Numerical computation of eigenvalues of the Mathieu equation

The Mathieu equation is

$$w'' + (\lambda - 2h^2 \cos(2t))w = 0.$$

For given $h^2 \in \mathbf{R}$ we are looking for nontrivial odd solutions w of period π. Setting

$$w(t) = \sum_{n=1}^{\infty} x_n \sin(2nt)$$

we obtain the eigenvalue problem

$$Ax = \lambda x, \quad x = (x_1, x_2, x_3, \ldots)^T \in l^2(\mathbf{N}),$$

where

$$A = \begin{pmatrix} d_1 & c_1 & 0 & 0 & 0 & \ldots \\ c1 & d_2 & c_2 & 0 & 0 & \ldots \\ 0 & c_2 & d_3 & c_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$c_n = h^2, \quad d_n = 4n^2.$$

To compute the eigenvalues of A denoted by

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

use the Ritz method. For given $r \in \mathbf{N}$, let

$$\mu_{1,r} < \mu_{2,r} < \dots < \mu_{r,r}$$

be the eigenvalues of the r by r submatrix of A consisting of the first r rows and columns of A. We expect $\mu_{k,r}$ to approximate λ_k. Is there an error estimate

$$|\lambda_k - \mu_{k,r}| \leq ???$$

First observe that

$$\mu_{k,k} \geq \mu_{k,k+1} \geq \mu_{k,k+2} \geq \dots, \quad \lim_{r \to \infty} \mu_{k,r} = \lambda_k.$$

How can we obtain bounds of the form

$$\mu_{k,r} - \lambda_k \leq ???$$

Let $\lambda_i \leq \Lambda_i$, $\lambda_k - \lambda_i \leq \Lambda_{i,k}$ with Λ_i, $\Lambda_{i,k}$ computable. For $i \in \mathbf{N}$ choose $\ell_i \geq 2$ so large that

$$d_n > |c_{n-1}| + |c_n| + \Lambda_i \quad \text{for } n \geq \ell_i.$$

For $n \geq \ell_i$, define

$$f_{i,n} := \frac{|c_{n-1}|}{d_n - \Lambda_i - |c_n|},$$

$$a_{i,n} := \prod_{j=\ell_i}^{n} f_{i,j},$$

$$b_{i,n} := (1 - f_{i,n+2}^2)^{-1/2} a_{i,n+1}.$$

Then, for $r \geq \ell_k$,

$$\mu_{k,r} - \lambda_k \leq \frac{M}{1-L} \quad \text{if } L < 1,$$

where

$$L := \sum_{i=1}^{k} b_{i,r}^2,$$

$$M : = |c_r| \left(\sum_{i=1}^{k} a_{i,r}^2 \sum_{j=1}^{k} a_{j,r+1}^2 \right)^{1/2} + \left(\sum_{i=1}^{k-1} \Lambda_{i,k}^2 b_{i,r}^2 \sum_{j=1}^{k} b_{j,r}^2) \right)^{1/2}.$$

Consequence:

$$\mu_{k,r} - \lambda_k = O\left(\frac{h^{4r}}{4^{2r+1}r!^2(r+1)!^2}\right) \quad \text{as } r \to \infty.$$

The O-constant is computable. For example, for $r \geq 3$, $-1 \leq h^2 \leq 1$,

$$\mu_{1,r} - \lambda_1 \leq \frac{94h^{4r}}{4^{2r+1}r!^2(r+1)!^2}.$$

This estimate is quite sharp. For example, if $r = 10$, $h^2 = 1$, the left hand side of this inequality is $\approx 0.56 \cdot 10^{-39}$ whereas the estimate on the right hand side gives $1.01 \cdot 10^{-39}$.