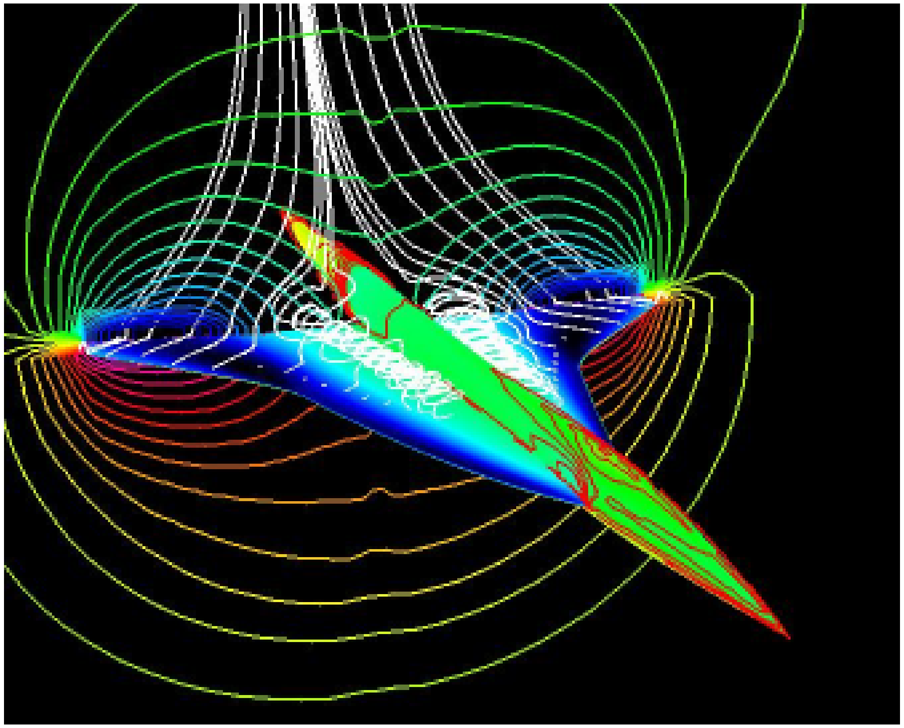
Other *SENSE*™ Applications



SENSITIVITIES FOR 3D SHAPE OPTIMIZATION

WITH ...

COMPLEX GEOMETRIES



General Comments

- **THERE ARE MANY VARIATIONS THAT CAN IMPROVE THE BASIC IDEA**
  - **COMBINING AUTOMATIC DIFFERENTIATION AND SEM**
  - **SMOOTHING AND GRADIENT PROJECTIONS**
  - **ADAPTIVE GRID GENERATION**
- **THE ORDER OF THINGS MATTER**
  - **DIFFERENTIATE-THEN-APPROXIMATE**
  - **DERIVE SENSITIVITY EQUATION BEFORE MAPPING TO A "COMPUTATIONAL DOMAIN"**
    - **DOES NOT REQUIRE MESH DERIVATIVES**
    - **REQUIRES A MORE SOPHISTICATED MATHEMATICAL FRAMEWORK**
    - **NEEDS A "DIFFERENT THEORY"**

J. A. Burns and L. G. Stanley, "A Note on the Use of Transformations in Sensitivity Computations for Elliptic Systems", Journal of Mathematical & Computer Modeling, in press.

Virginia Tech

Methods of Mappings

ICAM

$$w(x,q) = z(T(x,q),q) = z(\xi,q)$$

$$-\Delta_x w(x) = f(x)$$

$$-\Delta_x [\tfrac{\partial}{\partial q} w(x)] = 0$$

$\Omega(q)$

$\xi = T(x,q)$

$$-\Delta_\xi z(\xi) = g(\xi,q)$$

$$-\Delta_\xi [\tfrac{\partial}{\partial q} z(\xi)] = \tfrac{\partial}{\partial q} g(\xi,q)$$

$$-\Delta_\xi r(\xi) = 0$$

$$s(M(\xi,q)) = r(\xi,q)$$

"SOLVE"

$z^h(\xi)$

$[\tfrac{\partial}{\partial q} z]^h(\xi)$

$r^{h,k}(\xi)$

$x = M(\xi,q)$

$w^h(x)$

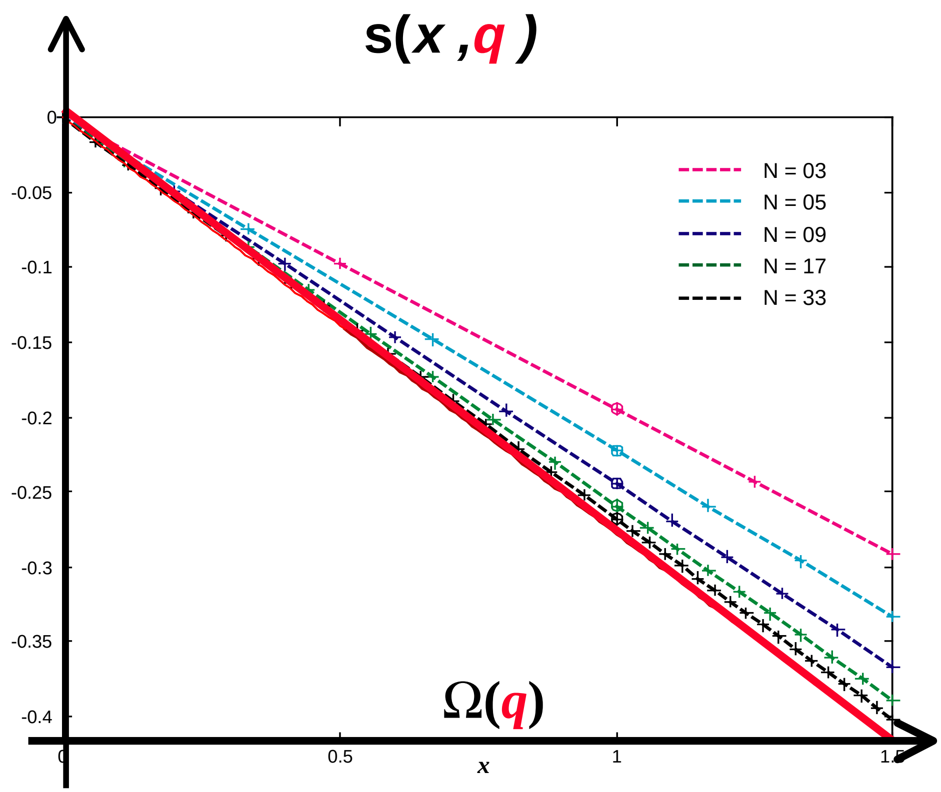
$[\tfrac{\partial}{\partial q} w]^h(x)$

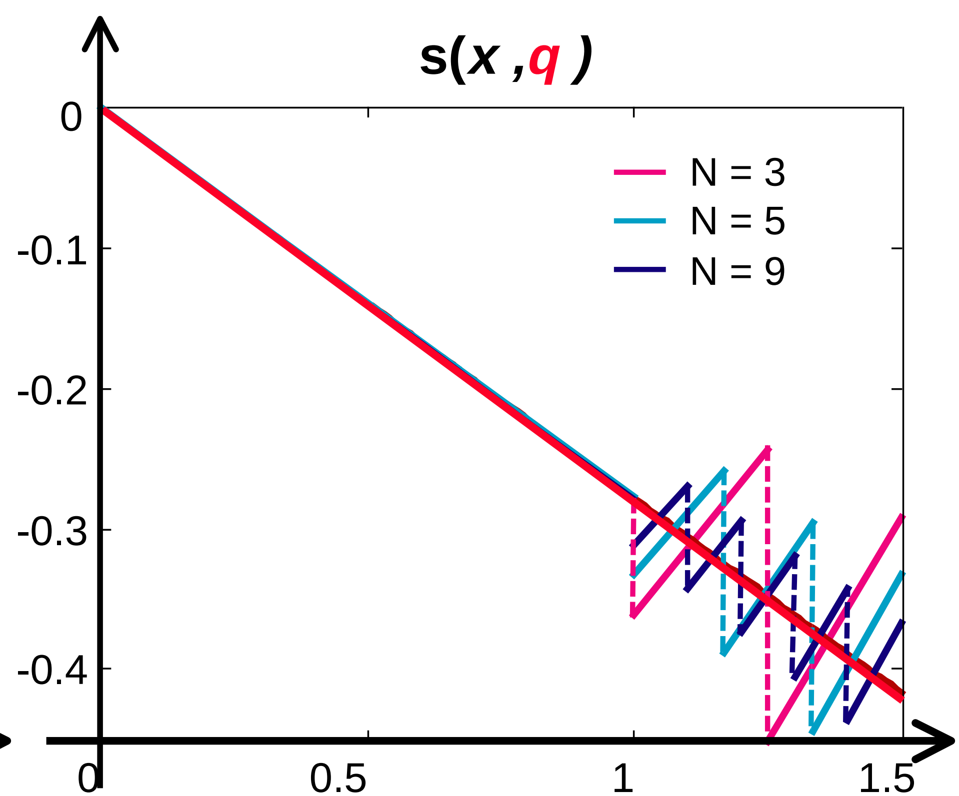
$[\tfrac{\partial}{\partial q} w]^{h,k}(\xi)$

Order Matters

DIFFERENTIATE THEN APPROXIMATE -- HYBRID SEM --

APPROXIMATE THEN DIFFERENTIATE -- "STANDARD" SEM --





Closing Remarks

- THIS IS NOT THE ONLY METHOD FOR OPTIMAL DESIGN
  - DERIVATIVE FREE METHODS
  - AUTOMATIC DIFFERENTIATION
  - ADJOINT METHODS
- OPTIMAL DESIGN IS NOT THE ONLY WAY TO DESIGN
- THIS IS NOT ALWAYS THE BEST WAY TO DO OPTIMAL DESIGN
- A GOOD THEORY CAN LEAD TO GREAT ALGORITHMS