```
> restart;
```

Requires algolib.

```
> libname;
```

"/backup/2002-07-19/CopieExportAlgolib6", "/usr/local/lib/maple/6/lib"

```
>
```

Clausen's Identity *(univariate closures)*

It gives the evaluation at 1 of a 4F3:

$$\sum_{k=0}^{n} \frac{(a+k-1)!\,(b+k-1)!\,(n-k)!\left(a+b+n-k-\frac{1}{2}\right)!}{k!\left(a+b+k-\frac{1}{2}\right)!\,(a+n-k-1)!\,(b+n-k-1)!} = \frac{(2\,a)_n\,(2\,b)_n\,(a+b)_n}{(a)_n\,(b)_n\,(2\,a+2\,b)_n}.$$

Under a generating function form, it becomes:

$$\mathrm{F}\left(a,\,b,\,a+b+\frac{1}{2},\,x\right)^2 = \mathrm{F}\left(2\,a,\,2\,b,\,a+b,\,a+b+\frac{1}{2},\,2\,(a+b),\,x\right)$$

(2F1 squared equals 3F2).

```
> restart;
> with(gfun);
```

[*Laplace, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtable, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, listtoseriestable, maxdegcoeff, maxdegeqn, maxordereqn, mindegcoeff, mindegeqn, minordereqn, optionsgf, pade2, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*]

$$> \ expr := \mathrm{hypergeom}\left([a,\,b],\,\left[a+b+\frac{1}{2}\right],\,x\right)^2$$

```
> gfun[holexprtodiffeq](expr,f(x));
```

$$\left\{\mathrm{f}(0)=1,\ \mathrm{D}(f)(0)=4\,\frac{a\,b}{2\,a+2\,b+1},\ (8\,a\,b^2+8\,a^2\,b)\,\mathrm{f}(x) + (4\,x\,b^2+2\,x+6\,x\,a+16\,a\,b\,x+6\,x\,b+4\,x\,a^2-2\,b-2\,a-4\,b^2-8\,a\,b-4\,a^2)\left(\frac{\partial}{\partial x}\mathrm{f}(x)\right) + (6\,x^2+6\,x^2\,a+6\,x^2\,b-6\,x\,a-3\,x-6\,x\,b)\left(\frac{\partial^2}{\partial x^2}\mathrm{f}(x)\right) + (2\,x^3-2\,x^2)\left(\frac{\partial^3}{\partial x^3}\mathrm{f}(x)\right),\ (\mathrm{D}^{(2)})(f)(0)=8\,\frac{a\,b\,(2\,a^2+4\,a^2\,b+4\,a\,b^2+3\,a+8\,a\,b+1+2\,b^2+3\,b)}{8\,a^3+20\,a^2+24\,a^2\,b+14\,a+40\,a\,b+24\,a\,b^2+20\,b^2+3+14\,b+8\,b^3}\right\}$$

```
> gfun[diffeqtorec](%,f(x),u(n));
```

$$\{u(0)=1, u(1)=4\frac{a\,b}{2\,a+2\,b+1},$$
$$(2\,n^3+(6\,b+6\,a)\,n^2+(4\,b^2+16\,a\,b+4\,a^2)\,n+8\,a\,b^2+8\,a^2\,b)\,u(n)+(-2\,n^3$$
$$+(-6\,a-3-6\,b)\,n^2+(-8\,b-8\,a-4\,b^2-8\,a\,b-4\,a^2-1)\,n-2\,b-2\,a-4\,b^2-8\,a\,b-4\,a^2)$$
$$u(n+1), u(2)=4\frac{a\,b\,(2\,a^2+4\,a^2\,b+4\,a\,b^2+3\,a+8\,a\,b+1+2\,b^2+3\,b)}{8\,a^3+20\,a^2+24\,a^2\,b+14\,a+40\,a\,b+24\,a\,b^2+20\,b^2+3+14\,b+8\,b^3}\}$$

(Order 1!)

```
> simplify(sum(rsolve(%,u(n))*x^n,n=0..infinity),GAMMA);
```

$$2\frac{\mathrm{hypergeom}\left([2\,a, 2\,b, a+b], \left[2\,b+2\,a, a+b+\frac{1}{2}\right], x\right)4^{(a+b)}}{2^{(2\,a+2\,b+1)}}$$

```
>
>
```

Mehler Formula for Hermite Polynomials *(definite sum by univariate closures)*

```
> restart;
```

The formula is:

$$\sum_{n=0}^{\infty}\frac{H_n(x)\,H_n(y)\,u^n}{n!}=\frac{\mathbf{e}^{\left(\frac{4\,u\,(u\,(x^2+y^2)-x\,y)}{4\,u^2-1}\right)}}{\sqrt{1-4\,u^2}}.$$

Maple fails to solve it:

$$>\ \sum_{n=0}^{\infty}\frac{\mathit{orthopoly}_H(n, x)\,\mathit{orthopoly}_H(n, y)\,u^n}{n!}$$

$$\sum_{n=0}^{\infty}\frac{\mathit{orthopoly}_H(n, x)\,\mathit{orthopoly}_H(n, y)\,u^n}{n!}$$

```
>
```

Using univariate closures (and gfun)

Using a (yet unpublished) version of the Mgfun package:

```
> with(Mgfun);
```

[*MG_Internals, creative_telescoping, dfinite_expr_to_diffeq, dfinite_expr_to_rec, dfinite_expr_to_sys, diag_of_sys, int_of_sys, pol_to_sys, sum_of_sys, sys*sys, sys+sys*]

```
> dfinite_expr_to_rec(orthopoly[H](n,x),h(n));
```

$$\{(2+2\,n)\,h(n)-2\,x\,h(n+1)+h(n+2)\}$$

```
> 'sys*sys'(%,subs(x=y,%));
```

$$\{(16\,n^4+112\,n^3+272\,n^2+272\,n+96)\,h(n)$$

$+(-8n^2+8x^2n-44n+8ny^2+24x^2-60+24y^2)\,h(n+2)-4xy\,h(n+3)+h(n+4)$
$+(-16xyn^2-80xyn-96xy)\,h(n+1)\}$

```
> with(gfun);
```

[*Laplace, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtable, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, listtoseriestable, maxdegcoeff, maxdegeqn, maxordereqn, mindegcoeff, mindegeqn, minordereqn, optionsgf, pade2, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*]

```
> {seq(h(i)=orthopoly[H](i,x)*orthopoly[H](i,y),i=0..3)};
```

$$\{h(0)=1,\ h(1)=4xy,\ h(2)=(4x^2-2)(4y^2-2),\ h(3)=(8x^3-12x)(8y^3-12y)\}$$

```
> rectodiffeq(%%% union %,h(n),h(u));
```

$$(96u^4-32xyu^3+(-4+8x^2+8y^2)u^2-4xyu+1)\,h(u)$$
$$+(672u^5-64xyu^4+(-20+8y^2+8x^2)u^3)\left(\frac{\partial}{\partial u}h(u)\right)$$
$$+(720u^6-16xyu^5-8u^4)\left(\frac{\partial^2}{\partial u^2}h(u)\right)+208u^7\left(\frac{\partial^3}{\partial u^3}h(u)\right)+16u^8\left(\frac{\partial^4}{\partial u^4}h(u)\right)+32xyu^3$$
$$-4(-4+8x^2+8y^2)xyu^3-(-4+8x^2+8y^2)u^2-4(-20+8y^2+8x^2)u^3xy$$
$$+2xy(8+32x^2y^2-16x^2-16y^2)u^3+16x^2y^2u^2-1$$
$$-\frac{1}{2}(8+32x^2y^2-16x^2-16y^2)u^2-\frac{1}{6}(864xy+384x^3y^3-576x^3y-576xy^3)u^3$$

```
> borel(%,h(u),diffeq);
```

$$\{h(0)=1,\ D(h)(0)=4xy,\ (16u^4-16xyu^3+(-4+8x^2+8y^2)u^2-4xyu)\,h(u)$$
$$+(16u^5-8u^3+u)\left(\frac{\partial}{\partial u}h(u)\right)+(-4(-4+8x^2+8y^2)xy$$
$$+2xy(16x^2y^2-8x^2-8y^2+4)-24xy-32x^3y^3+48x^3y+48xy^3)u^3$$
$$-(2xy(16x^2y^2-8x^2-8y^2+4)-4xy(8x^2y^2-4x^2-4y^2+2))u^3,$$
$$(D^{(2)})(h)(0)=16x^2y^2-8x^2-8y^2+4,\ (D^{(3)})(h)(0)=64x^3y^3-96x^3y-96xy^3+144xy$$
$$\}$$

```
> dsolve(%);
```

$$h(u)=\frac{I\,\mathbf{e}^{\left(\frac{y^2+x^2-4xyu}{(2u+1)(2u-1)}\right)}}{\sqrt{2u+1}\,\sqrt{2u-1}\,\mathbf{e}^{(-y^2-x^2)}}$$

>

>

Using multivariate closures (and Mgfun as a hammer)

```
> restart;
```

Using a (yet unpublished) version of the Mgfun package:

```
> with(Mgfun);
```

[*MG_Internals*, *creative_telescoping*, *dfinite_expr_to_diffeq*, *dfinite_expr_to_rec*, *dfinite_expr_to_sys*, *diag_of_sys*, *int_of_sys*, *pol_to_sys*, *sum_of_sys*, *sys*sys*, *sys+sys*]

```
> infolevel[Mgfun]:=5:
```

$$> expr := \frac{\mathrm{HermiteH}(n, x)\,\mathrm{HermiteH}(n, y)\,u^n}{n!}$$

```
> ct:=creative_telescoping(expr,u::diff,n::shift);
Mgfun/chyzak97:   "Describe explicit expression as d-finite"
Mgfun/chyzak97:   "Dimension is 4"
Mgfun/chyzak97:   "Start of Chyzak's algorithm"
Mgfun/chyzak97:   "Preparation of the system: .679 seconds."
Mgfun/uncoupling:   "Uncoupling of the LOF system"
Mgfun/chyzak97:   "Uncoupling of the system: .90e-1 seconds."
Mgfun/rational_sys_solve:   "Look for rational solutions of a system"
`Mgfun/denominator_bound`[shift]   "Intermediate bound on dispersion: infin
ity"
`Mgfun/denominator_bound`[shift]   "Bound on dispersion: 0"
`Mgfun/denominator_bound`[shift]   "Computing a resultant: .41e-1 seconds."
`Mgfun/rational_solve`[shift]   "Computing denominator bound: .41e-1 second
s."
Mgfun/rational_sys_solve:   "Equations solved: 1"
Mgfun/rational_sys_solve:   "Solving equation: .131 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 2"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 3"
Mgfun/rational_sys_solve:   "Solving equation: .29e-1 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 4"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/chyzak97:   "LOFE of order 1 found!"
```

$$ct := \Bigg[(16\,u^3 - 4\,u - 4\,y\,x + 8\,u\,y^2 + 8\,u\,x^2 - 16\,u^2\,y\,x)\,\_f(u, n) + (16\,u^4 + 1 - 8\,u^2)\left(\frac{\partial}{\partial u}\,\_f(u, n)\right),$$
$$-\frac{(-n + 8\,u^2\,n + 16\,u^3\,y\,x + 4\,u^2 - 8\,u^2\,y^2 - 8\,u^2\,x^2 + 4\,u\,y\,x)\,\_f(u, n)}{u}$$
$$-\frac{(-n + 8\,u^2\,n - 8\,u^2\,y^2 - 1 + 4\,u\,y\,x + 12\,u^2 - 8\,u^2\,x^2)\,\_f(u, n+1)}{u}$$
$$+\frac{(n + 2 - 4\,u\,y\,x)\,\_f(u, n+2)}{u} + \frac{(n+3)\,\_f(u, n+3)}{u}\Bigg]$$

Due to "natural boundaries":

```
> subs(_f(u,n)=_f(u),ct[1])=0;
```

$$(16\,u^3 - 4\,u - 4\,y\,x + 8\,u\,y^2 + 8\,u\,x^2 - 16\,u^2\,y\,x)\,\_f(u) + (16\,u^4 + 1 - 8\,u^2)\left(\frac{\partial}{\partial u}\,\_f(u)\right) = 0$$

```
> simplify(expand(dsolve(%)));
```

$$\_f(u) = \frac{e^{\left(-\frac{-x^2 - y^2 + 4\,u\,y\,x}{(2\,u-1)(2\,u+1)}\right)}\,\_C1}{\sqrt{2\,u+1}\,\sqrt{2\,u-1}}$$

\>

\>

\>

-> Automatically Generated Encyclopedia of Special Functions

\>

Indefinite Sum of the Harmonic Product binomial(n, m) H(n) *(extended Gosper algorithm)*

```
> restart;
> with(gfun);
```

[*Laplace, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtable, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, listtoseriestable, maxdegcoeff, maxdegeqn, maxordereqn, mindegcoeff, mindegeqn, minordereqn, optionsgf, pade2, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*]

$$> \; gfun_{rectohomrec}\left(\mathrm{u}(n+1) - \mathrm{u}(n) = \frac{1}{n+1}, \mathrm{u}(n)\right)$$

$$(-n-1)\,\mathrm{u}(n) + (2\,n+3)\,\mathrm{u}(n+1) + (-n-2)\,\mathrm{u}(n+2)$$

```
> u(n+1)-normal(binomial(n+1,m)/binomial(n,m),expanded)*u(n);
```

$$\mathrm{u}(n+1) - \frac{(-n-1)\,\mathrm{u}(n)}{-n-1+m}$$

```
> rec:=gfun['rec*rec'](%%,%,u(n));
```

$$rec := (n^2+2\,n+1)\,\mathrm{u}(n) + (-2\,n^2 - 5\,n + 2\,n\,m - 3 + 3\,m)\,\mathrm{u}(n+1) + (n^2 + 3\,n - 2\,n\,m + 2 - 3\,m + m^2)\,\mathrm{u}(n+2)$$

```
> with(Mgfun);
```

[*MG_Internals, creative_telescoping, dfinite_expr_to_diffeq, dfinite_expr_to_rec, dfinite_expr_to_sys, diag_of_sys, int_of_sys, pol_to_sys, sum_of_sys, sys*sys, sys+sys*]

```
> extended_gosper:=proc(expr,u,n) local ct,p; global LFSol;
      ct:=creative_telescoping(

  LFSol({eval(subs(u=unapply(u(n,p),n),expr)),u(n,p+1)-u(n,p)}),
      p::shift,n::shift);
      eval(subs(_f=unapply(_f(n),p,n),ct[2]/ct[1]*_f(p,n)))
  end:
> collect(extended_gosper(rec,u,n),_f,factor);
```

$$\frac{(m-n)(-n-1+m)\_f(n+1)}{(m+1)^2}+\frac{(m+2+n)(m-n)\_f(n)}{(m+1)^2}$$

```
> Sum(binomial(n,m)*H(n),n)=applyop(collect,3,factor(expand(eval(subs(_f=unapply(binomial(n,m)*H(n),n),%)))),H,factor);
```

$$\sum_n \mathrm{binomial}(n,m)\,\mathrm{H}(n)=-\frac{(m-n)\left((-m-2-n)\,\mathrm{H}(n)+(n+1)\,\mathrm{H}(n+1)\right)\mathrm{binomial}(n,m)}{(m+1)^2}$$

```
>
>
```

Indefinite Integral of the Cosine Integral Ci(z) *(extended Gosper algorithm)*

```
> restart;
> with(gfun);
```

[*Laplace, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtable, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, listtoseriestable, maxdegcoeff, maxdegeqn, maxordereqn, mindegcoeff, mindegeqn, minordereqn, optionsgf, pade2, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*]

For $\mathrm{Ci}(z)=\int \frac{\cos(z)}{z}\,dz$, we have:

```
> deq:=holexprtodiffeq(Ci(z),f(z));
```

$$deq := \left(\frac{\partial}{\partial z}\,\mathrm{f}(z)\right)z+2\left(\frac{\partial^2}{\partial z^2}\,\mathrm{f}(z)\right)+z\left(\frac{\partial^3}{\partial z^3}\,\mathrm{f}(z)\right)$$

```
> with(Mgfun);
```

[*MG_Internals, creative_telescoping, dfinite_expr_to_diffeq, dfinite_expr_to_rec, dfinite_expr_to_sys, diag_of_sys, int_of_sys, pol_to_sys, sum_of_sys, sys*sys, sys+sys*]

```
> extended_gosper:=proc(expr,f,z) local ct,p; global LFSol;
      ct:=creative_telescoping(

  LFSol({eval(subs(f=unapply(f(z,p),z),expr)),diff(f(z,p),p)}),
      p::diff,z::diff);
      eval(subs(_f=unapply(_f(z),p,z),ct[2]/ct[1]*_f(p,z)))
  end:
> extended_gosper(deq,f,z);
```

$$-z\,\_f(z)-z\left(\frac{\partial^2}{\partial z^2}\,\_f(z)\right)-\left(\frac{\partial}{\partial z}\,\_f(z)\right)$$

```
> Int(Ci(z),z)=collect(eval(subs(_f=Ci,%)),{Ci,cos,sin},normal);
```

$$\int \mathrm{Ci}(z)\,dz = -z\,\mathrm{Ci}(z) + \sin(z)$$

\>

\>

Neumann's Addition Theorem: $J(0,z)^2 + 2\left(\sum_{k=1}^{\infty} J(k,z)^2\right) = 1$ *(extended Zeilberger algorithm)*

\>

Using user-level tools

```
> restart;
> with(Mgfun);
```

$$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, dfinite\_expr\_to\_rec, dfinite\_expr\_to\_sys, diag\_of\_sys, int\_of\_sys, pol\_to\_sys, sum\_of\_sys, sys{*}sys, sys{+}sys]$$

```
> dfinite_expr_to_sys(BesselJ(k,z)^2,t(k::shift,z::diff));
```

$$\{(4k^2 - z^2 + 4k)\,t(k,z) + z^2\,t(k+2,z) + (-8k - 4 - 4k^2)\,t(k+1,z) + (-2kz - 2z)\left(\frac{\partial}{\partial z} t(k,z)\right), -2k\,t(k,z) + (2k+2)\,t(k+1,z) + z\left(\frac{\partial}{\partial z} t(k,z)\right) + z\,D_2(t)(k+1,z), 2z\,t(k,z) - 2z\,t(k+1,z) + (-2k+1)\left(\frac{\partial}{\partial z} t(k,z)\right) + z\left(\frac{\partial^2}{\partial z^2} t(k,z)\right)\}$$

```
> ct:=creative_telescoping(BesselJ(k,z)^2,z::diff,k::shift);
```

$$ct := \left[\frac{\partial}{\partial z}\_f(z,k), \frac{k\,\_f(z,k)}{z} + \frac{1}{2}\left(\frac{\partial}{\partial z}\_f(z,k)\right)\right]$$

Viewed in the ring of formal power series, the next expression goes to 0 when k goes to infinity.

```
> eval(subs(_f(z,k)=BesselJ(k,z)^2,ct[2]));
```

$$\frac{k\,\mathrm{BesselJ}(k,z)^2}{z} + \mathrm{BesselJ}(k,z)\left(-\mathrm{BesselJ}(k+1,z) + \frac{k\,\mathrm{BesselJ}(k,z)}{z}\right)$$

When k=1, it is:

```
> simplify(eval(subs(k=1,%)));
```

$$\mathrm{BesselJ}(0,z)\,\mathrm{BesselJ}(1,z)$$

```
> subs(_f(z,k)=Sum(J(k,z)^2,k=1..infinity),ct[1])+%=0;
```

$$\frac{\partial}{\partial z}\left(\sum_{k=1}^{\infty} J(k,z)^2\right) + \mathrm{BesselJ}(0,z)\,\mathrm{BesselJ}(1,z) = 0$$

```
> int(BesselJ(0,z)*BesselJ(1,z),z);
```

$$-\frac{1}{2}\,\mathrm{BesselJ}(0,z)^2$$

Getting the "1" requires to compare a few (= a single, here) initial conditions.

```
>
```

```
>
```

Using lower-level tools

Preparing the input and computing its associated module

*Bug in Mgfun['sys*sys'] (Skip this section: it's only to report a bug to me.)*

```
> with(Mgfun);
```

[*MG_Internals*, *creative_telescoping*, *dfinite_expr_to_diffeq*, *dfinite_expr_to_rec*, *dfinite_expr_to_sys*, *diag_of_sys*, *int_of_sys*, *pol_to_sys*, *sum_of_sys*, *sys*sys*, *sys+sys*]

```
> dfinite_expr_to_sys(BesselJ(k,z),t(k::shift,z::diff));
```

$$\left\{-k\,\mathrm{t}(k,z)+z\,\mathrm{t}(k+1,z)+z\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right),\ (z^2-k^2)\,\mathrm{t}(k,z)+z^2\left(\frac{\partial^2}{\partial z^2}\mathrm{t}(k,z)\right)+z\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)\right\}$$

The output to 'sys*sys' is inconsistent, and inconsistent with what I compute manually below.

```
> 'sys*sys'(%,%);
```

$$\{2\,z\,\mathrm{t}(k,z)-2\,z\,\mathrm{t}(k+1,z)+(-2\,k+1)\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)+z\left(\frac{\partial^2}{\partial z^2}\mathrm{t}(k,z)\right),$$
$$-2\,k\,\mathrm{t}(k,z)+(2+2\,k)\,\mathrm{t}(k+1,z)+z\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)+z\,\mathrm{D}_2(t)(k+1,z),$$
$$(4\,k^2+4\,k-z^2)\,\mathrm{t}(k,z)+(-4\,k^2-8\,k-4)\,\mathrm{t}(k+1,z)+(-2\,z\,k-2\,z)\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)$$
$$+z^2\,\mathrm{t}(k+2,z)\}$$

```
> dfinite_expr_to_sys(BesselJ(k,z)^2,t(k::shift,z::diff))
  ;
```

$$\{2\,z\,\mathrm{t}(k,z)-2\,z\,\mathrm{t}(k+1,z)+(-2\,k+1)\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)+z\left(\frac{\partial^2}{\partial z^2}\mathrm{t}(k,z)\right),$$
$$-2\,k\,\mathrm{t}(k,z)+(2+2\,k)\,\mathrm{t}(k+1,z)+z\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)+z\,\mathrm{D}_2(t)(k+1,z),$$
$$(4\,k^2+4\,k-z^2)\,\mathrm{t}(k,z)+(-4\,k^2-8\,k-4)\,\mathrm{t}(k+1,z)+(-2\,z\,k-2\,z)\left(\frac{\partial}{\partial z}\mathrm{t}(k,z)\right)$$
$$+z^2\,\mathrm{t}(k+2,z)\}$$

```
> pol_to_sys(f(k,z)=t(k,z),{[t(k,z),%]});
```

$$\{-2\,k\,\mathrm{f}(k,z)+(2+2\,k)\,\mathrm{f}(k+1,z)+z\left(\frac{\partial}{\partial z}\mathrm{f}(k,z)\right)+z\,\mathrm{D}_2(f)(k+1,z),$$
$$(4\,k^2+4\,k-z^2)\,\mathrm{f}(k,z)+(-4\,k^2-8\,k-4)\,\mathrm{f}(k+1,z)+(-2\,z\,k-2\,z)\left(\frac{\partial}{\partial z}\mathrm{f}(k,z)\right)$$
$$+z^2\,\mathrm{f}(k+2,z),\ 2\,z\,\mathrm{f}(k,z)-2\,z\,\mathrm{f}(k+1,z)+(-2\,k+1)\left(\frac{\partial}{\partial z}\mathrm{f}(k,z)\right)+z\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(k,z)\right)\}$$

```
>
>
> with(Ore_algebra);
```

[*OA_Internals*, *Ore_to_DESol*, *Ore_to_RESol*, *Ore_to_diff*, *Ore_to_shift*, *annihilators*, *applyopr*, *diff_algebra*, *dual_algebra*, *dual_polynomial*, *poly_algebra*, *qshift_algebra*, *rand_skew_poly*, *reverse_algebra*, *reverse_polynomial*, *shift_algebra*, *skew_algebra*, *skew_elim*, *skew_gcdex*, *skew_pdiv*, *skew_power*, *skew_prem*, *skew_product*]

```
> Alg:=shift_algebra(shift=[Sk,k],diff=[Dz,z]);
```

$$Alg := Ore\_algebra$$

```
> with(Groebner);
```

[*GB_Internals*, *fglm_algo*, *gbasis*, *gsolve*, *hilbertdim*, *hilbertpoly*, *hilbertseries*, *inter_reduce*, *is_finite*, *is_solvable*, *leadcoeff*, *leadmon*, *leadterm*, *normalf*, *pretend_gbasis*, *reduce*, *spoly*, *termorder*, *testorder*, *univpoly*]

```
> TOrd:=termorder(Alg,tdeg(Sk,Dz));
```

$$TOrd := term\_order$$

> $G := \{z^2\,Dz^2+z\,Dz+z^2-k^2,\ z\,Sk+z\,Dz-k\}$

```
> GB:=gbasis(G,TOrd);
```

$$GB := [z\,Sk+z\,Dz-k,\ z^2\,Dz^2+z\,Dz+z^2-k^2]$$

```
> with(Holonomy);
```

[*HO_Internals*, *algeq_to_dfinite*, *dfinite_add*, *dfinite_mul*, *holon_closure*, *holon_defint*, *holon_defqsum*, *holon_defsum*, *holon_diagonal*, *hypergeom_to_dfinite*, *takayama_algo*]

```
> PR:=dfinite_mul([[GB,TOrd],[GB,TOrd]],TOrd);
```

$$PR := [2\,z+z\,Dz^2-2\,z\,Sk+Dz-2\,k\,Dz,\ z\,Sk\,Dz+z\,Dz-2\,k+2\,Sk+2\,Sk\,k,$$
$$-z^2+z^2\,Sk^2-2\,Dz\,k\,z-2\,z\,Dz-4\,Sk\,k^2-4\,Sk-8\,Sk\,k+4\,k^2+4\,k]$$

```
>
> map(leadterm,PR,TOrd);
```

$$[Dz^2,\ Sk\,Dz,\ Sk^2]$$

Search for a first-order ODE for the sum

> $N := 1$

```
> t:=add(eta[i]*Dz^i,i=0..N);
```

$$t := \eta_0+\eta_1\,Dz$$

> $T := \phi_0(k) + \phi_1(k)\,Dz + \phi_2(k)\,Sk$

```
> collect(subs(k=k+1,T)*Sk-T-t,[Dz,Sk],distributed);
```

$$(-\phi_1(k) - \eta_1)\,Dz + \phi_1(k+1)\,Sk\,Dz + \phi_2(k+1)\,Sk^2 + (\phi_0(k+1) - \phi_2(k))\,Sk - \eta_0 - \phi_0(k)$$

```
> cf:=coeffs(%,[Dz,Sk],'tm');
```

$$cf := -\eta_0 - \phi_0(k),\ -\phi_1(k) - \eta_1,\ \phi_1(k+1),\ \phi_2(k+1),\ \phi_0(k+1) - \phi_2(k)$$

```
> tm;
```

$$1,\ Dz,\ Sk\,Dz,\ Sk^2,\ Sk$$

```
> add(cf[i]*normalf(tm[i],PR,TOrd),i=1..nops([tm]));
```

$$-\eta_0 - \phi_0(k) + (-\phi_1(k) - \eta_1)\,Dz + \phi_1(k+1)\left(\frac{(-2\,k-2)\,Sk}{z} - Dz + \frac{2\,k}{z}\right)$$
$$+ \phi_2(k+1)\left(\frac{(4\,k^2 + 4 + 8\,k)\,Sk}{z^2} + \frac{(2\,k+2)\,Dz}{z} + \frac{z^2 - 4\,k^2 - 4\,k}{z^2}\right)$$
$$+ (\phi_0(k+1) - \phi_2(k))\,Sk$$

```
> collect(%,[Dz,Sk],distributed,p->collect(p,[seq(phi[i],i=0..N)],normal));
```

$$\left(-\phi_1(k+1) - \phi_1(k) + \frac{-\eta_1\,z + 2\,\phi_2(k+1)\,k + 2\,\phi_2(k+1)}{z}\right)Dz +$$
$$\left(\phi_0(k+1) - \frac{2\,(k+1)\,\phi_1(k+1)}{z} + \frac{4\,\phi_2(k+1)\,k^2 + 4\,\phi_2(k+1) + 8\,\phi_2(k+1)\,k - \phi_2(k)\,z^2}{z^2}\right)$$
$$Sk - \phi_0(k) + \frac{2\,\phi_1(k+1)\,k}{z} + \frac{-\eta_0\,z^2 + \phi_2(k+1)\,z^2 - 4\,\phi_2(k+1)\,k^2 - 4\,\phi_2(k+1)\,k}{z^2}$$

```
> eq0:=coeff(coeff(%,Dz,0),Sk,0);
```

$$eq0 := -\phi_0(k) + \frac{2\,\phi_1(k+1)\,k}{z} + \frac{-\eta_0\,z^2 + \phi_2(k+1)\,z^2 - 4\,\phi_2(k+1)\,k^2 - 4\,\phi_2(k+1)\,k}{z^2}$$

```
> eq1:=coeff(coeff(%%,Dz,1),Sk,0);
```

$$eq1 := -\phi_1(k+1) - \phi_1(k) + \frac{-\eta_1\,z + 2\,\phi_2(k+1)\,k + 2\,\phi_2(k+1)}{z}$$

```
> eq2:=coeff(coeff(%%%,Dz,0),Sk,1);
```

$eq2 :=$

$$\phi_0(k+1) - \frac{2\,(k+1)\,\phi_1(k+1)}{z} + \frac{4\,\phi_2(k+1)\,k^2 + 4\,\phi_2(k+1) + 8\,\phi_2(k+1)\,k - \phi_2(k)\,z^2}{z^2}$$

```
>
> eq3:=collect(eval(subs(phi[0]=unapply(solve(eq0,phi[0](k)),k),eq2)),[phi[1],phi[2]],normal);
```

$$eq3 := -2\,\frac{(k+1)\,\phi_1(k+1)}{z} + \frac{2\,(k+1)\,\phi_1(k+2)}{z} + \frac{4\,(k^2 + 1 + 2\,k)\,\phi_2(k+1)}{z^2} - \phi_2(k)$$

$$+\frac{(-12\,k-8+z^2-4\,k^2)\,\phi_2(k+2)}{z^2}-\eta_0$$

```
> eq4:=collect(eval(subs(phi[2]=unapply(subs(k=k-1,solve(eq1,phi[2](k+1))),k),eq3)),phi[1],normal);
```

$$eq4 := \frac{1}{2}\,\frac{(-12\,k-8+z^2-4\,k^2)\,\phi_1(k+1)}{z\,(k+2)}-\frac{1}{2}\,\frac{(z^2-4\,k^2-4\,k)\,\phi_1(k)}{k\,z}+\frac{\frac{1}{2}\,z\,\phi_1(k+2)}{k+2}-\frac{1}{2}\,\frac{z\,\phi_1(k-1)}{k}-\frac{\eta_0\,k^2+2\,\eta_0\,k+\eta_1\,z}{(k+2)\,k}$$

```
> eq4:=collect(numer(eq4),phi[1],factor);
```

$$eq4 := k\,(-12\,k-8+z^2-4\,k^2)\,\phi_1(k+1)-(k+2)\,(z^2-4\,k^2-4\,k)\,\phi_1(k)+z^2\,\phi_1(k+2)\,k-z^2\,(k+2)\,\phi_1(k-1)-2\,(\eta_0\,k^2+2\,\eta_0\,k+\eta_1\,z)\,z$$

```
> LREtools[ratpolysols](eq4,phi[1](k),{},output=basis);
```

(Stupid convention of Maple: NULL is returned, to mean "no solution (provably).")

This is no contradiction!

```
>
> indets(eq4,function);
```

$$\{\phi_1(k+1), \phi_1(k), \phi_1(k+2), \phi_1(k-1)\}$$

```
> subs(k=k-2,coeff(eq4,phi[1](k+2)));
```

$$z^2\,(k-2)$$

```
> subs(k=k+1,coeff(eq4,phi[1](k-1)));
```

$$-z^2\,(k+3)$$

```
> gcd(%%,%);
```

$$z^2$$

All solutions have to be polynomial.

```
> M:= 1
> eval(subs(phi[1]=unapply(add(c[i]*k^i,i=0..M),k),eq4));
```

$$k\,(-12\,k-8+z^2-4\,k^2)\,(c_0+c_1\,(k+1))-(k+2)\,(z^2-4\,k^2-4\,k)\,(c_0+c_1\,k)+z^2\,(c_0+c_1\,(k+2))\,k-z^2\,(k+2)\,(c_0+c_1\,(k-1))-2\,(\eta_0\,k^2+2\,\eta_0\,k+\eta_1\,z)\,z$$

```
> sys:={coeffs(collect(%,k,expand),k)};
```

$$sys := \{-4\,z^2\,c_0-2\,\eta_1\,z^2+2\,z^2\,c_1, -4\,c_1, -2\,\eta_0\,z-12\,c_1, -4\,\eta_0\,z-8\,c_1\}$$

```
> sol:=`solve/linear`(sys,{seq(c[i],i=0..M),seq(eta[i],i=0..N)});
```

$$sol := \{c_1=0, \eta_1=-2\,c_0, c_0=c_0, \eta_0=0\}$$

```
> [seq(eta[i],i=0..N)]=subs(sol,[seq(eta[i],i=0..N)]);
```

$$[\eta_0, \eta_1]=[0, -2\,c_0]$$

```
> psi[1]:=subs(sol,add(c[i]*k^i,i=0..M));
```

$$\psi_1 := c_0$$

```
>
```

```
> eq3;
```

$$-2\,\frac{(k+1)\,\phi_1(k+1)}{z}+\frac{2\,(k+1)\,\phi_1(k+2)}{z}+\frac{4\,(k^2+1+2\,k)\,\phi_2(k+1)}{z^2}-\phi_2(k)+\frac{(-12\,k-8+z^2-4\,k^2)\,\phi_2(k+2)}{z^2}-\eta_0$$

```
> collect(numer(normal(eval(subs(phi[1]=unapply(psi[1],k),sol,eq3)))),phi[2],factor);
```

$$4\,(k+1)^2\,\phi_2(k+1)-\phi_2(k)\,z^2+(-12\,k-8+z^2-4\,k^2)\,\phi_2(k+2)$$

All solutions have to be polynomial.

```
> M:=5:
> sys2:={coeffs(collect(eval(subs(phi[2]=unapply(add(d[i]*k^i,i=0..M),k),%%)),k,expand),k)}:
> sol2:=`solve/linear`(sys2,{seq(d[i],i=0..M)});
```

$$sol2 := \{d_5=0,\,d_0=0,\,d_4=0,\,d_3=0,\,d_1=0,\,d_2=0\}$$

> $\psi_2 := 0$

```
>
> indets(eq2,function);
```

$$\{\phi_1(k+1),\,\phi_2(k+1),\,\phi_0(k+1),\,\phi_2(k)\}$$

```
> subs(k=k-1,eval(subs(phi[1]=unapply(psi[1],k),phi[2]=0,eq2)));
```

$$\phi_0(k)-\frac{2\,k\,c_0}{z}$$

```
> psi[0]:=solve(%,phi[0](k));
```

$$\psi_0 := 2\,\frac{k\,c_0}{z}$$

```
>
> T_calc:=eval(subs(seq(phi[i]=unapply(psi[i],k),i=0..2),sol,T)):
> T=T_calc;
```

$$\phi_0(k)+\phi_1(k)\,Dz+\phi_2(k)\,Sk=2\,\frac{k\,c_0}{z}+c_0\,Dz$$

```
> t_calc:=eval(subs(seq(phi[i]=unapply(psi[i],k),i=0..2),sol,t)):
> t=t_calc;
```

$$\eta_0+\eta_1\,Dz=-2\,c_0\,Dz$$

```
>
```

> $c_0 := 2\,z$

```
> applyopr(T_calc,BesselJ(k,z)^2,Alg);
```

$$4\,k\,\mathrm{BesselJ}(k,z)^2+4\,z\,\mathrm{BesselJ}(k,z)\left(-\mathrm{BesselJ}(k+1,z)+\frac{k\,\mathrm{BesselJ}(k,z)}{z}\right)$$

```
> nonhom:=collect(simplify(subs(k=k+1,%)-subs(k=0,%)),BesselJ,factor);
```

$$nonhom := 4\,z\,\mathrm{BesselJ}(k+1, z)\,\mathrm{BesselJ}(k, z) + 4\,z\,\mathrm{BesselJ}(0, z)\,\mathrm{BesselJ}(1, z)$$

As a series, BesselJ(k, z) goes to 0 when k goes to infinity, thus:

```
> nonhom:=remove(has,nonhom,k);
```

$$nonhom := 4\,z\,\mathrm{BesselJ}(0, z)\,\mathrm{BesselJ}(1, z)$$

```
> diff(BesselJ(0,z)^2,z);
```

$$-2\,\mathrm{BesselJ}(0, z)\,\mathrm{BesselJ}(1, z)$$

```
>
> t_calc&.Sum(BesselJ(k,z)^2,k=0..infinity)=nonhom;
```

$$(-4\,z\,Dz)\ \&.\left(\sum_{k=0}^{\infty} \mathrm{BesselJ}(k, z)^2\right) = 4\,z\,\mathrm{BesselJ}(0, z)\,\mathrm{BesselJ}(1, z)$$

```
>
> %;
```

$$(-4\,z\,Dz)\ \&.\left(\sum_{k=0}^{\infty} \mathrm{BesselJ}(k, z)^2\right) = 4\,z\,\mathrm{BesselJ}(0, z)\,\mathrm{BesselJ}(1, z)$$

This is the announced equation in disguise.

```
>
>
>
```

What Differential System Do Appel's F_4 Generalized Hypergeometric Functions Satisfy?

(Weyl algebras and Ore algebras; Groebner bases in Ore algebras; Hilbert dimension; Groebner-based elimination)

```
> restart;
```

By definition, $F_4(a, b, c, d, x, y)$ is the sum over all nonnegative m and n of

$$> expr := \frac{\mathrm{pochhammer}(a, m+n)\,\mathrm{pochhammer}(b, m+n)\,x^m\,y^n}{m!\,n!\,\mathrm{pochhammer}(c, m)\,\mathrm{pochhammer}(d, n)}$$

where $\mathrm{pochhammer}(x, k) = \dfrac{\Gamma(x+k)}{\Gamma(x)}$.

```
> with(Ore_algebra);
```

[*OA_Internals, Ore_to_DESol, Ore_to_RESol, Ore_to_diff, Ore_to_shift, annihilators, applyopr, diff_algebra, dual_algebra, dual_polynomial, poly_algebra, qshift_algebra, rand_skew_poly, reverse_algebra, reverse_polynomial, shift_algebra, skew_algebra, skew_elim, skew_gcdex, skew_pdiv, skew_power, skew_prem, skew_product*]

```
> A:=skew_algebra(diff=[Dx,x],diff=[Dy,y],shift=[Sm,m],shift=[Sn,n],polynom={m,n},comm={a,b,c,d});
```

$$A := Ore\_algebra$$

```
> with(Mgfun);
```

$$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, dfinite\_expr\_to\_rec, dfinite\_expr\_to\_sys, diag\_of\_sys, int\_of\_sys, pol\_to\_sys, sum\_of\_sys, sys{*}sys, sys{+}sys]$$

```
> G:=dfinite_expr_to_sys(expr,f(m::shift,n::shift,x::diff,y::diff)
);
```

$$G := \{(-x\,m^2 - 2\,x\,m\,n - x\,b\,m - x\,m\,a - x\,n^2 - x\,b\,n - x\,n\,a - x\,b\,a)\,f(m, n, x, y) + (m\,c + m^2 + c + m)\,f(m+1, n, x, y), -n\,f(m, n, x, y) + y\left(\frac{\partial}{\partial y} f(m, n, x, y)\right), -m\,f(m, n, x, y) + x\left(\frac{\partial}{\partial x} f(m, n, x, y)\right), (-y\,m^2 - 2\,y\,m\,n - y\,b\,m - y\,m\,a - y\,n^2 - y\,n\,a - y\,b\,n - y\,b\,a)\,f(m, n, x, y) + (n\,d + n^2 + d + n)\,f(m, n+1, x, y)\}$$

```
> G:=subs([f(m,n,x,y)=1,f(m+1,n,x,y)=Sm,f(m,n+1,x,y)=Sn,diff(f(m,n
,x,y),x)=Dx,diff(f(m,n,x,y),y)=Dy],G);
```

$$G := \{-x\,m^2 - 2\,x\,m\,n - x\,b\,m - x\,m\,a - x\,n^2 - x\,b\,n - x\,n\,a - x\,b\,a + (m\,c + m^2 + c + m)\,Sm, -y\,m^2 - 2\,y\,m\,n - y\,b\,m - y\,m\,a - y\,n^2 - y\,n\,a - y\,b\,n - y\,b\,a + (n\,d + n^2 + d + n)\,Sn, -n + y\,Dy, -m + x\,Dx\}$$

```
> with(Groebner);
```

$$[GB\_Internals, fglm\_algo, gbasis, gsolve, hilbertdim, hilbertpoly, hilbertseries, inter\_reduce, is\_finite, is\_solvable, leadcoeff, leadmon, leadterm, normalf, pretend\_gbasis, reduce, spoly, termorder, testorder, univpoly]$$

```
> T:=termorder(A,tdeg(Dx,Dy,Sn,Sm,n,m));
```

$$T := term\_order$$

```
> GB:=gbasis(G,T);
```

$$GB := [-n + y\,Dy, -m + x\,Dx, -x\,m^2 - 2\,x\,m\,n - x\,b\,m - x\,m\,a - x\,n^2 - x\,b\,n - x\,n\,a - x\,b\,a + Sm\,m\,c + Sm\,m^2 + Sm\,c + Sm\,m, -y\,m^2 - 2\,y\,m\,n - y\,b\,m - y\,m\,a - y\,n^2 - y\,n\,a - y\,b\,n - y\,b\,a + Sn\,n\,d + Sn\,n^2 + Sn\,d + Sn\,n]$$

```
> hilbertdim(GB,T);
```

$$2$$

```
> T:=termorder(A,lexdeg([n,m],[Dx,Dy,Sn,Sm]));
```

$$T := term\_order$$

```
> GB:=gbasis(G,T);
```

$$GB := [-Dx^2\,x^2 - x\,Dx\,b - x\,Dx - 2\,x\,Dx\,Dy\,y + x\,Sm\,Dx^2 - x\,Dx\,a - b\,a - Dy^2\,y^2 - y\,Dy - Dy\,y\,b - Dy\,y\,a + Dx\,c\,Sm, -Dy^2\,y^2 - 2\,x\,Dx\,Dy\,y - Dy\,y\,b - Dy\,y\,a - y\,Dy + y\,Sn\,Dy^2 - Dx^2\,x^2 - x\,Dx - x\,Dx\,b - x\,Dx\,a + Dy\,d\,Sn - b\,a, m - x\,Dx, n - y\,Dy]$$

```
> GB:=subs([Sn=1,Sm=1],remove(has,GB,[m,n]));
```

$$GB := [-Dx^2\,x^2 - x\,Dx\,b - x\,Dx - 2\,x\,Dx\,Dy\,y + x\,Dx^2 - x\,Dx\,a - b\,a - Dy^2\,y^2 - y\,Dy - Dy\,y\,b - Dy\,y\,a + Dx\,c, -Dy^2\,y^2 - 2\,x\,Dx\,Dy\,y - Dy\,y\,b - Dy\,y\,a - y\,Dy + y\,Dy^2 - Dx^2\,x^2 - x\,Dx$$

$- x\,Dx\,b - x\,Dx\,a + Dy\,d - b\,a]$

```
> map(applyopr,GB,f(x,y),A);
```

$$\Big[-b\,a\,f(x,y) - y^2\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) + (-x^2+x)\left(\frac{\partial^2}{\partial x^2} f(x,y)\right) + (-x\,b - x\,a - x + c)\left(\frac{\partial}{\partial x} f(x,y)\right) + (-y - b\,y - a\,y)\left(\frac{\partial}{\partial y} f(x,y)\right) - 2\,x\,y\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y)\right), -b\,a\,f(x,y) + (-y^2+y)\left(\frac{\partial^2}{\partial y^2} f(x,y)\right) - x^2\left(\frac{\partial^2}{\partial x^2} f(x,y)\right) + (-x\,a - x - x\,b)\left(\frac{\partial}{\partial x} f(x,y)\right) + (-b\,y - y + d - a\,y)\left(\frac{\partial}{\partial y} f(x,y)\right) - 2\,x\,y\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y)\right)\Big]$$

Going to a differential equation in a single derivative is too long for generic a,b,c,d (over 6 minutes in Maple8).

```
> A:=skew_algebra(diff=[Dx,x],diff=[Dy,y]);
```

$$A := Ore\_algebra$$

```
> T:=termorder(A,plex(Dy,Dx));
```

$$T := term\_order$$

```
> collect(gbasis(subs([a=1/2,b=1/3,c=1/5,d=1/7],GB),T),[Dx,Dy],distributed,factor);
```

$$[1271871875\,y + 74040625\,x - 1271871875 + 420\,x\,(-37090116\,y^2 + 37090116\,y - 12363372 - 60534510\,x^2 + 66756407\,x + 12363372\,y^3 - 1833090\,x\,y + 6141475\,x^3 + 46418470\,y\,x^2 - 64923317\,x\,y^2)\,Dx^3 + 44100\,x^2\,(6235\,x - 49061 + 49061\,y)\,(y^2 - 2\,x\,y - 2\,y + x^2 - 2\,x + 1)\,Dx^4 + (1557784872\,y - 1557784872\,y^2 - 60344578775\,x^2 + 33453981210\,x - 519261624 - 31547673605\,x\,y^2 + 5787482875\,x^3 + 519261624\,y^3 + 54883124050\,y\,x^2 - 1906307605\,x\,y)\,Dx^2 + (-486847505\,y - 33317864875\,x - 1904834785\,y^2 + 2391682290 + 33067747850\,x\,y + 2765378375\,x^2)\,Dx, -24185 + 24185\,y - 25025\,x - 88200\,x^2\,(y^2 - 2\,x\,y - 2\,y + x^2 - 2\,x + 1)\,Dx^3 + 210\,x\,(-2325\,x^2 + 2138\,x\,y + 187 + 187\,y^2 - 374\,y + 2138\,x)\,Dx^2 + (-58044\,y + 29022\,y^2 + 29022 - 451675\,x^2 + 154525\,x\,y - 59605\,x)\,Dx + 5\,y\,(6235\,x - 49061 + 49061\,y)\,Dy]$$

```
> applyopr(op(1,%),F4(x,y),A);
```

$$(1271871875\,y + 74040625\,x - 1271871875)\,F4(x,y) + 420\,x\,(-37090116\,y^2 + 37090116\,y - 12363372 - 60534510\,x^2 + 66756407\,x + 12363372\,y^3 - 1833090\,x\,y + 6141475\,x^3 + 46418470\,y\,x^2 - 64923317\,x\,y^2)\left(\frac{\partial^3}{\partial x^3} F4(x,y)\right)$$

$$+44100\, x^2 (6235\, x-49061+49061\, y)(y^2-2\, x\, y-2\, y+x^2-2\, x+1)\left(\frac{\partial^4}{\partial x^4}\mathrm{F4}(x,y)\right)+(1557784872\, y-1557784872\, y^2-60344578775\, x^2+33453981210\, x-519261624-31547673605\, x\, y^2+5787482875\, x^3+519261624\, y^3+54883124050\, y\, x^2-1906307605\, x\, y)\left(\frac{\partial^2}{\partial x^2}\mathrm{F4}(x,y)\right)+(-486847505\, y-33317864875\, x-1904834785\, y^2+2391682290+33067747850\, x\, y+2765378375\, x^2)\left(\frac{\partial}{\partial x}\mathrm{F4}(x,y)\right)$$

when $a=\frac{1}{2}$, $b=\frac{1}{3}$, $c=\frac{1}{5}$, and $d=\frac{1}{7}$.

\>

\>

A Relation Related to Apery's Proof of the Irrationality of $\zeta(3)$

$$: \sum_{k=0}^{n} \mathrm{binomial}(n,k)^2\, \mathrm{binomial}(n+k,k)^2 = \sum_{k=0}^{n} \mathrm{binomial}(n,k)\, \mathrm{binomial}(n+k,k) \left(\sum_{j=0}^{k} \mathrm{binomial}(k,j)^3\right)$$

```
> restart;
>
```

Recurrence for the Right-Hand Side

```
> with(Mgfun);
```

[*MG_Internals*, *creative_telescoping*, *dfinite_expr_to_diffeq*, *dfinite_expr_to_rec*, *dfinite_expr_to_sys*, *diag_of_sys*, *int_of_sys*, *pol_to_sys*, *sum_of_sys*, *sys*sys*, *sys+sys*]

Perform creative telescoping on the summand of the inner sum.

\> $sys_1 := \mathrm{dfinite\_expr\_to\_sys}(\mathrm{binomial}(k,j)^3, \mathrm{f}(k{::}shift, j{::}shift))$

$$sys_1 := \{(j^3-3\, k\, j^2+3\, k^2\, j-k^3)\, \mathrm{f}(k,j)+(j^3+3\, j^2+3\, j+1)\, \mathrm{f}(k,j+1), (-1-k^3-3\, k^2-3\, k)\, \mathrm{f}(k,j)+(k^3-3\, k^2\, j+3\, k^2+3\, k\, j^2-6\, k\, j+3\, k-j^3+3\, j^2-3\, j+1)\, \mathrm{f}(k+1,j)\}$$

```
> sys[2]:=sum_of_sys(sys[1],j=0..k,_natural_boundaries);
```

$$sys_2 := \{(8\, k^2+16\, k+8)\, \mathrm{f}(k)+(7\, k^2+21\, k+16)\, \mathrm{f}(k+1)+(-k^2-4\, k-4)\, \mathrm{f}(2+k)\}$$

\> $sys_3 := \{\mathrm{f}(n+1,k)-\mathrm{f}(n,k)\}$ union eval(subs($f=\mathrm{unapply}(\mathrm{f}(n,k),k)$, sys_2))

$$sys_3 := \{(8\, k^2+16\, k+8)\, \mathrm{f}(n,k)+(7\, k^2+21\, k+16)\, \mathrm{f}(n,k+1)+(-k^2-4\, k-4)\, \mathrm{f}(n,2+k),$$

$f(n+1,k) - f(n,k)\}$

Describe the binomial weight of the outer sum.

```
> sys[4] := dfinite_expr_to_sys(binomial(n, k) binomial(n+k, k), f(n::shift, k::shift))
```

$$sys_4 := \{(-n^2 - n + k^2 + k)\, f(n,k) + (k^2 + 2k + 1)\, f(n, k+1), (n+k+1)\, f(n,k) + (-n-1+k)\, f(n+1,k)\}$$

Multiply both to get a description of the summand of the outer summation.

```
> sys[5]:=`sys*sys`(sys[3],sys[4]);
```

$$sys_5 := \{(16k^2 n + 16k^2 n^2 - 40k^2 + 32nk - 16n^3 - 8n^4 + 8n^2 + 32kn^2 - 16k - 32k^3 - 8k^4 + 16n)\, f(n,k) + (32 - 16n + 93k^2 + 90k - 7k^2 n - 21nk + 42k^3 - 7k^2 n^2 - 16n^2 + 7k^4 - 21kn^2)\, f(n, k+1) + (k^4 + 8k^3 + 24k^2 + 32k + 16)\, f(n, 2+k), (n+k+1)\, f(n,k) + (-n-1+k)\, f(n+1,k)\}$$

Performing the summation yields a recurrence equation of order two which describes the right-hand side.

```
> sys[6]:=sum_of_sys(sys[5],k=0..infinity,_natural_boundaries);
```

$$sys_6 := \{(n^3 + 3n^2 + 3n + 1)\, f(n) + (-34n^3 - 153n^2 - 231n - 117)\, f(n+1) + (6n^2 + 12n + n^3 + 8)\, f(n+2)\}$$

```
>
```

```
>
```

Recurrence for the Left-Hand Side

Summation by two traightforward calls.

```
> sys[7]:=dfinite_expr_to_sys(binomial(n,k)^2*binomial(n+k,k)^2
  ,f(n::shift,k::shift));
```

$$sys_7 := \{(-2k - k^2 - n^2 - 1 - 2nk - 2n)\, f(n,k) + (n^2 + 2n - 2nk + 1 - 2k + k^2)\, f(n+1,k), (-2n^3 - k^4 - 2k^3 + 2k^2 n^2 - n^4 - k^2 + 2k^2 n + 2kn^2 + 2nk - n^2)\, f(n,k) + (k^4 + 4k^3 + 6k^2 + 4k + 1)\, f(n, k+1)\}$$

```
> sys[8]:=sum_of_sys(sys[7],k=0..n,_natural_boundaries);
```

$$sys_8 := \{(-n^3 - 3n^2 - 3n - 1)\, f(n) + (153n^2 + 34n^3 + 231n + 117)\, f(n+1) + (-6n^2 - 12n - n^3 - 8)\, f(n+2)\}$$

This is the same equation as for the right-hand side.

```
>
```

```
>
```

Proving the Identity

Both sides satisfy the same second-order recurrence; there only remains to verify that sufficiently many initial conditions agree.

The recurrence rewrites

```
> collect(solve(subs(n=n-2,sys[8]),f(n)),f,factor);
```

$$\{f(n) = -\frac{(n-1)^3 f(n-2)}{n^3} + \frac{(2n-1)(17n^2-17n+5) f(n-1)}{n^3}\}$$

Thus, a solution sequence u is completely determined on the non-negative integers by its first two values u(0) and u(1).

Initial conditions for the left-hand side are:

```
> seq(add(binomial(n,k)^2*binomial(n+k,k)^2,k=0..n),n=0..1);
```

$$1, 5$$

And for the right-hand side:

```
> seq(add(binomial(n,k)*binomial(n+k,k)*add(binomial(k,j)^3,j=0..k),k=0..n),n=0..1);
```

$$1, 5$$

The identity is now proved by induction.

```
>
>
```

Bessel Integral:

$$\int_0^\infty x\, e^{\left(-\frac{x^2}{p}\right)} J_n(b\,x)\, I_n(c\,x)\, dx = \frac{1\, p\, e^{\left(\frac{1\,(c^2-b^2)\,p}{4}\right)} J_n\left(\frac{b\,c\,p}{2}\right)}{2}$$

```
> restart;
> expr:=x*exp(-x^2/p)*BesselJ(n,b*x)*BesselI(n,c*x);
```

$$expr := x\, e^{\left(-\frac{x^2}{p}\right)} \mathrm{BesselJ}(n, b\,x)\, \mathrm{BesselI}(n, c\,x)$$

```
> with(Mgfun);
```

[*MG_Internals*, *creative_telescoping*, *dfinite_expr_to_diffeq*, *dfinite_expr_to_rec*, *dfinite_expr_to_sys*, *diag_of_sys*, *int_of_sys*, *pol_to_sys*, *sum_of_sys*, *sys*sys*, *sys+sys*]

```
> infolevel[Mgfun]:=5:
> ct:=creative_telescoping(expr,p::diff,x::diff):
Mgfun/chyzak97:   "Describe explicit expression as d-finite"
Mgfun/chyzak97:   "Dimension is 4"
Mgfun/chyzak97:   "Start of Chyzak's algorithm"
Mgfun/chyzak97:   "Preparation of the system: .419 seconds."
Mgfun/uncoupling:   "Uncoupling of the LOF system"
Mgfun/chyzak97:   "Uncoupling of the system: .109 seconds."
Mgfun/rational_sys_solve:   "Look for rational solutions of a system"
`Mgfun/rational_solve`[diff]   "Computing denominator bound: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 1"
Mgfun/rational_sys_solve:   "Solving equation: .60e-1 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 2"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 3"
Mgfun/rational_sys_solve:   "Solving equation: .40e-1 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 4"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
```

```
Mgfun/chyzak97:   "No LOFE of order 1 :-("
Mgfun/chyzak97:   "Preparation of the system: .150 seconds."
Mgfun/uncoupling:   "Uncoupling of the LOF system"
Mgfun/chyzak97:   "Uncoupling of the system: .111 seconds."
Mgfun/rational_sys_solve:   "Look for rational solutions of a system"
‘Mgfun/rational_solve‘[diff]   "Computing denominator bound: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 1"
Mgfun/rational_sys_solve:   "Solving equation: .89e-1 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 2"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 3"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 4"
Mgfun/rational_sys_solve:   "Solving equation: .11e-1 seconds."
Mgfun/chyzak97:   "LOFE of order 2 found!"
> ct[1];
```

$$(2\,p^2\,b^2\,c^2+p^2\,b^4+p^2\,c^4-4\,p\,b^2+4\,p\,c^2-16\,n^2+16)\,\_f(p,x)$$
$$+(-8\,p^2\,c^2+8\,p^2\,b^2-16\,p)\left(\frac{\partial}{\partial p}\,\_f(p,x)\right)+16\,p^2\left(\frac{\partial^2}{\partial p^2}\,\_f(p,x)\right)$$

```
> normal(eval(subs(_f(p,x)=expr,ct[2])));
```

$$-\mathbf{e}^{\left(-\frac{x^2}{p}\right)}\,x\,(2\,x\,\mathrm{BesselJ}(n,b\,x)\,p^2\,c^2\,\mathrm{BesselI}(n,c\,x)$$
$$+4\,p^2\,x\,\mathrm{BesselJ}(n+1,b\,x)\,b\,\mathrm{BesselI}(n+1,c\,x)\,c+\mathrm{BesselI}(n,c\,x)\,p^3\,b^3\,\mathrm{BesselJ}(n+1,b\,x)$$
$$-2\,x\,\mathrm{BesselI}(n,c\,x)\,p^2\,b^2\,\mathrm{BesselJ}(n,b\,x)-4\,x^2\,\mathrm{BesselJ}(n,b\,x)\,p\,\mathrm{BesselI}(n+1,c\,x)\,c$$
$$+8\,x\,\mathrm{BesselJ}(n,b\,x)\,\mathrm{BesselI}(n,c\,x)\,p-8\,x^3\,\mathrm{BesselJ}(n,b\,x)\,\mathrm{BesselI}(n,c\,x)$$
$$+p^3\,b^2\,\mathrm{BesselJ}(n,b\,x)\,\mathrm{BesselI}(n+1,c\,x)\,c+p^3\,\mathrm{BesselJ}(n+1,b\,x)\,b\,c^2\,\mathrm{BesselI}(n,c\,x)$$
$$+\mathrm{BesselJ}(n,b\,x)\,p^3\,c^3\,\mathrm{BesselI}(n+1,c\,x)+4\,x^2\,\mathrm{BesselI}(n,c\,x)\,p\,\mathrm{BesselJ}(n+1,b\,x)\,b$$
$$-8\,x\,\mathrm{BesselJ}(n,b\,x)\,p\,n\,\mathrm{BesselI}(n,c\,x)+4\,p^2\,\mathrm{BesselJ}(n+1,b\,x)\,b\,n\,\mathrm{BesselI}(n,c\,x)$$
$$-4\,p^2\,n\,\mathrm{BesselJ}(n,b\,x)\,\mathrm{BesselI}(n+1,c\,x)\,c)\,/\,p$$

```
> collect(subs(_f(p,x)=_f(p),ct[1]),{diff,_f},factor);
```

$$-8\,p\,(p\,c^2-p\,b^2+2)\left(\frac{\partial}{\partial p}\,\_f(p)\right)$$
$$+(2\,p^2\,b^2\,c^2+p^2\,b^4+p^2\,c^4-4\,p\,b^2+4\,p\,c^2-16\,n^2+16)\,\_f(p)+16\,p^2\left(\frac{\partial^2}{\partial p^2}\,\_f(p)\right)$$

```
> dsolve(%);
```

$$\_f(p)=\_C1\,p\,\mathrm{BesselY}\left(n,\frac{1}{2}\sqrt{b^2\,c^2}\,p\right)\mathbf{e}^{(1/4\,p\,(c-b)\,(c+b))}$$
$$+\_C2\,p\,\mathrm{BesselJ}\left(n,\frac{1}{2}\sqrt{b^2\,c^2}\,p\right)\mathbf{e}^{(1/4\,p\,(c-b)\,(c+b))}$$

```
> 
> 
> 
> 
```

Closed Form Evaluation of $\int_0^\infty u^n e^{\left(-\frac{u z \left(1-\frac{1}{u^2}\right)}{2}\right)} (J_n(z) + u J_{n-1}(z))\, dz$

```
> restart;
> with(Mgfun);
```

$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, dfinite\_expr\_to\_rec, dfinite\_expr\_to\_sys, diag\_of\_sys, int\_of\_sys, pol\_to\_sys, sum\_of\_sys, sys*sys, sys+sys]$

We study this integral when parameter values make it well-defined, that is to say for $1 \le n$ and $|u| < 1$.
Here again, we first input the integrand and make our choice of variables with respect to which we want to study the dependency. This has to contain the integration variable *z*. Here, we made the choice to leave no parameter.

> $expr := u^n \mathbf{e}^{\left(-\frac{u z \left(1-\frac{1}{u^2}\right)}{2}\right)} (\mathrm{BesselJ}(n, z) + u\, \mathrm{BesselJ}(n-1, z))$

```
> sys[1]:=dfinite_expr_to_sys(expr,f(n::shift,z::diff,u::diff));
```

$$sys_1 := \{(-2\, n\, u + u^2\, z)\, f(n, z, u) + u\, z \left(\frac{\partial}{\partial z} f(n, z, u)\right) + u^2 \left(\frac{\partial}{\partial u} f(n, z, u)\right) + z\, f(n+1, z, u),$$
$$(-12\, n^2\, u^2 + 4\, u^3\, z\, n + 4\, n\, u^2 + 4\, z\, n\, u + u^4\, z^2 + 2\, z^2\, u^2 - 4\, u\, z + z^2)\, f(n, z, u)$$
$$+ (4\, z^2\, u^3 + 4\, u^2\, z - 4\, u\, z^2) \left(\frac{\partial}{\partial z} f(n, z, u)\right) + (8\, n\, u^3 - 4\, u^3) \left(\frac{\partial}{\partial u} f(n, z, u)\right)$$
$$+ 4\, z^2\, u^2 \left(\frac{\partial^2}{\partial z^2} f(n, z, u)\right),$$
$$(4\, n^2\, u^2 - 4\, u^3\, z\, n - 4\, u\, z + 4\, n\, u^2 - 4\, z\, n\, u + u^4\, z^2 + z^2 + 2\, z^2\, u^2)\, f(n, z, u) + 4\, u^4 \left(\frac{\partial^2}{\partial u^2} f(n, z, u)\right)$$
$$+ (-8\, n\, u^3 + 4\, u^2\, z + 4\, u^4\, z) \left(\frac{\partial}{\partial u} f(n, z, u)\right)\}$$

The integration itself takes place in the following call. The justification that we may use the option *takayama_algo*, and thus, that we may perform an integration over natural boundaries is the following. We will consider this integral for *u* in a neighbourhood of 0, so that the exponential term is exponentially small when *z* tends to infinity. At 0, the integrand has valuation $2\,n$ in *u*, so that any linear combination of the integrand and its derivatives with polynomial function coefficients will be 0 at $u = 0$, provided *n* becomes sufficiently large.

```
> sys[2]:=int_of_sys(sys[1],z=0..infinity,_takayama_algo);
```

$$sys_2 := \{(-4\,n\,u^4 - u^4\,n^2 - u^6\,n - u^6\,n^2)\,f(n,u) + (-u^4\,n^2 + u^6\,n - 2\,n^2\,u^2 + u^6\,n^2 - 11\,n\,u^2 - n\,u^4 - 12\,u^2)\,f(n+1,u) + (n^2\,u^2 + n^2 + 4\,n\,u^2 + 7\,n + 12)\,f(n+3,u) + (2\,u^4\,n^2 - 7\,n - n^2 + 7\,n\,u^2 - 12 + n^2\,u^2 + 12\,u^2 + 5\,n\,u^4)\,f(n+2,u),$$
$$(3\,n^2\,u^2 + 2\,n^2 + 4\,n\,u^2 + 7\,n + 3)\,f(n,u) + (-n\,u^3 - n\,u - 3\,u)\left(\frac{\partial}{\partial u} f(n,u)\right) + (-3\,n^2\,u^2 - n^2 - 4\,n\,u^2 - 4\,n - 3)\,f(n+1,u) + (n\,u^3 + n\,u + 3\,u)\left(\frac{\partial}{\partial u} f(n+1,u)\right) + (-n^2 - 3\,n)\,f(n+2,u),$$
$$(9 - 8\,n^2 - 3\,u^2 - 12\,n^3\,u^2 - 4\,n^3 - n\,u^4 + 15\,n - 24\,n^2\,u^2 - 10\,n\,u^2)\,f(n,u) + (u^5\,n - 3\,n\,u - 2\,n\,u^3 + 3\,u^3 - 9\,u)\left(\frac{\partial}{\partial u} f(n,u)\right) + (12\,n^3\,u^2 + 12\,n\,u^2 + 24\,n^2\,u^2 - 24\,n - 20\,n^2 - 4\,n^3)\,f(n+1,u) + (u^6\,n + 2\,n\,u^4 + n\,u^2 + 3\,u^4 + 3\,u^2)\left(\frac{\partial^2}{\partial u^2} f(n,u)\right) + (8\,n^3 + 28\,n^2 + 12\,n)\,f(n+2,u)\}$$

The above system cancels the integral for sufficiently large n. (But unfortunately, we do not know at this point any lower bound for the valid n.)

At this point, we face a non-deterministic behaviour of Maple: the above output of *int_of_sys* varies, according to the system on which this session is run. Avoiding this non-determinism would require that the user inputs some constraints that are not expressable at the level of the Mgfun package, but only at the lower-level of the Holonomy package (this relates to choices of term orders in Groebner bases, see the Groebner package). Due to this, we provide two ways to solve the system, depending on the output of the following two Maple commands:

```
> map(indets,sys[2],function);
```

$$\{\{f(n+1,u),\, f(n+2,u),\, D_2(f)(n+1,u),\, f(n,u),\, \frac{\partial}{\partial u} f(n,u)\},\ \{f(n+1,u),\, f(n+2,u),\, f(n+3,u),\, f(n,u)\},\ \{f(n+1,u),\, f(n+2,u),\, \frac{\partial^2}{\partial u^2} f(n,u),\, f(n,u),\, \frac{\partial}{\partial u} f(n,u)\}\}$$

```
> if nops(remove(has,%,[D,diff]))=1 then "Case A" else "Case B"
  fi;
```

"Case A"

Please, directly enter the section with title indicated by the last output above.

Case A

In this situation, a recurrence is already available in the system. It is readily solved.

```
> eq[0]:=collect(eval(subs(f=unapply(f(n),n,u),op(remove(has,sy
  s[2],diff)))),f,factor);
```

$$eq_0 := -n\,u^4\,(n\,u^2 + 4 + n + u^2)\,f(n)$$

$$+u^2(-n^2u^2+nu^4-2n^2+u^4n^2-11n-nu^2-12)\,f(n+1)$$
$$+(n+4)(nu^2+n+3)\,f(n+3)$$
$$+(2u^4n^2-7n-n^2+7nu^2-12+n^2u^2+12u^2+5nu^4)\,f(n+2)$$

```
> normal(LREtools[hypergeomsols](eq[0],f(n),{},output=basis),expanded);
```

$$\frac{\_C_1(-u^2)^n n^2+\_C_1(-u^2)^n n+2\,\_C_2(-u^2)^n+2\,\_C_3n^2+2\,\_C_3n}{n+n^2}$$

```
> solve(expand({subs(n=1,%)=2*u,subs(n=2,%)=2*u,subs(n=3,%)=2*u}),{_C[1],_C[2],_C[3]});
```

$$\{\_C_2=0,\ \_C_1=0,\ \_C_3=u\}$$

```
> rsol:=normal(subs(%,%%),expanded);
```

$$rsol := 2\,u$$

```
>
```

Case B

In this situation, we proceed to eliminate all shifts in n so as to obtain a purely differential relation.

```
> eq[1]:=op(select(has,sys[2],diff(f(n,u),u$3)));
```

$$eq_1 := (-9+u^2-19n^2-29n+5n^2u^2+3nu^2+8n^3+4n^4)\,f(n,u)$$
$$+(2u^3n+2u^5n+u^3+u^5)\left(\frac{\partial^3}{\partial u^3}f(n,u)\right)+(-8n^3+8n-4n^4+4n^2)\,f(n+1,u)$$
$$+(-5u^4n^2+3u^4n-4u^2+2u^4-5n^2u^2-9nu^2)\left(\frac{\partial^2}{\partial u^2}f(n,u)\right)$$
$$+(-5n^2u^3-u^3+9u-3u^3n+15n^2u+21nu)\left(\frac{\partial}{\partial u}f(n,u)\right)$$

```
> eq[2]:=op(select(has,sys[2],D[2](f)(n+1,u)));
```

$$eq_2 := (-u^2+20n+7+12n^2)\,f(n,u)+(-20n-12n^2-4)\,f(n+1,u)$$
$$+(8nu+4u)\,D_2(f)(n+1,u)+(u^4+u^2)\left(\frac{\partial^2}{\partial u^2}f(n,u)\right)+(u^3-7u-8nu)\left(\frac{\partial}{\partial u}f(n,u)\right)$$

```
> eq[3]:=op(select(has,sys[2],f(n+2,u)));
```

$$eq_3 := (9-3u^2-8n^2+15n-24n^2u^2-4n^3-10nu^2-u^4n-12n^3u^2)\,f(n,u)$$
$$+(24n^2u^2+12nu^2-4n^3+12n^3u^2-24n-20n^2)\,f(n+1,u)$$
$$+(nu^2+2u^4n+3u^4+nu^6+3u^2)\left(\frac{\partial^2}{\partial u^2}f(n,u)\right)+(28n^2+12n+8n^3)\,f(n+2,u)$$
$$+(3u^3-3nu-2u^3n+u^5n-9u)\left(\frac{\partial}{\partial u}f(n,u)\right)$$

[Procedure to replace var in expr with its value solved from relation.]

```
> plug:=proc(relation,expr,var)
      eval(subs(var=solve(relation,var),expr))
  end:
```

[Procedure that collects in each function call.]

```
> normalize:=proc(expr)
      collect(expr,indets(expr,function),distributed,factor)
  end:
> eq[4]:=normalize(plug(diff(eq[1],u),eq[2],D[2](f)(n+1,u)));
```

$$eq_4 := -((-23\,n^2\,u^2 - 12\,n\,u^2 - 18\,n^3\,u^2 - 2\,u^2 + n^4\,u^2 - 35\,n^4 + 47\,n^2 - 44\,n^5 + 30\,n^3 + 14\,n - 12\,n^6)\,\mathrm{f}(n,u))/(n\,(n-1)\,(n+2)\,(n+1)) + (-20\,n - 12\,n^2 - 4)\,\mathrm{f}(n+1,u) + u^2\,(-48\,n^3\,u^2 + 21\,n\,u^2 + n^4\,u^2 - 8\,n^2\,u^2 + 12\,n^3 + n^4 + 10\,n^2 + 3\,n + 1 + 7\,u^2)\left(\frac{\partial^2}{\partial u^2}\,\mathrm{f}(n,u)\right)/(n\,(n-1)\,(n+2)\,(n+1)) - \frac{u^3\,(2\,n+1)\,(1 - 7\,u^2 - 13\,n\,u^2 + 3\,n + 5\,n^2 + 5\,n^2\,u^2)\left(\frac{\partial^3}{\partial u^3}\,\mathrm{f}(n,u)\right)}{n\,(n-1)\,(n+2)\,(n+1)} + u\,(-12\,n\,u^2 + 3\,n^2 - 3\,n^4 - 18\,n^3\,u^2 - 23\,n^2\,u^2 - 2\,u^2 + 6\,n + n^4\,u^2 - 6\,n^3)\left(\frac{\partial}{\partial u}\,\mathrm{f}(n,u)\right)/(n\,(n-1)\,(n+2)\,(n+1)) + \frac{u^4\,(2\,n+1)^2\,(1+u^2)\left(\frac{\partial^4}{\partial u^4}\,\mathrm{f}(n,u)\right)}{n\,(n-1)\,(n+2)\,(n+1)}$$

```
> eq[5]:=normalize(plug(eq[1],eq[4],f(n+1,u)));
```

$$eq_5 := -\frac{(2\,n+1)^3\,(-6\,n + 2\,n\,u^2 - u^2 - 9)\,\mathrm{f}(n,u)}{n\,(n-1)\,(n+2)\,(n+1)} + \frac{(2\,n+1)^2\,(4\,n^2 + 4\,n^2\,u^2 + 12\,n - 12\,n\,u^2 + 5\,u^2 + 5)\,u^2\left(\frac{\partial^2}{\partial u^2}\,\mathrm{f}(n,u)\right)}{n\,(n-1)\,(n+2)\,(n+1)} - \frac{2\,(2\,n+1)^2\,(2\,n + 2\,n\,u^2 + 1 - 3\,u^2)\,u^3\left(\frac{\partial^3}{\partial u^3}\,\mathrm{f}(n,u)\right)}{n\,(n-1)\,(n+2)\,(n+1)} + \frac{u^4\,(2\,n+1)^2\,(1+u^2)\left(\frac{\partial^4}{\partial u^4}\,\mathrm{f}(n,u)\right)}{n\,(n-1)\,(n+2)\,(n+1)} + \frac{(2\,n+1)^3\,(-6\,n + 2\,n\,u^2 - u^2 - 9)\,u\left(\frac{\partial}{\partial u}\,\mathrm{f}(n,u)\right)}{n\,(n-1)\,(n+2)\,(n+1)}$$

The following is a purely differential equation satisfied by the integral.

```
> eq[5]:=normalize(primpart(subs(f(n,u)=f(u),eq[5]),diff(f(u),u
  $4)));
```

$$eq_5 := u^2 (4n^2 + 4n^2u^2 + 12n - 12nu^2 + 5u^2 + 5)\left(\frac{\partial^2}{\partial u^2} f(u)\right) + u(2n+1)(-6n + 2nu^2 - u^2 - 9)\left(\frac{\partial}{\partial u} f(u)\right) - 2u^3(2n + 2nu^2 + 1 - 3u^2)\left(\frac{\partial^3}{\partial u^3} f(u)\right) + u^4(1+u^2)\left(\frac{\partial^4}{\partial u^4} f(u)\right) - (2n+1)(-6n + 2nu^2 - u^2 - 9) f(u)$$

There only remains to solve it. First, a general solution is obtained by a call to Maple's general dsolve.

```
> dsol:=op(2,dsolve(eq[5],f(u)));
```

$$dsol := \frac{\_C1\, u}{1+u^2} + \frac{\_C2\, u^3}{1+u^2} + \frac{\_C3\, u^{(2n+3)}}{1+u^2} + \frac{\_C4\, u^{(2n+1)}}{1+u^2}$$

Next, we use initial conditions.

```
> solve({seq(subs(n=i,dsol)=simplify(int(subs(n=i,expr),z=0..in
  finity),symbolic),i=1..4)},{_C1,_C2,_C3,_C4});
```

$$\{\_C4 = \_C4,\ \_C3 = -\frac{\_C4}{u^2},\ \_C1 = 2 - \_C2\, u^2 + 2u^2,\ \_C2 = \_C2\}$$

```
> normal(subs(%,dsol),expanded);
```

$$2u$$

```
>
```

We have just obtained: $\int_0^{\infty} u^n \, \mathbf{e}^{\left(-\frac{u z \left(1 - \frac{1}{u^2}\right)}{2}\right)} (\mathrm{BesselJ}(n, z) + u\, \mathrm{BesselJ}(n-1, z))\, dz = 2u$ whenever $1 \le n$.

```
>
```

The Identity

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \mathrm{binomial}(n,j)\right)^2 = \frac{n\,4^n}{2} + 4^n - \frac{n\,\mathrm{binomial}(2n,n)}{2}$$, **and**

Calkin's "Curious Identity,"

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \mathrm{binomial}(n,j)\right)^3 = \frac{n\,8^n}{2} + 8^n - \frac{3n\,\mathrm{binomial}(2n,n)}{4}$$

```
> restart;
```

```
>
> with(Mgfun);
```

$$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, dfinite\_expr\_to\_rec,$$
$$dfinite\_expr\_to\_sys, diag\_of\_sys, int\_of\_sys, pol\_to\_sys, sum\_of\_sys, sys*sys, sys+sys]$$

```
>
```

Set $p = 2$ for the simple case; requires a few seconds.
Set $p = 3$ for Calkin's identity; requires 2 large minutes due to a weakness in the generic implementation; much less by expert manipulations with lower-level packages.

```
> p:=3:
>
```

Describe the indefinite sum of the binomial coefficients.

$$> sys := [(k-2\,n-1)\,\mathrm{u}(n, k) + (n-k)\,\mathrm{u}(n+1, k) + (k+1)\,\mathrm{u}(n, k+1),$$
$$(-k-1+n)\,\mathrm{u}(n, k) + (-n-1)\,\mathrm{u}(n, k+1) + (k+2)\,\mathrm{u}(n, k+2)]$$

Raise the internal sum to the power p.

```
> sys:=pol_to_sys(h(n,k)=u(n,k)^p,{[u(n,k),sys]});
```

$$sys := \{(-72 + 24\,k - 432\,n + 18\,k^2 + 44\,k\,n - 1048\,n^2 + 1104\,n^7 + 12\,k^4 - 704\,k\,n^3 + 370\,k^2\,n^2$$
$$- 72\,k\,n^2 + 42\,k^2\,n + 10\,k^3 - 672\,n^4 - 1304\,n^3 - 20\,k^5 + 228\,k^4\,n - 946\,k^3\,n^2 + 4388\,k^4\,n^4$$
$$- 2432\,n^7\,k + 1302\,k^6\,n^3 - 438\,k^7\,n^2 + 88\,k^8\,n - 2556\,k^5\,n^4 + 3444\,k^4\,n^5 - 3192\,k^3\,n^6$$
$$- 84\,k^3\,n + 138\,k^6\,n - 6\,k^7 + 128\,n^9 + 1288\,k^4\,n^2 + 2832\,k^4\,n^3 - 5356\,k^3\,n^4 + 6100\,k^2\,n^5$$
$$- 3960\,n^6\,k - 2124\,k^5\,n^3 + 638\,k^6\,n^2 - 108\,k^7\,n + 4912\,k^2\,n^6 - 5828\,k^3\,n^5 - 880\,k^5\,n^2$$
$$+ 1968\,n^7\,k^2 - 736\,n^8\,k + 8\,k^8 + 544\,n^8 - 8\,k^9 - 3964\,n^5\,k - 3220\,n^3\,k^3 + 2002\,n^3\,k^2$$
$$- 2312\,n^4\,k + 4748\,n^4\,k^2 - 300\,k^5\,n + 1248\,n^6 + 504\,n^5 + 34\,k^6)\,\mathrm{h}(n, k) + (-2\,k^2 + 4\,k\,n - 2\,n^2$$
$$+ 18\,n^7 - 2\,k^4 + 52\,k\,n^3 - 52\,k^2\,n^2 + 25\,k\,n^2 - 20\,k^2\,n + 5\,k^3 - 18\,n^4 - 10\,n^3 - 3\,k^5 + 10\,k^4\,n$$
$$- k^3\,n^2 + 380\,k^4\,n^4 - 90\,n^7\,k + 120\,k^6\,n^3 - 45\,k^7\,n^2 + 10\,k^8\,n - 209\,k^5\,n^4 + 246\,k^4\,n^5$$
$$- 195\,k^3\,n^6 + 20\,k^3\,n + 16\,k^6\,n - k^7 + 4\,n^9 + 68\,k^4\,n^2 + 210\,k^4\,n^3 - 295\,k^3\,n^4 + 236\,k^2\,n^5$$
$$- 101\,n^6\,k - 234\,k^5\,n^3 + 90\,k^6\,n^2 - 20\,k^7\,n + 250\,k^2\,n^6 - 392\,k^3\,n^5 - 83\,k^5\,n^2 + 100\,n^7\,k^2$$
$$- 30\,n^8\,k + 2\,k^8 + 14\,n^8 - k^9 - 38\,n^5\,k - 110\,n^3\,k^3 - 28\,n^3\,k^2 + 34\,n^4\,k + 92\,n^4\,k^2 - 20\,k^5\,n$$
$$+ 6\,n^6 - 12\,n^5 + 2\,k^6)\,\mathrm{h}(n+2, k) + (-72 - 144\,k - 336\,n - 94\,k^2 - 556\,k\,n - 744\,n^2 - 48\,n^7$$
$$+ 72\,k^4 - 1072\,k\,n^3 - 262\,k^2\,n^2 - 952\,k\,n^2 - 330\,k^2\,n + 4\,k^3 - 960\,n^4 - 1032\,n^3 + 52\,k^5$$
$$+ 16\,k^4\,n - 280\,k^3\,n^2 - 188\,k^4\,n^4 - 96\,n^7\,k - 170\,k^6\,n^3 + 56\,k^7\,n^2 - 8\,k^8\,n + 292\,k^5\,n^4$$
$$- 300\,k^4\,n^5 + 176\,k^3\,n^6 - 180\,k^3\,n + 50\,k^6\,n - 8\,k^7 - 100\,k^4\,n^2 + 68\,k^4\,n^3 - 236\,k^3\,n^4$$
$$+ 340\,k^2\,n^5 - 304\,n^6\,k + 208\,k^5\,n^3 - 126\,k^6\,n^2 + 48\,k^7\,n + 112\,k^2\,n^6 + 20\,k^3\,n^5 - 56\,k^5\,n^2$$
$$- 48\,n^7\,k^2 - 8\,k^8 - 580\,n^5\,k - 176\,n^3\,k^3 + 94\,n^3\,k^2 - 904\,n^4\,k + 300\,n^4\,k^2 + 80\,k^5\,n - 240\,n^6$$
$$- 600\,n^5 + 6\,k^6)\,\mathrm{h}(n, k+1) + (54\,k^2 - 108\,k\,n + 54\,n^2 - 276\,n^7 - 15\,k^4 - 624\,k\,n^3 + 333\,k^2\,n^2$$
$$- 459\,k\,n^2 + 270\,k^2\,n - 27\,k^3 + 300\,n^4 + 216\,n^3 + 21\,k^5 - 167\,k^4\,n + 449\,k^3\,n^2 - 3613\,k^4\,n^4$$
$$+ 1036\,n^7\,k - 1087\,k^6\,n^3 + 394\,k^7\,n^2 - 84\,k^8\,n + 1949\,k^5\,n^4 - 2361\,k^4\,n^5 + 1937\,k^3\,n^6 + 6\,k^3\,n$$

$$- 133 k^6 n + 10 k^7 - 48 n^9 - 576 k^4 n^2 - 1836 k^4 n^3 + 2909 k^3 n^4 - 2693 k^2 n^5 + 1341 n^6 k$$
$$+ 2208 k^5 n^3 - 861 k^6 n^2 + 196 k^7 n - 2628 k^2 n^6 + 3862 k^3 n^5 + 678 k^5 n^2 - 1040 n^7 k^2$$
$$+ 332 n^8 k - 20 k^8 - 180 n^8 + 8 k^9 + 872 n^5 k + 1368 n^3 k^3 - 373 n^3 k^2 - 38 n^4 k - 1595 n^4 k^2$$
$$+ 112 k^5 n - 174 n^6 + 108 n^5 - 7 k^6) \mathrm{h}(n + 1, k) + (144 + 120 k + 768 n - 228 k^2 + 1120 k n$$
$$+ 1488 n^2 + 164 k^4 + 3440 k n^3 + 556 k^2 n^2 + 3096 k n^2 - 592 k^2 n - 118 k^3 + 384 n^4$$
$$+ 1248 n^3 - 8 k^5 + 452 k^4 n - 2302 k^3 n^2 + 384 k^4 n^4 - 116 k^6 n^3 + 110 k^7 n^2 - 48 k^8 n$$
$$+ 48 k^5 n^4 - 1064 k^3 n - 148 k^6 n - 2 k^7 - 932 k^4 n^2 - 1652 k^4 n^3 + 1200 k^3 n^4 - 772 k^5 n^3$$
$$+ 616 k^6 n^2 - 228 k^7 n + 824 k^5 n^2 + 32 k^8 + 8 k^9 - 508 n^3 k^3 + 2680 n^3 k^2 + 1344 n^4 k$$
$$+ 1824 n^4 k^2 + 604 k^5 n - 112 k^6) \mathrm{h}(n, k + 2), (-36 + 84 k - 252 n - 84 k^2 + 484 k n - 716 n^2$$
$$- 8 k^4 + 1228 k n^3 - 652 k^2 n^2 + 1100 k n^2 - 388 k^2 n + 44 k^3 - 864 n^4 - 1060 n^3 - 16 k^4 n$$
$$+ 148 k^3 n^2 + 140 k^3 n - 8 k^4 n^2 + 144 n^5 k + 52 n^3 k^3 - 476 n^3 k^2 + 672 n^4 k - 128 n^4 k^2$$
$$- 64 n^6 - 368 n^5) \mathrm{h}(n, k) + (9 - 6 k + 48 n - 12 k^2 - 2 k n + 93 n^2 - k^4 + 98 k n^3 - 116 k^2 n^2$$
$$+ 54 k n^2 - 70 k^2 n + 14 k^3 + 24 n^4 + 78 n^3 - 8 k^5 + 30 k^4 n - 22 k^3 n^2 + 34 k^3 n + 55 k^4 n^2$$
$$- 58 n^3 k^3 - 38 n^3 k^2 + 48 n^4 k + 24 n^4 k^2 - 24 k^5 n + 4 k^6) \mathrm{h}(n + 1, k + 1) + (-k^2 + 2 k n - n^2$$
$$- 6 k^4 + 33 k n^3 - 45 k^2 n^2 + 14 k n^2 - 13 k^2 n + 4 k^3 - 9 n^4 - 5 n^3 + 4 k^5 - 23 k^4 n + 52 k^3 n^2$$
$$+ 27 k^3 n - 20 k^4 n^2 + 11 n^5 k + 30 n^3 k^3 - 58 n^3 k^2 + 32 n^4 k - 25 n^4 k^2 + 7 k^5 n - 2 n^6 - 7 n^5$$
$$- k^6) \mathrm{h}(n + 2, k) + (-36 k + 28 k^2 - 268 k n^2 - 4 k^2 n^2 - 188 k n^3 - 132 n + 68 k^3 n - 180 n^2$$
$$+ 76 k^3 n^2 + 52 k^2 n - 164 k n + 28 n^3 k^3 - 8 k^4 - 24 n^4 k^2 - 24 n^4 + 20 k^3 - 52 n^3 k^2 - 36$$
$$- 48 n^4 k - 16 k^4 n - 8 k^4 n^2 - 108 n^3) \mathrm{h}(n, k + 1) + (27 k^2 - 54 k n + 27 n^2 + 51 k^4 - 492 k n^3$$
$$+ 558 k^2 n^2 - 270 k n^2 + 216 k^2 n - 54 k^3 + 159 n^4 + 108 n^3 - 24 k^5 + 158 k^4 n - 422 k^3 n^2$$
$$- 276 k^3 n + 107 k^4 n^2 - 106 n^5 k - 192 n^3 k^3 + 568 n^3 k^2 - 382 n^4 k + 195 n^4 k^2 - 32 k^5 n$$
$$+ 24 n^6 + 102 n^5 + 4 k^6) \mathrm{h}(n + 1, k), (-4032 + 14016 k - 40224 n - 24192 k^2 + 126880 k n$$
$$- 176784 n^2 - 307608 n^7 - 18624 k^4 + 1133816 k n^3 - 681264 k^2 n^2 + 501456 k n^2$$
$$- 196160 k^2 n + 26240 k^3 - 739920 n^4 - 451032 n^3 + 8640 k^5 - 110752 k^4 n + 545616 k^3 n^2$$
$$- 228080 k^4 n^4 + 344168 n^7 k - 6016 k^6 n^3 + 512 k^7 n^2 + 29664 k^5 n^4 - 80720 k^4 n^5$$
$$+ 132288 k^3 n^6 + 128 k^7 n^3 + 184768 k^3 n - 8192 k^6 n + 256 k^7 - 18624 n^9 - 1536 n^{10}$$
$$- 266880 k^4 n^2 - 333648 k^4 n^3 + 818416 k^3 n^4 - 1161536 k^2 n^5 + 906432 n^6 k + 68064 k^5 n^3$$
$$- 10688 k^6 n^2 + 640 k^7 n - 524192 k^2 n^6 + 447680 k^3 n^5 + 75744 k^5 n^2 - 131688 n^7 k^2$$
$$+ 6976 n^9 k + 74400 n^8 k + 4992 k^5 n^5 - 1216 k^6 n^4 - 11536 k^4 n^6 + 16280 k^3 n^7$$
$$- 14096 n^8 k^2 - 99408 n^8 + 1498560 n^5 k + 873832 n^3 k^3 - 1322648 n^3 k^2 + 1614096 n^4 k$$
$$- 1568288 n^4 k^2 + 40992 k^5 n - 611280 n^6 - 815472 n^5 - 2304 k^6) \mathrm{h}(n, k) + (1176 k - 1176 n$$
$$- 6440 k^2 + 16604 k n - 10164 n^2 - 53130 n^7 - 23848 k^4 + 242425 k n^3 - 268934 k^2 n^2$$

$$
\begin{aligned}
&+87402\,k\,n^2-65352\,k^2\,n+15960\,k^3-80871\,n^4-38010\,n^3+23848\,k^5-161816\,k^4\,n\\
&+439532\,k^3\,n^2-449287\,k^4\,n^4+114547\,n^7\,k-70324\,k^6\,n^3+23824\,k^7\,n^2-4720\,k^8\,n\\
&+133968\,k^5\,n^4-170994\,k^4\,n^5+146346\,k^3\,n^6-2248\,k^8\,n^2+7492\,k^7\,n^3+400\,k^9\,n\\
&+131228\,k^3\,n-32\,k^{10}-70552\,k^6\,n+7752\,k^7-3822\,n^9-336\,n^{10}-437616\,k^4\,n^2\\
&-604060\,k^4\,n^3+796120\,k^3\,n^4-632176\,k^2\,n^5+279732\,n^6\,k+278075\,k^5\,n^3-108110\,k^6\,n^2\\
&+24116\,k^7\,n-305261\,k^2\,n^6+467159\,k^3\,n^5+276314\,k^5\,n^2-81016\,n^7\,k^2+2594\,n^9\,k\\
&+26322\,n^8\,k+24771\,k^5\,n^5-16422\,k^6\,n^4-26071\,k^4\,n^6+18923\,k^3\,n^7-9071\,n^8\,k^2-2360\,k^8\\
&-18879\,n^8+416\,k^9+420870\,n^5\,k+780648\,n^3\,k^3-595990\,n^3\,k^2+402408\,n^4\,k\\
&-784828\,n^4\,k^2+131268\,k^5\,n-93870\,n^6-107982\,n^5-16472\,k^6)\,\mathrm{h}(n+2,k)+(-72\,k+72\,n\\
&+444\,k^2-1080\,k\,n+636\,n^2+3654\,n^7+2021\,k^4-16916\,k\,n^3+20086\,k^2\,n^2-5914\,k\,n^2\\
&+4714\,k^2\,n-1230\,k^3+5277\,n^4+2430\,n^3-2178\,k^5+14066\,k^4\,n-36048\,k^3\,n^2\\
&+41470\,k^4\,n^4-8682\,n^7\,k+7264\,k^6\,n^3-2560\,k^7\,n^2+526\,k^8\,n-13240\,k^5\,n^4+16068\,k^4\,n^5\\
&-12976\,k^3\,n^6+259\,k^8\,n^2-828\,k^7\,n^3-48\,k^9\,n-10468\,k^3\,n+4\,k^{10}+6974\,k^6\,n-792\,k^7\\
&+270\,n^9+24\,n^{10}+38878\,k^4\,n^2+54732\,k^4\,n^3-68186\,k^3\,n^4+50728\,k^2\,n^5-20856\,n^6\,k\\
&-26942\,k^5\,n^3+10917\,k^6\,n^2-2524\,k^7\,n+24929\,k^2\,n^6-40740\,k^3\,n^5-26254\,k^5\,n^2\\
&+6720\,n^7\,k^2-202\,n^9\,k-2024\,n^8\,k-2498\,k^5\,n^5+1737\,k^6\,n^4+2493\,k^4\,n^6-1704\,k^3\,n^7\\
&+763\,n^8\,k^2+255\,k^8+1317\,n^8-48\,k^9-30792\,n^5\,k-65528\,n^3\,k^3+45784\,n^3\,k^2-28806\,n^4\,k\\
&+61724\,n^4\,k^2-12232\,k^5\,n+6354\,n^6+7182\,n^5+1596\,k^6)\,\mathrm{h}(n+3,k)+(4032+4032\,k\\
&+22176\,n-6016\,k^2+28704\,k\,n+51408\,n^2+504\,n^7+4288\,k^4+121944\,k\,n^3-27088\,k^2\,n^2\\
&+81552\,k\,n^2-23104\,k^2\,n-3968\,k^3+48384\,n^4+65016\,n^3+704\,k^5+14752\,k^4\,n\\
&-64112\,k^3\,n^2-12048\,k^4\,n^4+1512\,n^7\,k-3456\,k^6\,n^3+512\,k^7\,n^2+8096\,k^5\,n^4-7344\,k^4\,n^5\\
&+864\,k^3\,n^6+128\,k^7\,n^3-26304\,k^3\,n-4608\,k^6\,n+256\,k^7+15104\,k^4\,n^2-1456\,k^4\,n^3\\
&-43376\,k^3\,n^4+29696\,k^2\,n^5+13728\,n^6\,k+16544\,k^5\,n^3-6080\,k^6\,n^2+640\,k^7\,n\\
&+10944\,k^2\,n^6-9696\,k^3\,n^5+15392\,k^5\,n^2+1512\,n^7\,k^2+1472\,k^5\,n^5-704\,k^6\,n^4-1392\,k^4\,n^6\\
&+504\,k^3\,n^7+51744\,n^5\,k-75096\,n^3\,k^3+3416\,n^3\,k^2+104592\,n^4\,k+33680\,n^4\,k^2+6176\,k^5\,n\\
&+5040\,n^6+21168\,n^5-1280\,k^6)\,\mathrm{h}(n,k+1)+(-7056\,k+7056\,n+26880\,k^2-83160\,k\,n\\
&+56280\,n^2+229572\,n^7+63232\,k^4-1013746\,k\,n^3+954512\,k^2\,n^2-393596\,k\,n^2\\
&+248528\,k^2\,n-51744\,k^3+395682\,n^4+196812\,n^3-52560\,k^5+406304\,k^4\,n\\
&-1269372\,k^3\,n^2+970232\,k^4\,n^4-403938\,n^7\,k+98880\,k^6\,n^3-26880\,k^7\,n^2+4096\,k^8\,n\\
&-229104\,k^5\,n^4+350608\,k^4\,n^5-356948\,k^3\,n^6+1408\,k^8\,n^2-6976\,k^7\,n^3-128\,k^9\,n\\
&-399680\,k^3\,n+119680\,k^6\,n-11392\,k^7+15624\,n^9+1344\,n^{10}+1044208\,k^4\,n^2\\
&+1371520\,k^4\,n^3-2101936\,k^3\,n^4+1952296\,k^2\,n^5-1018092\,n^6\,k-510888\,k^5\,n^3
\end{aligned}
$$

$$+168288\,k^{6}\,n^{2}-31552\,k^{7}\,n+910788\,k^{2}\,n^{6}-1184660\,k^{3}\,n^{5}-541648\,k^{5}\,n^{2}+234320\,n^{7}\,k^{2}$$
$$-8696\,n^{9}\,k-90340\,n^{8}\,k-39104\,k^{5}\,n^{5}+20448\,k^{6}\,n^{4}+50632\,k^{4}\,n^{6}-44426\,k^{3}\,n^{7}$$
$$+25498\,n^{8}\,k^{2}+2560\,k^{8}+79170\,n^{8}-256\,k^{9}-1590060\,n^{5}\,k-2151234\,n^{3}\,k^{3}$$
$$+2002648\,n^{3}\,k^{2}-1590516\,n^{4}\,k+2520114\,n^{4}\,k^{2}-273192\,k^{5}\,n+420084\,n^{6}+503496\,n^{5}$$
$$+30080\,k^{6})\,\mathrm{h}(n+1,k)\}$$

Creative telescoping of the outer sum.

```
> infolevel[Mgfun]:=5: infolevel[syzygies]:=4:
  ct:=creative_telescoping(LFSol(sys),n::shift,k::shift,{"start"=p
  -1,"stop"=p-1}):
  infolevel[Mgfun]:=0: infolevel[syzygies]:=0:
term_order["gbasis"]   "introduce 3 polynomials one after another"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "remove syzygies"
term_order["gbasis"]   2   " syzygies left; "   1   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^2   dx^2   ") ="
dx^2
term_order["gbasis"]   "add polynomial with head term"   dx*dx
term_order["gbasis"]   2   " syzygies left; "   2   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^2   dx*dx   ") ="
dx*dx^2
term_order["gbasis"]   "add polynomial with head term"   dx^3
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^3   dx^2   ") ="
dx^3
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   1   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx*dx   dx^3   ") ="
dx^3*dx
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   "perform inter-reduction"
term_order["reduce_list"]   "inter-reduce 3 generators"
term_order["reduce_list"]   "step 1"
term_order["reduce_list"]   "step 2"
term_order["reduce_list"]   "step 3"
term_order["gbasis"]   "introduce 3 polynomials one after another"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "remove syzygies"
term_order["gbasis"]   2   " syzygies left; "   1   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^2   dx^2   ") ="
dx^2
term_order["gbasis"]   "add polynomial with head term"   dx*dx
term_order["gbasis"]   2   " syzygies left; "   2   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^2   dx*dx   ") ="
dx*dx^2
term_order["gbasis"]   "add polynomial with head term"   dx^3
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^3   dx^2   ") ="
dx^3
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   1   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx*dx   dx^3   ") ="
dx^3*dx
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   "perform inter-reduction"
term_order["reduce_list"]   "inter-reduce 3 generators"
term_order["reduce_list"]   "step 1"
term_order["reduce_list"]   "step 2"
term_order["reduce_list"]   "step 3"
term_order["gbasis"]   "introduce 3 polynomials one after another"
```

```
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "remove syzygies"
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^2   dx*dx   ") ="
dx^2*dx
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   1   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx^3   dx*dx   ") ="
dx*dx^3
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   "perform inter-reduction"
term_order["reduce_list"]   "inter-reduce 3 generators"
term_order["reduce_list"]   "step 1"
term_order["reduce_list"]   "step 2"
term_order["reduce_list"]   "step 3"
Mgfun/chyzak97:   "Suitable term order guessed"
term_order["gbasis"]   "introduce 3 polynomials one after another"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "remove syzygies"
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx*dx   dx^2   ") ="
dx^2*dx
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   1   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   dx*dx   dx^3   ") ="
dx*dx^3
term_order["gbasis"]   "reduction to zero"
term_order["gbasis"]   "perform inter-reduction"
term_order["reduce_list"]   "inter-reduce 3 generators"
term_order["reduce_list"]   "step 1"
term_order["reduce_list"]   "step 2"
term_order["reduce_list"]   "step 3"
Mgfun/chyzak97:   "Implicit d-finite expression recognized"
Mgfun/chyzak97:   "Dimension is 4"
Mgfun/chyzak97:   "Start of Chyzak's algorithm"
Mgfun/chyzak97:   "Preparation of the system: 3.890 seconds."
Mgfun/uncoupling:   "Uncoupling of the LOF system"
term_order["gbasis"]   "introduce 5 polynomials one after another"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "remove syzygies"
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[4]*Sk   s[4]*Sk   "
) ="   s[4]*Sk
term_order["gbasis"]   "add polynomial with head term"   s[4]
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[4]*Sk   s[4]   ") =
"   s[4]*Sk
term_order["gbasis"]   "add polynomial with head term"   s[2]*Sk^2
term_order["gbasis"]   1   " syzygies left; "   4   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[1]*Sk   s[1]   ") =
"   s[1]*Sk
term_order["gbasis"]   "add polynomial with head term"   Sk*s[2]
term_order["gbasis"]   1   " syzygies left; "   4   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[2]*Sk^2   Sk*s[2]
") ="   s[2]*Sk^2
term_order["gbasis"]   "add polynomial with head term"   s[2]
term_order["gbasis"]   1   " syzygies left; "   4   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   Sk*s[2]   s[2]   ") =
"   Sk*s[2]
term_order["gbasis"]   "add polynomial with head term"   s[3]*Sk^4
term_order["gbasis"]   "perform inter-reduction"
term_order["reduce_list"]   "inter-reduce 5 generators"
term_order["reduce_list"]   "step 1"
term_order["reduce_list"]   "step 2"
```

```
term_order["reduce_list"]   "step 3"
term_order["reduce_list"]   "step 4"
term_order["reduce_list"]   "step 5"
Mgfun/chyzak97:   "Uncoupling of the system: 55.250 seconds."
Mgfun/rational_sys_solve:   "Look for rational solutions of a system"
`Mgfun/denominator_bound`[shift]   "Intermediate bound on dispersion: infinity"
`Mgfun/denominator_bound`[shift]   "Bound on dispersion: 0"
`Mgfun/denominator_bound`[shift]   "Computing a resultant: .390 seconds."
`Mgfun/rational_solve`[shift]   "Computing denominator bound: .539 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 1"
Mgfun/rational_sys_solve:   "Solving equation: 5.969 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 2"
Mgfun/rational_sys_solve:   "Solving equation: .80e-1 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 3"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/rational_sys_solve:   "Equations solved: 4"
Mgfun/rational_sys_solve:   "Solving equation: 0. seconds."
Mgfun/chyzak97:   "No LOFE of order 1 :-("
Mgfun/chyzak97:   "Preparation of the system: 6.881 seconds."
Mgfun/uncoupling:   "Uncoupling of the LOF system"
term_order["gbasis"]   "introduce 5 polynomials one after another"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "next polynomial"
term_order["gbasis"]   "remove syzygies"
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[4]*Sk   s[4]*Sk   "
) ="   s[4]*Sk
term_order["gbasis"]   "add polynomial with head term"   s[4]
term_order["gbasis"]   2   " syzygies left; "   3   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[4]*Sk   s[4]   ") =
"   s[4]*Sk
term_order["gbasis"]   "add polynomial with head term"   s[2]*Sk^2
term_order["gbasis"]   1   " syzygies left; "   4   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[1]*Sk   s[1]   ") =
"   s[1]*Sk
term_order["gbasis"]   "add polynomial with head term"   Sk*s[2]
term_order["gbasis"]   1   " syzygies left; "   4   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   s[2]*Sk^2   Sk*s[2]
") ="   s[2]*Sk^2
term_order["gbasis"]   "add polynomial with head term"   s[2]
term_order["gbasis"]   1   " syzygies left; "   4   " polynomials"
term_order["gbasis"]   "deal with pair of head terms ("   Sk*s[2]   s[2]   ") =
"   Sk*s[2]
term_order["gbasis"]   "add polynomial with head term"   s[3]*Sk^4
term_order["gbasis"]   "perform inter-reduction"
term_order["reduce_list"]   "inter-reduce 5 generators"
term_order["reduce_list"]   "step 1"
term_order["reduce_list"]   "step 2"
term_order["reduce_list"]   "step 3"
term_order["reduce_list"]   "step 4"
term_order["reduce_list"]   "step 5"
Mgfun/chyzak97:   "Uncoupling of the system: 72.541 seconds."
Mgfun/rational_sys_solve:   "Look for rational solutions of a system"
`Mgfun/denominator_bound`[shift]   "Intermediate bound on dispersion: infinity"
`Mgfun/denominator_bound`[shift]   "Bound on dispersion: 0"
`Mgfun/denominator_bound`[shift]   "Computing a resultant: .500 seconds."
`Mgfun/rational_solve`[shift]   "Computing denominator bound: .680 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 1"
Mgfun/rational_sys_solve:   "Solving equation: 18.100 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 2"
Mgfun/rational_sys_solve:   "Solving equation: 11.000 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 3"
Mgfun/rational_sys_solve:   "Solving equation: .511 seconds."
Mgfun/rational_sys_solve:   "Equations solved: 4"
Mgfun/rational_sys_solve:   "Solving equation: .489 seconds."
Mgfun/chyzak97:   "LOFE of order 2 found!"
> ct[1];
```

$$(8n+4)\_f(n,k)+(-n-1)\_f(n+2,k)+(7n+12)\_f(n+1,k)$$

```
> ct[2];
```

$$2\,\frac{(2k^3-3k^2n-k^2+11kn+6kn^2+5k-26n^2-8-9n^3-25n)\_f(n,k)}{-5kn-3k+3n^2+2k^2+3n}-((2k^4-5k^3n-3k^3-k^2n^2-4k^2n-3k^2+12kn^3+34kn^2+32kn+10k-9n^4-35n^3-51n^2-33n-8)\_f(n+2,k))\Big/(4k^3-16k^2n-12k^2+23kn^2+38kn+15k-12n^3-33n^2-30n-9)+2\,(9n^2+17n+8+27kn^2+44kn+19k+27k^2n^2+30k^2n+11k^2+9k^3n^2-4k^3n-k^3-7k^4n+k^4+2k^5)\_f(n,k+1)\Big/(-3k+6k^2-5k^3+2k^4+3n-19kn+23k^2n-11k^3n+13n^2-36kn^2+23k^2n^2+18n^3-22kn^3+8n^4)-\frac{(k^2-5kn-4k+9n^2+17n+8)\_f(n+1,k)}{k-2n-1}$$

The identity obtained by creative telescoping is

$$ct_1(n,k)+ct_2(n,k+1)-ct_2(n,k)=0$$

so that upon summation with respect to k over $0\,..\,n+p-1$, we obtain

$$\left(\sum_{k=0}^{n+p-1} ct_1(n,k)\right)+ct_2(n,k+p)-ct_2(n,0)=0.$$

Thus, ct_2 induces a nonhomogeneous term which we now compute.

```
> subs(k=n+p,ct[2])-subs(k=0,ct[2]);
```

$$2\,((2(n+3)^3-3(n+3)^2n-(n+3)^2+11(n+3)n+6(n+3)n^2-20n+7-26n^2-9n^3)\_f(n,n+3))\Big/(-5(n+3)n-9+3n^2+2(n+3)^2)-((2(n+3)^4-5(n+3)^3n-3(n+3)^3-(n+3)^2n^2-4(n+3)^2n-3(n+3)^2+12(n+3)n^3+34(n+3)n^2+32(n+3)n-23n+22-9n^4-35n^3-51n^2)\_f(n+2,n+3))\Big/(4(n+3)^3-16(n+3)^2n-12(n+3)^2+23(n+3)n^2+38(n+3)n-15n+36-12n^3-33n^2)+2\,(9n^2+36n+65+27(n+3)n^2+44(n+3)n+27(n+3)^2n^2+30(n+3)^2n+11(n+3)^2+9(n+3)^3n^2-4(n+3)^3n-(n+3)^3-7(n+3)^4n+(n+3)^4+2(n+3)^5)\_f(n,n+4)\Big/(-9+6(n+3)^2-5(n+3)^3+2(n+3)^4-19(n+3)n+23(n+3)^2n-11(n+3)^3n+13n^2-36(n+3)n^2+23(n+3)^2n^2+18n^3-22(n+3)n^3+8n^4)-\frac{((n+3)^2-5(n+3)n+13n-4+9n^2)\_f(n+1,n+3)}{-n+2}-\frac{2(-9n^3-25n-26n^2-8)\_f(n,0)}{3n+3n^2}+\frac{(-33n-8-9n^4-35n^3-51n^2)\_f(n+2,0)}{-30n-9-12n^3-33n^2}-\frac{2(17n+8+9n^2)\_f(n,1)}{8n^4+18n^3+3n+13n^2}+\frac{(17n+8+9n^2)\_f(n+1,0)}{-2n-1}$$

```
> value(eval(subs(_f=unapply(Sum(binomial(n,j),j=0..k)^p,n,k),%)))
```

;

$$2\,((2\,(n+3)^3-3\,(n+3)^2\,n-(n+3)^2+11\,(n+3)\,n+6\,(n+3)\,n^2-20\,n+7-26\,n^2-9\,n^3)\,(2^n+\mathrm{binomial}(n,n+1)+\mathrm{binomial}(n,n+2)+\mathrm{binomial}(n,n+3))^3)\Big/\big(-5\,(n+3)\,n-9+3\,n^2+2\,(n+3)^2\big)-\Big((2\,(n+3)^4-5\,(n+3)^3\,n-3\,(n+3)^3-(n+3)^2\,n^2-4\,(n+3)^2\,n-3\,(n+3)^2+12\,(n+3)\,n^3+34\,(n+3)\,n^2+32\,(n+3)\,n-23\,n+22-9\,n^4-35\,n^3-51\,n^2)\,(2^{(n+2)}+\mathrm{binomial}(n+2,n+3))^3\Big)\Big/\big(4\,(n+3)^3-16\,(n+3)^2\,n-12\,(n+3)^2+23\,(n+3)\,n^2+38\,(n+3)\,n-15\,n+36-12\,n^3-33\,n^2\big)+2\,(9\,n^2+36\,n+65+27\,(n+3)\,n^2+44\,(n+3)\,n+27\,(n+3)^2\,n^2+30\,(n+3)^2\,n+11\,(n+3)^2+9\,(n+3)^3\,n^2-4\,(n+3)^3\,n-(n+3)^3-7\,(n+3)^4\,n+(n+3)^4+2\,(n+3)^5)\,(2^n+\mathrm{binomial}(n,n+1)+\mathrm{binomial}(n,n+2)+\mathrm{binomial}(n,n+3)+\mathrm{binomial}(n,n+4))^3\Big/\big(-9+6\,(n+3)^2-5\,(n+3)^3+2\,(n+3)^4-19\,(n+3)\,n+23\,(n+3)^2\,n-11\,(n+3)^3\,n+13\,n^2-36\,(n+3)\,n^2+23\,(n+3)^2\,n^2+18\,n^3-22\,(n+3)\,n^3+8\,n^4\big)-\Big(((n+3)^2-5\,(n+3)\,n+13\,n-4+9\,n^2)\,(2^{(n+1)}+\mathrm{binomial}(n+1,n+2)+\mathrm{binomial}(n+1,n+3))^3\Big)/(-n+2)-\frac{2\,(-9\,n^3-25\,n-26\,n^2-8)}{3\,n+3\,n^2}+\frac{-33\,n-8-9\,n^4-35\,n^3-51\,n^2}{-30\,n-9-12\,n^3-33\,n^2}-\frac{2\,(17\,n+8+9\,n^2)\,(n+1)^3}{8\,n^4+18\,n^3+3\,n+13\,n^2}+\frac{17\,n+8+9\,n^2}{-2\,n-1}$$

```
> nh:=simplify(normal(expand(%)));
```

$$nh := -2\,8^n\,(27\,n+62)$$

The following recurrence cancels *nh*.

```
> op(dfinite_expr_to_rec(nh,_g(n)));
```

$$(-216\,n-712)\,\_g(n)+(27\,n+62)\,\_g(n+1)$$

We now sum ct_1 over 0 .. $n+p-1$; mind that $\_f(n,n+1)$ is not zero, but $\_f(n,n)+(2^p)^n$.

```
> [seq(_f(n+i,k)=_f(n+i)+(2^p)^i*(2^p)^n*(p-1-i),i=0..p-1),seq(_f(
  n+i,0)=1,i=0..p-1)];
```

$$[\_f(n,k)=\_f(n)+2\,8^n,\ \_f(n+1,k)=\_f(n+1)+8\,8^n,\ \_f(n+2,k)=\_f(n+2),\ \_f(n,0)=1,\ \_f(n+1,0)=1,\ \_f(n+2,0)=1]$$

```
> expand(eval(subs(_g=unapply(subs(%,ct[1]),n),%%)));
```

$$-1728\,n^2\,\_f(n)-6560\,n\,\_f(n)+405\,\_f(n+2)\,n^2+1875\,\_f(n+2)\,n-1296\,n^2\,\_f(n+1)-6756\,n\,\_f(n+1)-2848\,\_f(n)+13248\,8^n+1890\,\_f(n+2)-7800\,\_f(n+1)-116\,\_f(n+3)\,n-27\,\_f(n+3)\,n^2-124\,\_f(n+3)$$

First get an equation with hypergeometric non-homogeneous part.

```
> rec:=collect(primpart(%,_f(n+p)),_f,factor);
```

$$rec := -(27\,n+62)\,(n+2)\,\_f(n+3)+(-6756\,n-1296\,n^2-7800)\,\_f(n+1)$$

$$-32\,(27\,n+89)\,(2\,n+1)\,\_f(n)+(1875\,n+405\,n^2+1890)\,\_f(n+2)+13248\,8^n$$

Petkovsek's algorithm solves it for hypergeometric term solutions, but the Maple implementation of it does not. We thus transform it into a homogeneous equation.

```
> remove(has,rec,(2^p)^n);
```

$$-(27\,n+62)\,(n+2)\,\_f(n+3)+(-6756\,n-1296\,n^2-7800)\,\_f(n+1)$$
$$-32\,(27\,n+89)\,(2\,n+1)\,\_f(n)+(1875\,n+405\,n^2+1890)\,\_f(n+2)$$

```
> hrec:=collect(subs(n=n+1,%)-(2^p)*%,_f(n+p),_f,factor);
```

$$hrec := (27\,n+89)\,(23\,n+58)\,\_f(n+3)+(27\,n+89)\,(-168\,\_f(n+2)\,n-\_f(n+4)\,n$$
$$+512\,n\,\_f(n)+320\,n\,\_f(n+1)-3\,\_f(n+4)+256\,\_f(n)-348\,\_f(n+2)+576\,\_f(n+1))$$

For the final resolution, we compute and input initial conditions.

```
> {seq(_f(n)=add(add(binomial(n,j),j=0..k)^p,k=0..n),n=0..p+1)};
```

$$\{\_f(0)=1,\ \_f(2)=92,\ \_f(1)=9,\ \_f(3)=920,\ \_f(4)=8928\}$$

```
> sol:=LREtools[hypergeomsols](hrec,_f(n),%,output=gensol);
```

$$sol := 8^n - \frac{3}{4}\,\frac{8^n \left(\frac{1}{2}\right)^n 2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)\sqrt{\pi}} + \frac{1}{2}\,8^n\,n$$

```
> sol:=subs([(1/2)^n=1/2^n,GAMMA(n+1/2)=GAMMA(n)*n*Pi^(1/2)*binomial(2*n,n)/4^n],sol);
```

$$sol := 8^n - \frac{3}{4}\,\frac{8^n\,n\,\mathrm{binomial}(2\,n,n)}{4^n} + \frac{1}{2}\,8^n\,n$$

```
>
>
```