

Integral relations for solutions of Hill's equation

Consider the Hill equation

$$u'' - g(x)u = 0,$$

where $g : \mathbf{R} \rightarrow \mathbf{C}$ is even, continuous and of period ω . We assume that there is a nontrivial solution u_1 of period 2ω . Let u_2 be a linearly independent solution. The product $v(x, y) = u_1(x)u_2(y)$ satisfies the partial differential equation

$$(\partial_x^2 - \partial_y^2)v - (g(x) - g(y))v = 0.$$

Let $A : \mathbf{R}^2 \rightarrow \mathbf{C}$ be the Riemann function of this equation, that is, for $(x_0, y_0) \in \mathbf{R}^2$ let $A(\cdot, \cdot, x_0, y_0)$ be the unique solution with

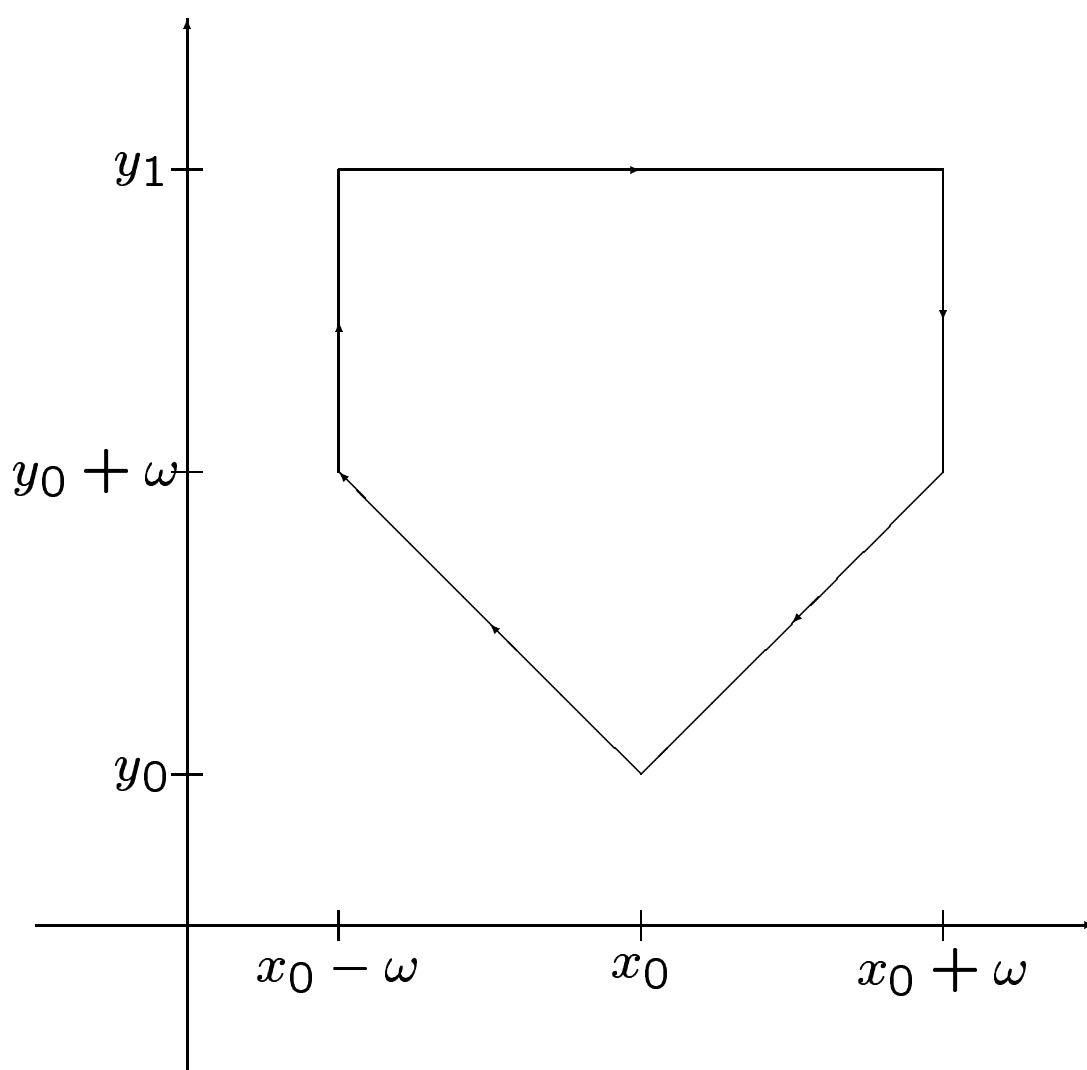
$$A(x, y, x_0, y_0) = 1 \quad \text{if } y - y_0 = \pm(x - x_0).$$

Then

$$\int_K (A\partial_y v - v\partial_y A) dx + (A\partial_x v - v\partial_x A) dy = 0$$

where K is any closed rectifiable curve in \mathbf{R}^2 .

We choose for K the following pentagon



Theorem. There is a constant μ such that, for all $x_1, x_2, x_3 \in \mathbf{R}$,

$$\mu u_1(x_1)u_1(x_2)u_1(x_3) = \int_{-\omega}^{\omega} A(x, x_1, x_2, x_3)u_1(x) dx.$$

The constant μ can be expressed by

$$\mu = \frac{2\sigma\tau}{W[u_1, u_2]}$$

where σ, τ are determined by

$$u_1(x+\omega) = \sigma u_1(x), \quad u_2(x+\omega) = \tau u_1(x) + \sigma u_2(x).$$

For the Mathieu equation

$$u'' + (\lambda - 2h^2 \cos(2x))u = 0$$

we have

$$A(x, y, x_0, y_0) = J_0((4h^2(F_1^2 - F_2^2))^{1/2}),$$

$$F_1 = \cos x \cos y - \cos x_0 \cos y_0,$$

$$F_2 = \sin x \sin y - \sin x_0 \sin y_0.$$

For the Lamé equation

$$u'' + (\lambda - \nu(\nu + 1)k^2 \operatorname{sn}^2 x)u = 0$$

we have

$$A(x, y, x_0, y_0) = P_\nu(G_1 + G_2 + G_3),$$

$$G_1 = k^2 \operatorname{sn} x \operatorname{sn} x_0 \operatorname{sn} y \operatorname{sn} y_0,$$

$$G_2 = -\frac{k^2}{k'^2} \operatorname{cn} x \operatorname{cn} x_0 \operatorname{cn} y \operatorname{cn} y_0,$$

$$G_3 = \frac{1}{k'^2} \operatorname{dn} x \operatorname{dn} x_0 \operatorname{dn} y \operatorname{dn} y_0.$$