

The Mathieu equation with complex parameters

Consider the Mathieu equation

$$w'' + (\lambda - 2h^2 \cos(2z))w = 0.$$

For $h^2 \in \mathbf{R}$, let

$$\lambda_1(h^2) < \lambda_2(h^2) < \lambda_3(h^2) < \dots$$

denote the eigenvalues of λ such that the Mathieu equation admits nontrivial odd solutions with period π . The functions λ_m are real-analytic. What happens if we continue these functions analytically into the complex h^2 -plane?

The eigenvalue functions can be continued analytically throughout the complex h^2 -plane when we avoid a certain countable set that has no finite point of accumulation (h^2 -coordinates of branch points.) The eigenvalue functions approach a finite value at these branch points.

Since the eigenvalue function $\lambda_n(h^2)$ is analytic, it can be expanded in a power series at $h^2 = 0$. Let ρ_n be the radius of convergence of this expansion. What is the growth order of ρ_n as $n \rightarrow \infty$? It is easy to show that ρ_n grows at least linearly with n . F. W. Schäfke conjectured that the ρ_n grow like n^2 as $n \rightarrow \infty$. I proved that

$$\liminf \frac{\rho_n}{n^2} \geq kk'K^2 = 2.0418\dots,$$

where the modulus k of the complete elliptic integrals is determined by $2E = K$.

Call two positive integers m, n equivalent, if there exists a curve starting at $h^2 = 0$ and ending at $h^2 = 0$ such that $\lambda_m(h^2)$ when continued analytically along the curve becomes $\lambda_n(h^2)$. F. W. Schäfke proved that there is only one equivalence class for the Mathieu equation. Let us look at the proof.

For $k \in \mathbb{N}$, define

$$\tau_k = \min_{n > k} \rho_n.$$

Lemma 1 Let M be an equivalence class and $k \in \mathbb{N}$. Then the function

$$\sigma_k(h^2) = \sum_{n \in M, n \leq k} \lambda_n(h^2)$$

which is analytic in a neighborhood of $\mu = 0$ can be extended analytically onto the disk $\{h^2 : |h^2| < \tau_k\}$.

Proof: Let $\phi(t)$ be a curve starting at $h^2 = 0$ and ending at $h^2 = 0$ with $|\phi(t)| < \tau_k$ and which does not pass through branch points. Then analytic extension

along ϕ defines a bijection $\pi : M \rightarrow M$. We have $\pi(m) = m$ for $m > k$. This shows that σ_k is one-valued in the disk apart from the finitely many branch points. Since these points are removable singularities for σ_k , the lemma is proved.

Lemma 2 Let M, k, σ_k be as in Lemma 1. Then

$$|\sigma_k''(0)| \leq 8k \frac{1}{\tau_k}$$

Proof Note that eigenpairs (λ, μ) satisfy

$$|\operatorname{Im} \lambda| \leq 2|h^2|.$$

Therefore

$$|\operatorname{Im} \sigma_k(h^2)| \leq 2k\tau_k$$

for $|h^2| < \tau_k$. By Lemma 1 and a modified form of Cauchy's estimate, we obtain the statement of the lemma.

From Lemma 2, we obtain

Theorem 3 Let M be an equivalence class. Then

$$\sum_{n \in M} \lambda''_n(0) = 0.$$

Now $\lambda''_1(0) < 0$ and $\lambda''_n(0) > 0$ for all $n \geq 2$. Therefore \mathbb{N} is the only equivalence class.