Thermodynamically Valid Noise Models for Nonlinear Resistors

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ANALOG DEVICES

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Outline

- Introduction
- Thermodynamic Requirements
- Testing the Models:
  - Nyquist-Johnson Model
  - Nonlinear Gaussian Models
  - Poisson Models

Introduction

Goals of this research:

- Model exactly consistent with thermodynamics
- Device model, independent of load
- Valid even for large excursions caused by noise (small, but non-zero probability)

Thermodynamic Requirements

1. No isothermal conversion of heat to work

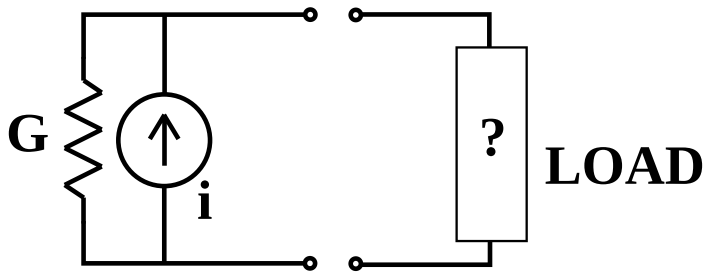
2. Gibbs distribution at equilibrium

3. Increasing entropy during transients (maximum entropy at equilibrium)

4. No heat transfer between two devices at the same temperature

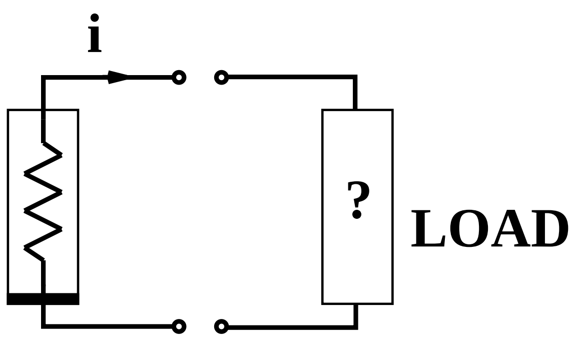
Device Noise Models

Linear:



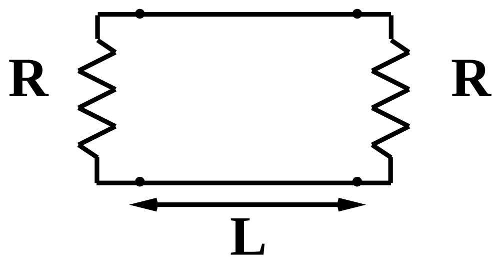
$S_{ii}(\omega) = 2kTG,$ for all frequencies ω

Nonlinear: ??



Question: what noise models are physically correct for a nonlinear resistor?

Nyquist's 4kTR noise (1928)

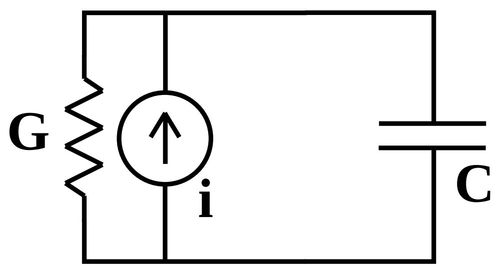


- energy of kT per mode
  (Equipartition Theorem)
- number of modes in $(f, f + df)$ is $2L\frac{df}{v}$
- energy transferred during time L/v
- power $= E/t = \frac{2L\frac{df}{v}kT}{L/v} = 2kTdf$
- power $= \frac{\Delta V^2}{2R}df$

$\Rightarrow \Delta V^2 df = 4kTR\,df$

a (linear) Fluctuation-Dissipation Theorem

Testing the Linear Resistor Noise Model



$$\frac{dq}{dt} = -G\frac{q}{C} + i(t)$$

TR1. Satisfied if $\overline{i(t)} = 0$ for all temperatures.

TR2. Equilibrium density?

Linear R Equilibrium Density

$$\frac{dq}{dt} = -G\frac{q}{C} + i(t)$$

$$q(t) = q(0)\, e^{-tG/C} + \int_0^t e^{-(t-\tau)G/C}\, i(\tau)\, d\tau$$

Could calculate many sample paths, average results (Monte-Carlo method)

OR could use Fokker-Planck Equation for probability density

$$\frac{\partial \rho(t,q)}{\partial t} = \frac{\partial}{\partial q}\left[\frac{qG}{C}\rho(t,q)\right] + kTG\frac{\partial^2 \rho(t,q)}{\partial q^2}$$

Thermodynamic Requirements for a Linear Resistor

At equilibrium,

$$\dot{\rho}_{eq} = 0 = \frac{\partial}{\partial q}\left[\frac{qG\rho_{eq}}{C} + kTG\frac{d\rho_{eq}}{dq}\right]$$

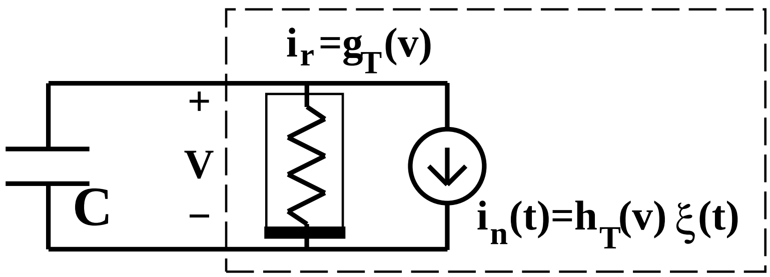
$$\Rightarrow \rho_{eq}(q) = N \exp\left(\frac{-q^2}{2CkT}\right) = N \exp\left(\frac{-E_C(q)}{kT}\right)$$

Equilibrium density is correct! (TR2)

(TR3 Entropy also increases during transients.)

(TR4 Heat transfer correct [Wyatt,Siebert,Tan 1984].)

Nonlinear Gaussian Models



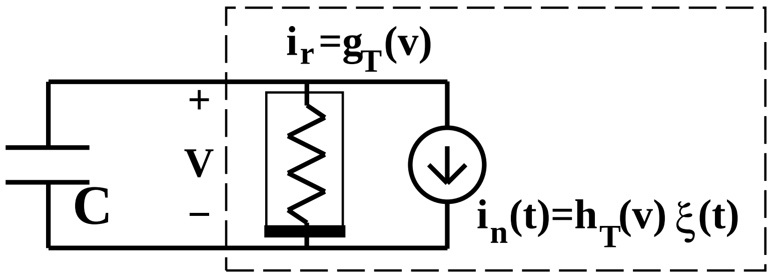
$$C\frac{dv}{dt} = -g_T(v) - h_T(v)w(t)$$

Itô FPE:

$$\frac{\partial \rho(t,v)}{\partial t} = \frac{\partial}{\partial v}\Big[g_T(v)\rho(t,v)\Big] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\Big[h_T^2(v)\rho(t,v)\Big]$$

TR1 satisfied for zero-mean Gaussian noise $w(t)$.

Nonlinear Gaussian Models



Equilibrium test simplifies to:

$$\frac{\partial h^2(v,T)}{\partial v} = C\left[\frac{v}{kT}h^2(v,T) - 2g(v)\right]$$

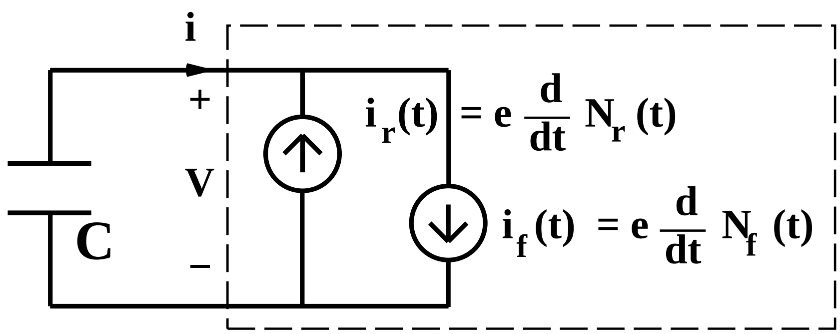
but LHS must be independent of C, so both sides are zero

LHS: $\Rightarrow h$ independent of v

RHS: $h^2 = \frac{2g(v)kT}{v} = 2kTG$

Nonlinear Gaussian model cannot satisfy TR2 regardless of form of $h_T(v)$.

Poisson Models

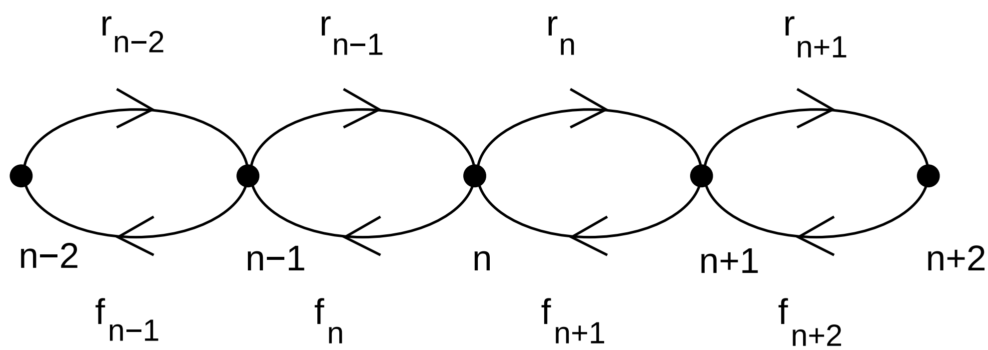


$$
\begin{aligned}
i(t) \;&=\; \frac{d}{dt}\Big\{ eN_f\left(\int_0^t f_T(v(\tau))\,d\tau\right) \\
&\qquad -eN_r\left(\int_0^t r_T(v(\tau))\,d\tau\right)\Big\}
\end{aligned}
$$

constitutive relation $\overline{i(v)} = e\,[f_T(v) - r_T(v)]$
TR1 satisfied if device is passive

power spectral density (fixed V)
$S_{ii}(\omega; T, V) = e^2\,[f_T(V) + r_T(V)]\,, \quad \forall \omega \neq 0$

Poisson Models, con't



Gibbs distribution:

$$\frac{p_{n+1}}{p_n} = \frac{\exp\left[-\frac{(n+1)^2e^2}{2CkT}\right]}{\exp\left[-\frac{n^2e^2}{2CkT}\right]} = \exp\left[-\frac{(n+1/2)e^2}{CkT}\right]$$

Detailed balance (reversibility)

$$p_n\ r_n = p_{n+1}\ f_{n+1} \Rightarrow \frac{p_{n+1}}{p_n} = \frac{r_T(v_\beta(ne))}{f_T(v_\alpha((n+1)e))}$$

Poisson Models, con't

Suppose $\overline{i(v)} = I_S\,[\exp(v/v_T) - 1]$

$f_T(v) = I_S\ \exp(v/v_T) \qquad r_T(v) = I_S$

$$\begin{aligned} \exp\left[-\frac{(n+1/2)e}{C\,v_T}\right] &= \frac{r_T(v_\beta(ne))}{f_T(v_\alpha((n+1)e))} \\ &= \exp\left(-\frac{v_\alpha((n+1)e)}{v_T}\right) \end{aligned}$$

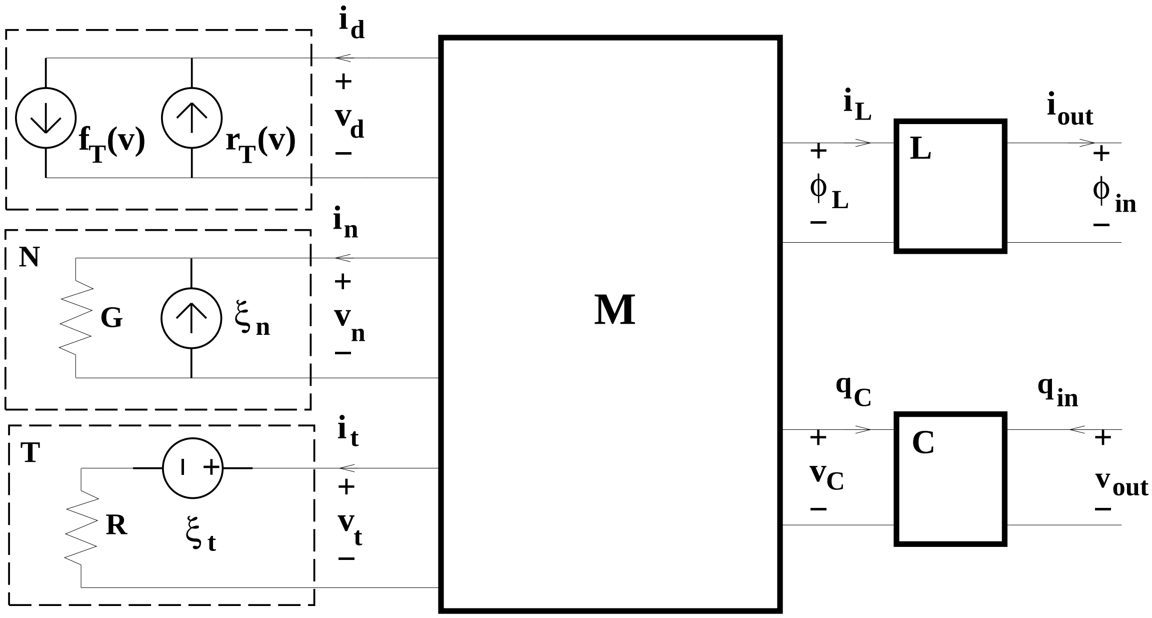
$$\Rightarrow v_\alpha((n+1)e) = \frac{(n+1/2)e}{C}$$

Similarly, $v_\beta(ne) = \frac{(n+1/2)e}{C}$

TR2 satisfied iff

$$\frac{f_T(v)}{r_T(v)} = \exp\left(v/v_T\right), \qquad \text{for all } v$$

Poisson and Gaussian together



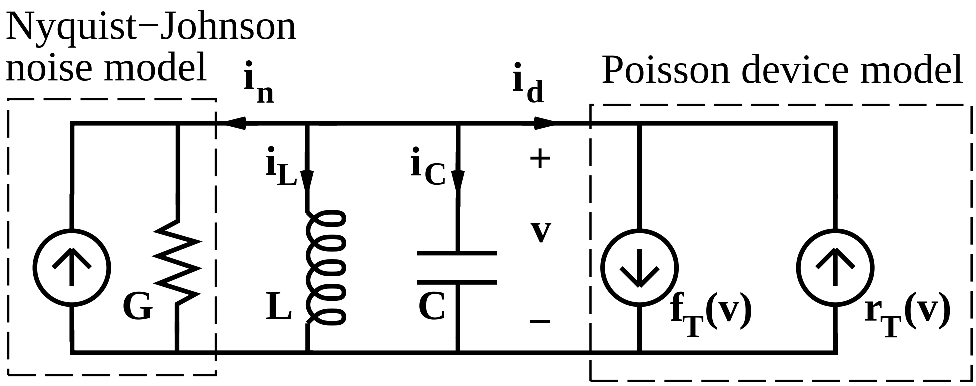
Multiport inductor L, capacitor C

Interconnection box M is linear, memoryless, lossless

Poisson model and Linear Gaussian model (Nyquist-Johnson) Norton or Thévenin form

... TR1-3 satisfied.

Examples



$$\begin{bmatrix} \mathbf{i}_C \\ \mathbf{v}_L \\ \mathbf{v}_d \\ \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \\ \mathbf{i}_d \\ \mathbf{i}_n \end{bmatrix}$$

$$\rho_{eq} \propto \exp\left[-\frac{q^2}{2CkT} - \frac{\phi^2}{2LkT}\right]$$

Nonlinear:

$$\rho_{eq} \propto \exp\left[-\frac{1}{kT}\int_0^q h(\tilde{q})\, d\tilde{q} - \frac{1}{kT}\int_0^\phi g(\tilde{\phi})\, d\tilde{\phi}\right]$$

Examples, con't

$$
\begin{aligned}
&\frac{d}{dt}\rho(t,q,\phi) = \\
&\quad \frac{\partial}{\partial q}\left[\left(g(\phi)+\frac{h(q)}{R}\right)\rho(t,q,\phi)\right] \\
&\quad -h(q)\frac{\partial}{\partial \phi}[\rho(t,q,\phi)] + \frac{\partial^2}{\partial q^2}\left[\frac{kT}{R}\,\rho(t,q,\phi)\right] \\
&\quad + f_T(v_\alpha(q+e))\;\rho(t,q+e,\phi) \\
&\quad + r_T(v_\beta(q-e))\;\rho(t,q-e,\phi) \\
&\quad - \left[f_T(v_\alpha(q)) + r_T(v_\beta(q))\right]\;\rho(t,q,\phi)
\end{aligned}
$$

$$
v_\alpha(q) \;=\; v_\beta(q-e) \;=\; \frac{1}{e}\int_{q-e}^{q} h(\tilde{q})\,d\tilde{q}
$$

$$
f_T(v_\alpha(q)) \;=\; r_T(v_\alpha(q))\;\exp\left[\frac{e}{kT}\,v_\alpha(q)\right]
$$

Examples, con't

Capacitor entropy:

$$\frac{d}{dt}S_{LC} = -k\frac{d}{dt}\iint \rho(t,q,\phi)\log\rho(t,q,\phi)\,dq\,d\phi$$

$$= -k\iint \dot{\rho}(t,q,\phi)\log\rho(t,q,\phi)\,dq\,d\phi$$

Reservoir entropy:

$$\dot{S}_R = -\frac{1}{T}\iint E_{LC}(q,\phi)\,\dot{\rho}(t,q,\phi)\,dq\,d\phi.$$

Total entropy:

$$\dot{S}_{tot} = -\iint\left[\frac{1}{T}E_{LC}(q,\phi) + k\,\log\rho(t,q,\phi)\right]$$

$$\times\quad \dot{\rho}(t,q,\phi)\,dq\,d\phi$$

Examples, con't

After integration by parts and changes of variables,

$$\dot{S}_{tot} =$$

$$\iint \Big[\, k\, r_T(v_\beta(q))\Big]$$

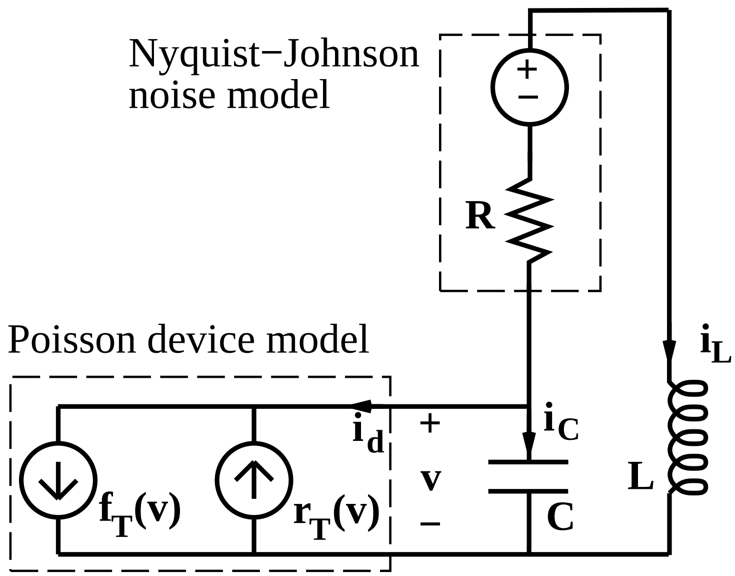
$$\times \left[\rho(t, q+e, \phi)\ \exp\left(\frac{v_\beta(q)}{v_T}\right) - \rho(t, q, \phi)\right]$$

$$\times \left[\log \rho(t, q+e, \phi) + \left(\frac{v_\beta(q)}{v_T}\right) - \log \rho(t, q, \phi)\right] dq\ d\phi$$

$$+ \iint \frac{G}{T\ \rho(t, q, \phi)} \left[\frac{q}{C}\, \rho(t, q, \phi) + kT \frac{\partial \rho(t, q, \phi)}{\partial q}\right]^2 dq\ d\phi$$

≥ 0

Examples, con't

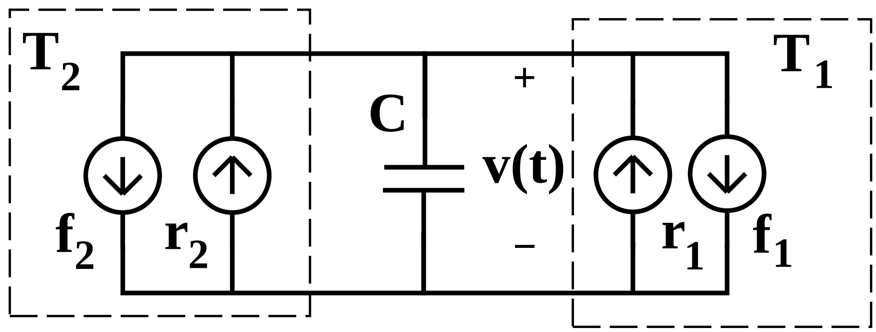


$$\begin{bmatrix} \mathbf{i}_C \\ \mathbf{v}_L \\ \mathbf{v}_d \\ \mathbf{i}_t \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \\ \mathbf{i}_d \\ \mathbf{v}_t \end{bmatrix}$$

Heat Transfer

Heat transfer must go from hotter device to colder, regardless of nonlinearity

Heat Transfer: Two Poisson Devices



Steady-state:

$$\frac{\rho_{ss}(q+e)}{\rho_{ss}(q)}$$

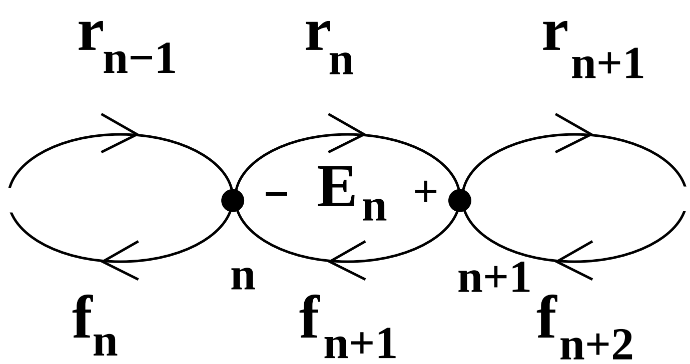
$$= \frac{r_1(v_\beta(q)) + r_2(v_\beta(q))}{r_1(v_\beta(q))\exp\left(\frac{ev_\beta(q)}{kT_1}\right) + r_2(v_\beta(q))\exp\left(\frac{ev_\beta(q)}{kT_2}\right)}$$

$$= \exp\left(-\frac{ev_\beta(q)}{kT_1}\right)$$

$$\times \frac{r_1(v_\beta(q)) + r_2(v_\beta(q))}{r_1(v_\beta(q)) + r_2(v_\beta(q))\exp\left[\frac{ev_\beta(q)}{k}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]}$$

$$\frac{f_1(v_\alpha(ne+e))}{r_1(v_\beta(ne))} = \exp\left(\frac{ev_\beta(q)}{kT_1}\right)$$

Heat Transfer, con't



$\rho_n^o\, r_n =$ steady-state rate at which transitions occur from state n to $n+1$

$\rho_n^o\, r_n\, E_n =$ steady-state rate at which the energy E_n is delivered by the reverse source for transitions from n to $n+1$

$\sum_n \rho_n^o\, r_n\, E_n =$ steady-state power delivered by the reverse source on all transitions

Heat Transfer, con't

Power supplied by device 1

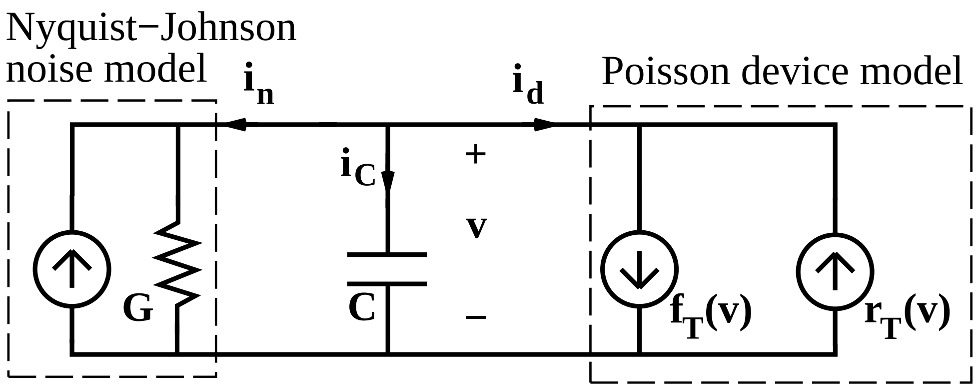
$$= \frac{e^2}{2C} \sum_n \rho_{ss}(ne)\, r_1(n)\, (2n+1)$$
$$\times \left[1 - \frac{r_1(n) + r_2(n)}{r_1(n) + r_2(n) EXP}\right]$$

$$EXP = \exp\left[\frac{e^2(n+1/2)}{Ck}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]$$

Power supplied $= 0$ when $T_1 = T_2$

> 0 when $T_1 > T_2$

Heat Transfer, con't



For $T_1 = T_2$, system is at equilibrium

$$\rho = A \exp\left(\frac{q^2}{2CkT}\right)$$

independent of noise models.

Since noise models themselves do not supply net heat, they are oblivious to the other.

No closed-form steady-state distribution for $T_1 \neq T_2$.

Conclusions

Four thermodynamic requirements for device noise models, presented as simple mathematical tests.

Usual nonlinear Gaussian model fails.

Poisson model works when $\frac{f_T(v)}{r_T(v)} = \exp(v/v_T)$

$\Rightarrow$ a nonlinear fluctuation-dissipation theorem

Accomplishment: the first nonlinear device noise model consistent with a variety of thermodynamic tests.

Questions

What devices may be described by the Poisson model? (Mesoscopic physics results: carrier mean free path longer than the device.)

What other noise models can we test?

Does this give any insight into the physical processes actually producing the noise?