A day in the life of a DLMF-author: Normalization of Lamé functions

Consider the Lamé equation

$$\frac{d^2w}{dx^2} + (h - \nu(\nu+1)k^2 \mathrm{sn}^2(x,k))w = 0.$$

The Lamé functions $Ec_\nu^{2m}(x,k^2)$ are solutions which are even and of period $2K$. With

$$\phi = \frac{1}{2}\pi - \mathrm{am}(x,k)$$

we set

$$\begin{aligned} Ec_\nu^{2m}(x,k^2) &= \frac{1}{2}A_0 + \sum_{p=1}^{\infty} A_{2p}\cos(2p\phi) \\ &= \mathrm{dn}(x,k)\left(\frac{1}{2}C_0 + \sum_{p=1}^{\infty} C_{2p}\cos(2p\phi)\right). \end{aligned}$$

The coefficients A_{2p}, C_{2p} satisfy three-term recursion formulas.

How to normalize $Ec_\nu^{2m}(x, k^2)$?

Note the orthogonality relations

$$\int_0^K Ec_\nu^{2m}(x, k^2) Ec_\nu^{2p}(x, k^2)\, dx = 0, \quad m \neq p.$$

Therefore, it appears natural to fix the value of

$$\int_0^K Ec_\nu^{2m}(x, k^2)^2\, dx.$$

However, such a condition is difficult to express in terms of the A_{2p} and C_{2p} because the expansions from the previous slide use functions that are orthogonal with respect to the inner product

$$\int_0^K \mathrm{dn}\, x f(x) g(x)\, dx = \int_0^{\pi/2} f(\mathrm{am}\, \phi) g(\mathrm{am}\, \phi)\, d\phi,$$

or

$$\int_0^K \frac{1}{\mathrm{dn}\, x} f(x) g(x)\, dx = \int_0^{\pi/2} \frac{f(\mathrm{am}\, \phi) g(\mathrm{am}\, \phi)}{(\mathrm{dn}(\mathrm{am}\, \phi))^2}\, d\phi.$$

Following Janzen (1977) we normalize by setting

$$\int_0^K \mathrm{dn}\, x \, Ec_\nu^{2m}(x,k^2)^2 \, dx = \frac{\pi}{4}.$$

Then

$$\frac{1}{2}A_0^2 + \sum_{p=1}^{\infty} A_{2p}^2 = 1.$$

What about the C_{2p}'s? Square both sides of the expansion with the C_{2p}, multiply by $\mathrm{dn}\, x$ and use addition formulas for trigonometric functions. We obtain that

$$\left(1-\frac{k^2}{2}\right)\left(\frac{1}{2}C_0^2 + \sum_{p=1}^{\infty} C_{2p}^2\right) - \frac{k^2}{2}\sum_{p=0}^{\infty} C_{2p}C_{2p+2} = 1.$$

Moreover,

$$\int_0^K Ec_\nu^{2m}(x,k^2)^2 \, dx = \frac{\pi}{4}\left(\frac{1}{2}A_0C_0 + \sum_{p=1}^{\infty} A_{2p}C_{2p}\right).$$