Possibilities and Limitations of Tomography

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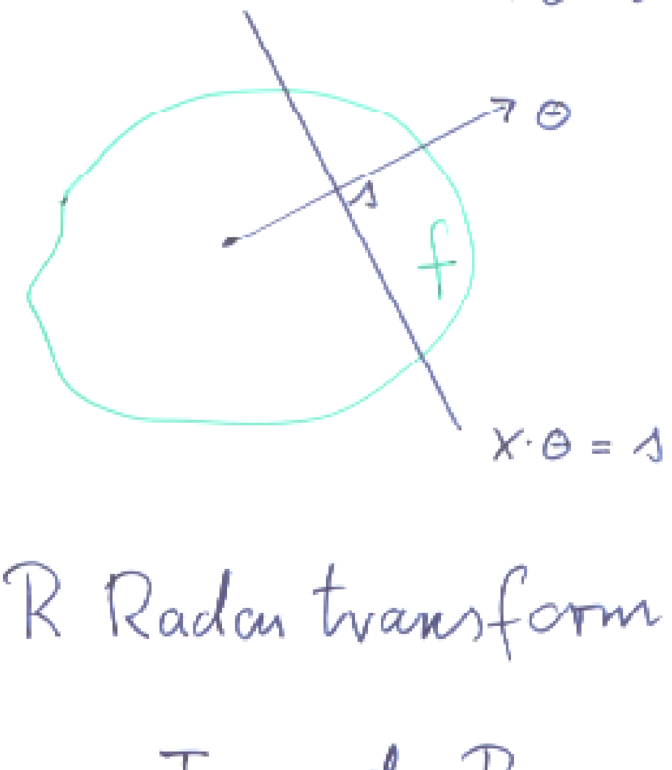
Tomography (narrow sense):

Reconstruction of a function from its line integrals.

Simplest case (2D):

$$(Rf)(\theta, s) = \int_{x\cdot\theta = s} f(x)\, dx$$

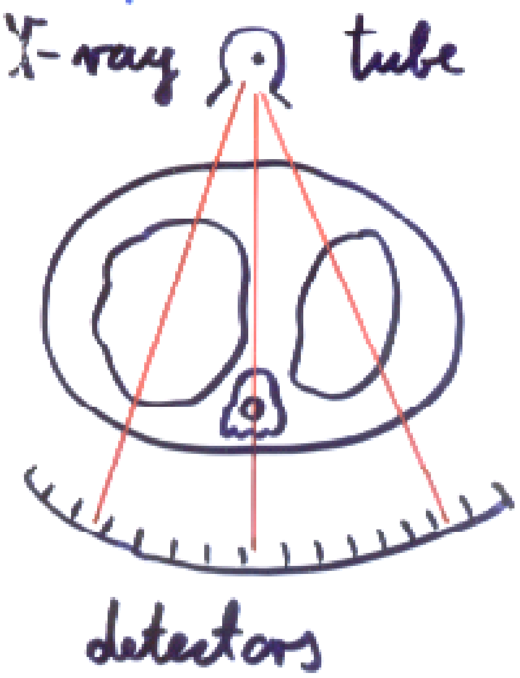


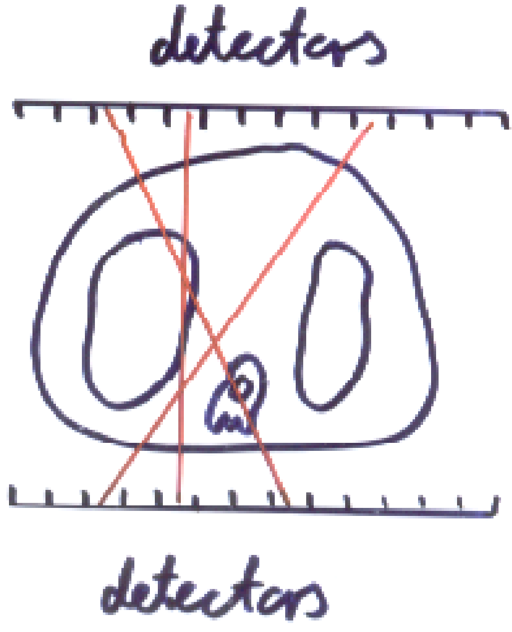
R Radon transform

Problem: Invert R

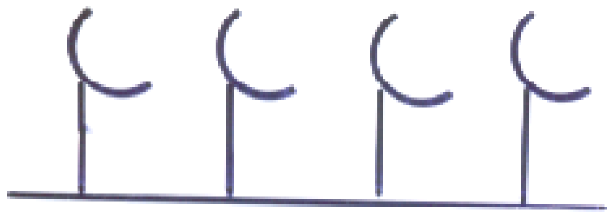
(given a sampled and noisy $g = Rf$)

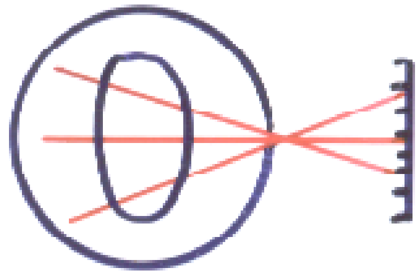
Applications:



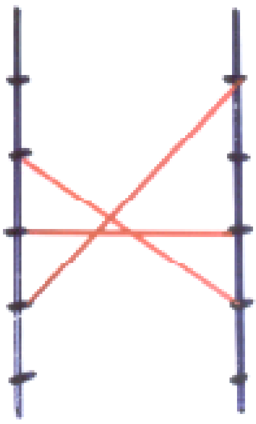


- Transmission tomography
- Emission





- Radio astronomy
- Plasma diagnostics



- Electron microscopy
- Radar
- Nondestructive testing
- ⋮

- Seismology

Tomography (wide sense):

Imaging techniques only remotely related to the straight line paradigm:

- Impedance tomography
- Optical tomography
- Ultrasound tomography
- Microwave tomography
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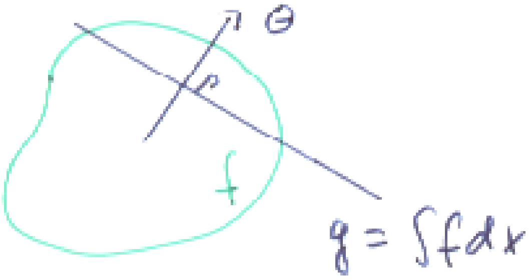
Inverse Problems of PDE's,

Nonlinear

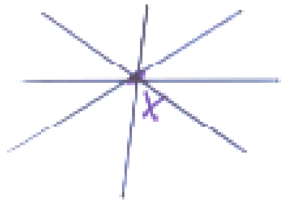
Reconstruction in 2D:



$$f(x) = -\frac{1}{4\pi}\int_{S^1} (Hg')(\theta, x\cdot\theta)\,d\theta$$

$$(Hg)(s) = \frac{1}{\pi}\int_{\mathbb{R}^1} \frac{g(t)}{s-t}\,dt$$

Conclusions: Inversion is

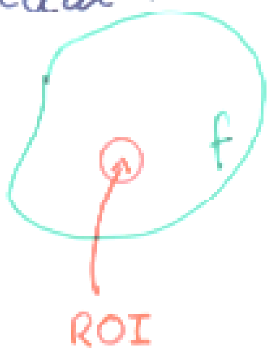


- (Slightly) unstable:

  $$g_\varepsilon - g \sim \varepsilon \implies f_\varepsilon - f \sim \varepsilon^{\alpha/(\alpha+1/2)}$$

  α smoothness class of f

- Not local:



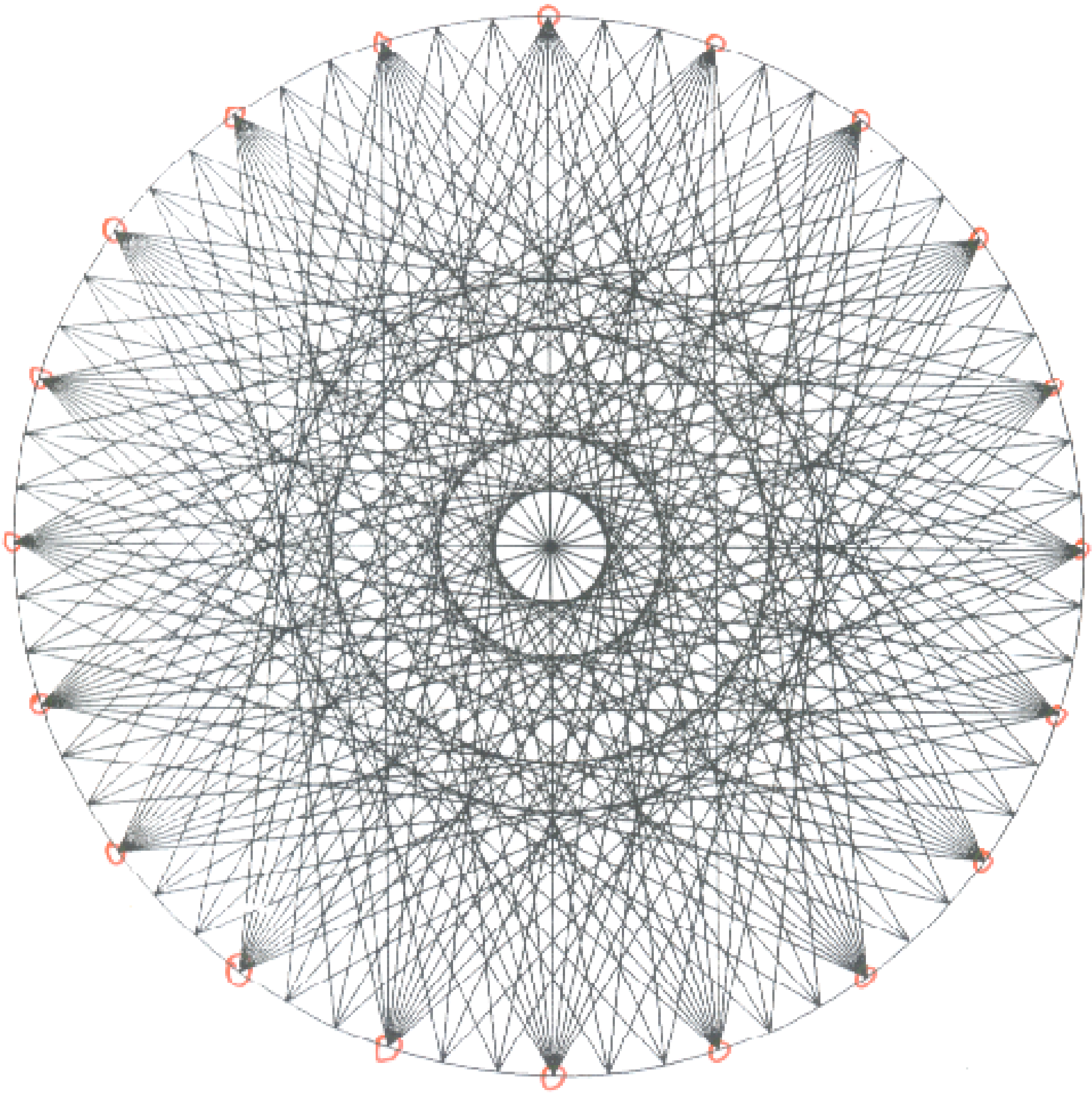
  Need all line integrals, not only those hitting region of interest (ROI)

- Resolution: p directions, q lines each
  f supported in circle of radius ρ
  f (essential) bandwidth Ω

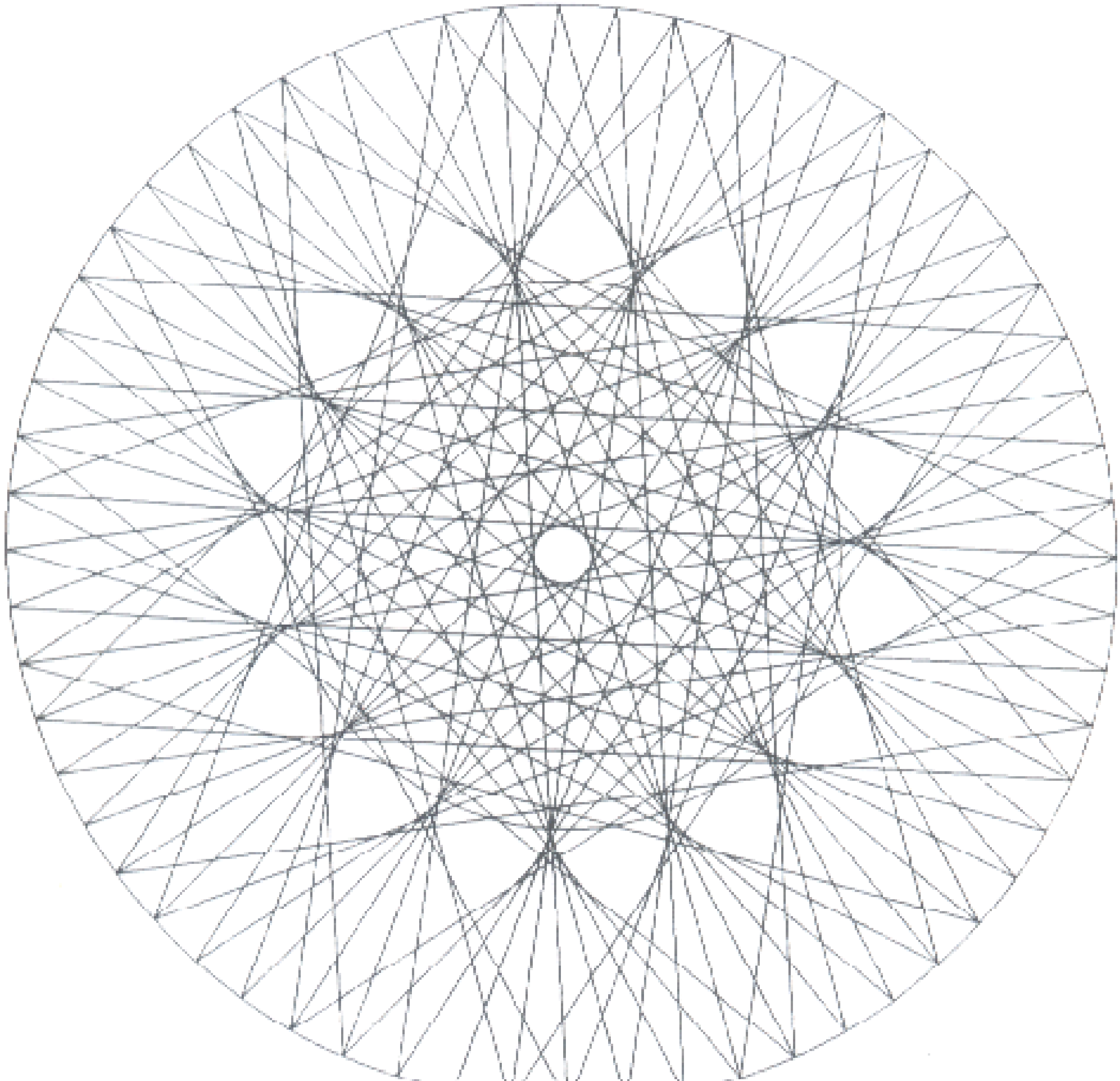
  $$p \geq \rho\Omega, \quad q \geq \frac{2}{\pi}\rho\Omega.$$

data $= \frac{2}{\pi}(\rho\Omega)^2$

not best possible!



Standard fan beam
scheme
for $\Omega = 15$,
$\rho = 1$, $r = 2$.
217 lines



Efficient fan beam scheme

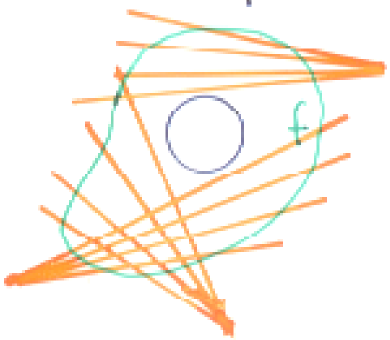
for $\Omega - 15$,

$\mu - 1$, $r - 2$.

75 lines

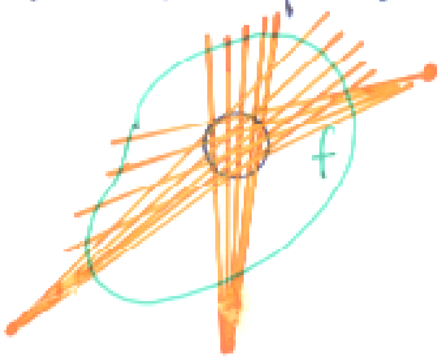
Incomplete data:

- Exterior problem: Only integrals outside $|x| \le a$ measured.



Uniquely solvable in $|x| \ge a$, but highly unstable!

- Interior problem: Only integrals hitting $|x| \le a$ measured.

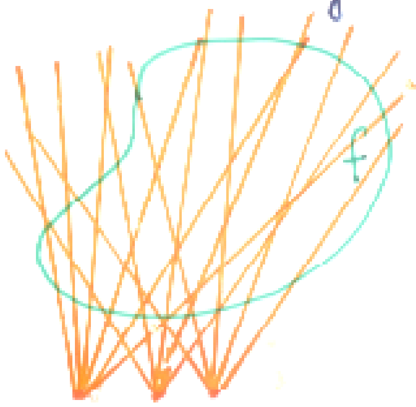


Not unique, but stable!

$$f = f_0 + f_1$$

f_0: almost constant; f_1: stably determined

- Limited angle: Only directions in an angular range $< \pi$ measured



Uniquely solvable, but highly unstable!

Rule of thumb: Tangents to curves of discontinuity must be measured!

Reconstruction algorithms:

- Radon inversion (filtered backprojection)

$$g \to Hg' \qquad \text{(filtering)}$$

$$Hg' \to \int_{S^1} (Hg')(\theta, x\cdot\theta)\, d\theta \quad \text{(backproj.)}$$

$O(N^3)$ operations for $N \times N$ grid

- Fourier inversion

$$(Rf)^{\wedge}(\theta,\sigma) \sim \hat{f}(\sigma\theta), \quad \theta \in S^1, \sigma \in \mathbb{R}^1$$

$$g \xrightarrow{\text{1D FFT's}} \hat{g} \xrightarrow{\text{interpolation}} \hat{f} \xrightarrow{\text{2D FFT}} f$$

polar - cartesian

Gridding, non-equidistant FFT

$O(N^2 \log N)$ operations

- Iterative methods (ART, EM)

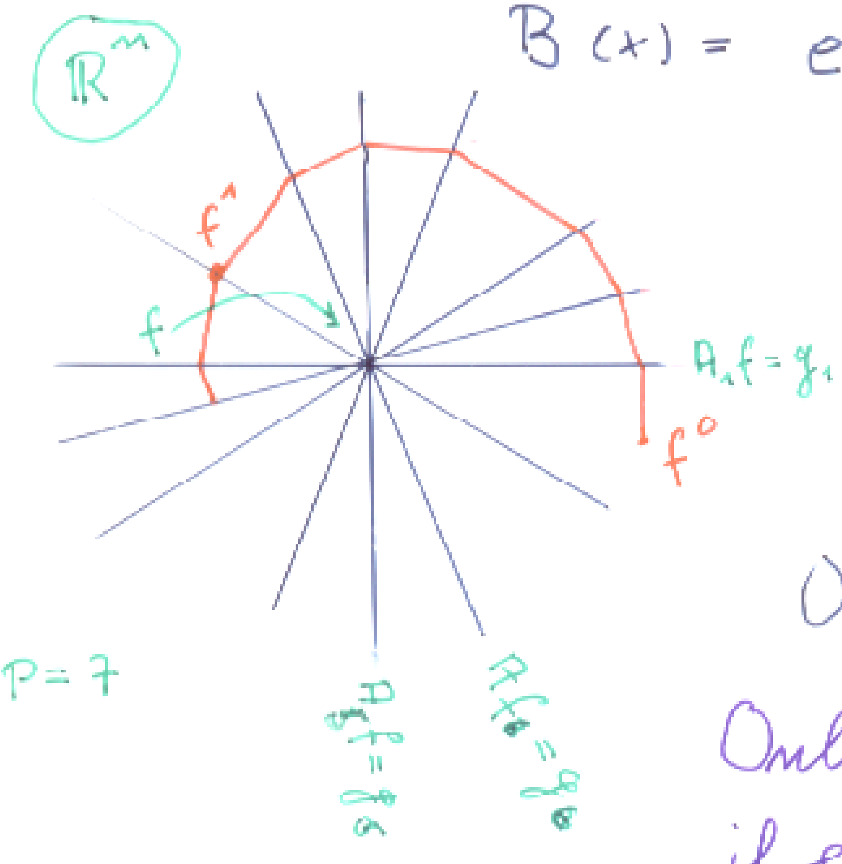
Discretize $Rf = g$ to yield a linear system $A_j f = g_j$, A (m,n)-matrix, $j = 1, \dots, p$

Method of choice:

$$f = \sum_{k=1}^{m} f_k \, B\left(\frac{x - x_k}{h}\right)$$

B „blob", i.e. radial symmetric fast decaying function, e.g.

$$B(x) = e^{-|x|^2/w}$$



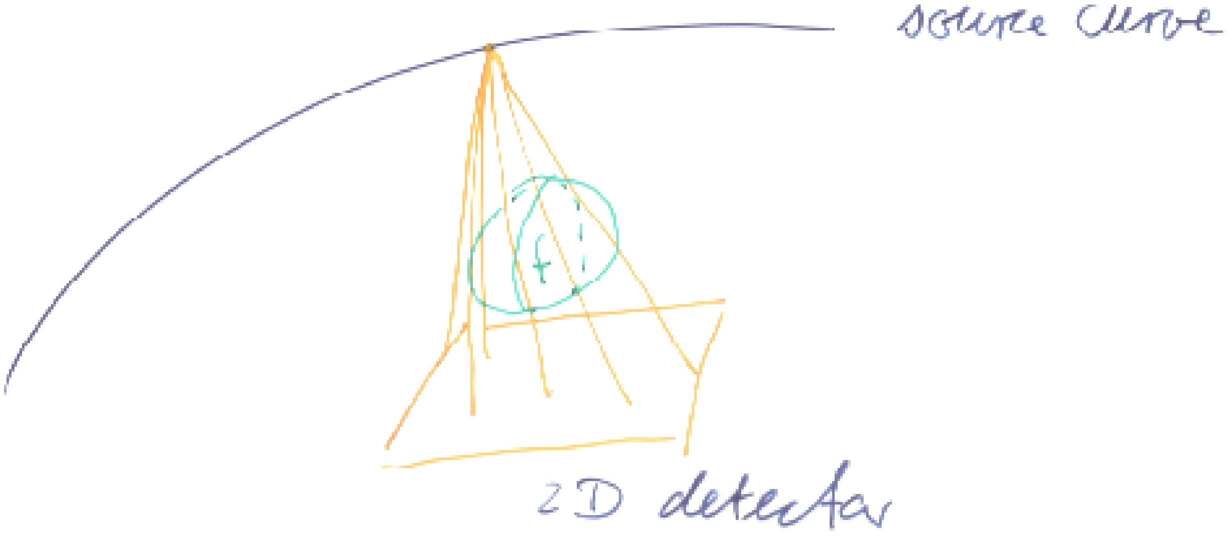
$$f^{t+1} = P_p \cdots P_1 f^t, \quad t = 0, 1, \dots$$

P_j orth. proj. on $A_j f = g_j$

$O(N^3)$ operations / step.

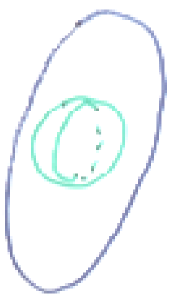
Only a few steps necessary, if equations are properly arranged!

Reconstruction in 3D:



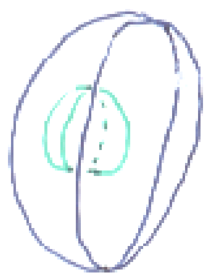
Uniqueness for an arbitrarily short source curve, but stability only if Kirillov-Tuy (KT)-condition holds:

Each plane hitting supp(f) must contain a source.



1 orbit: KT not true

Feldkamp approximate algorithm



2 orth. orbit: KT yes

Grangeat's method

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