

A Class of Conditional Independent Branching Particle Systems and Their Interacting Limit Superprocesses

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Outline

Super-Brownian motion as a special Dawson-Watanabe processes

A new model for interacting branching particle systems

State classification, a new class of SPDE for density processes

Purely-atomic superprocess and a degenerate SPDE

A generalized new model and SDSM

Location dependent state classification

*Singular and degenerate SPDE and coalescing Brownian motion

*Immigration SDSM and their excursion representation

(*) We will discuss if time available

Classical Model for Super-Brownian Motion

What is the super-Brownian motion?

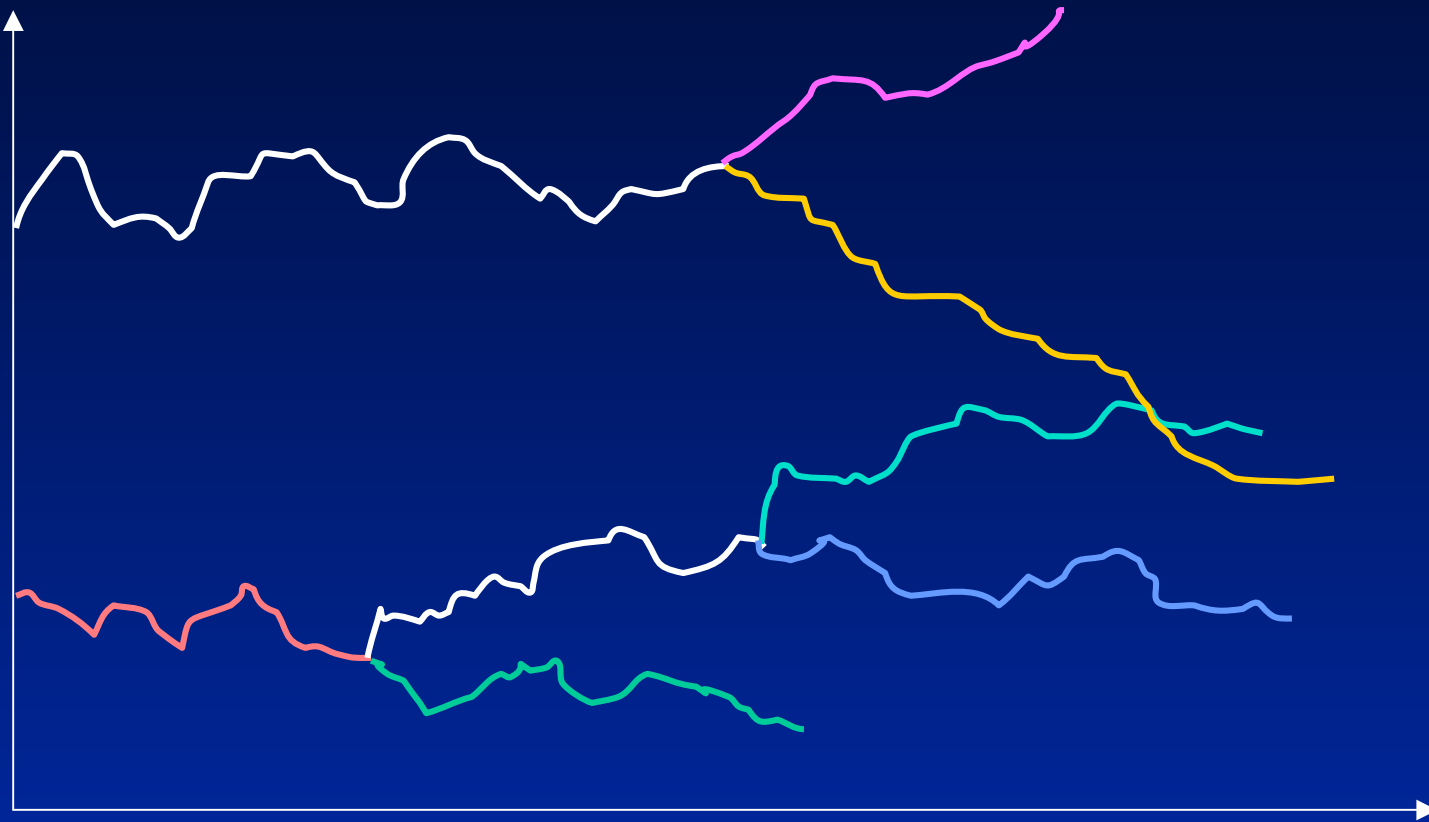
Starting from a simplest branching particle system

Intuitive ideas and graphs

Rigorously mathematical construction

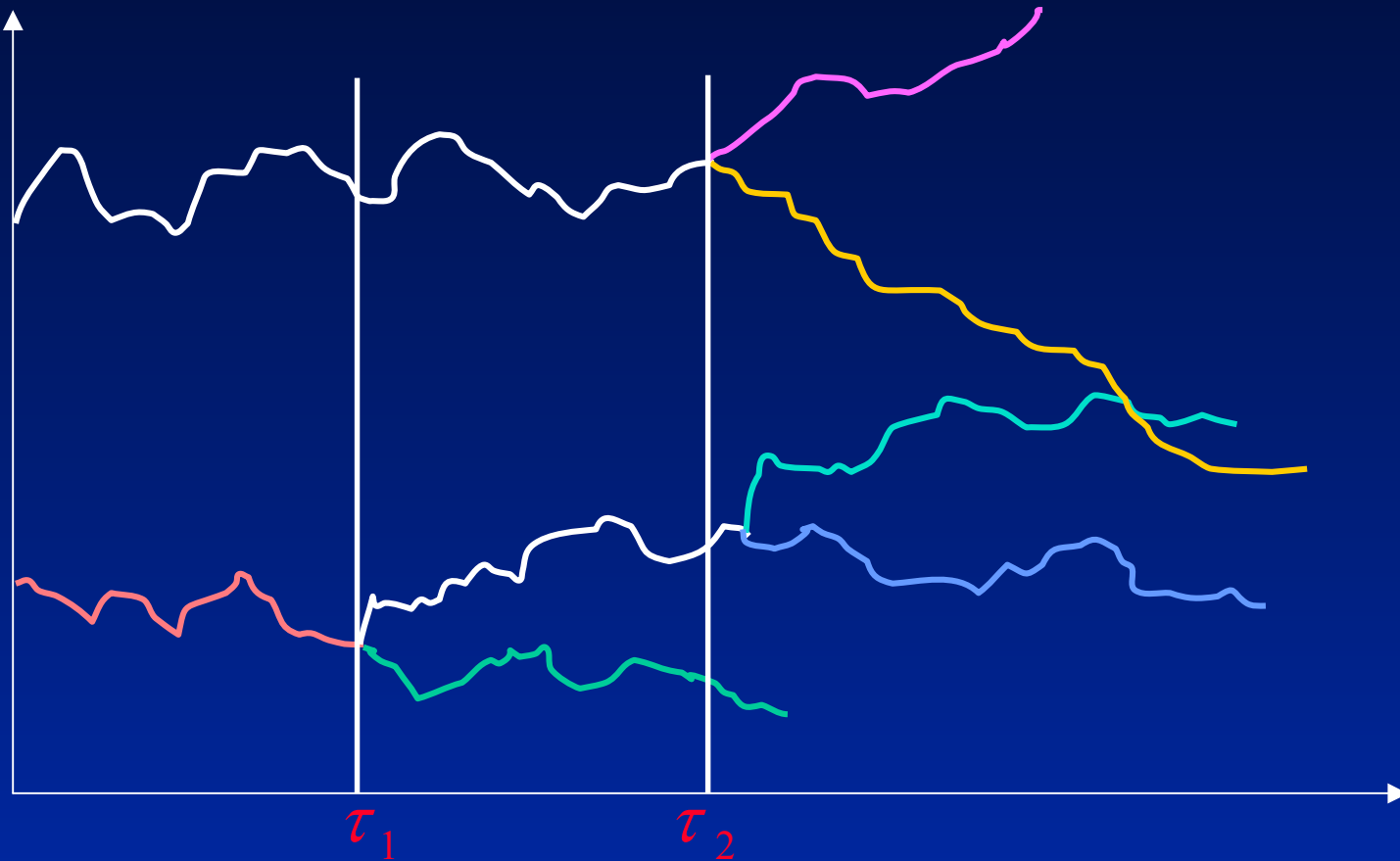
Brownian branching particle systems

Brownian binary branching particle trees



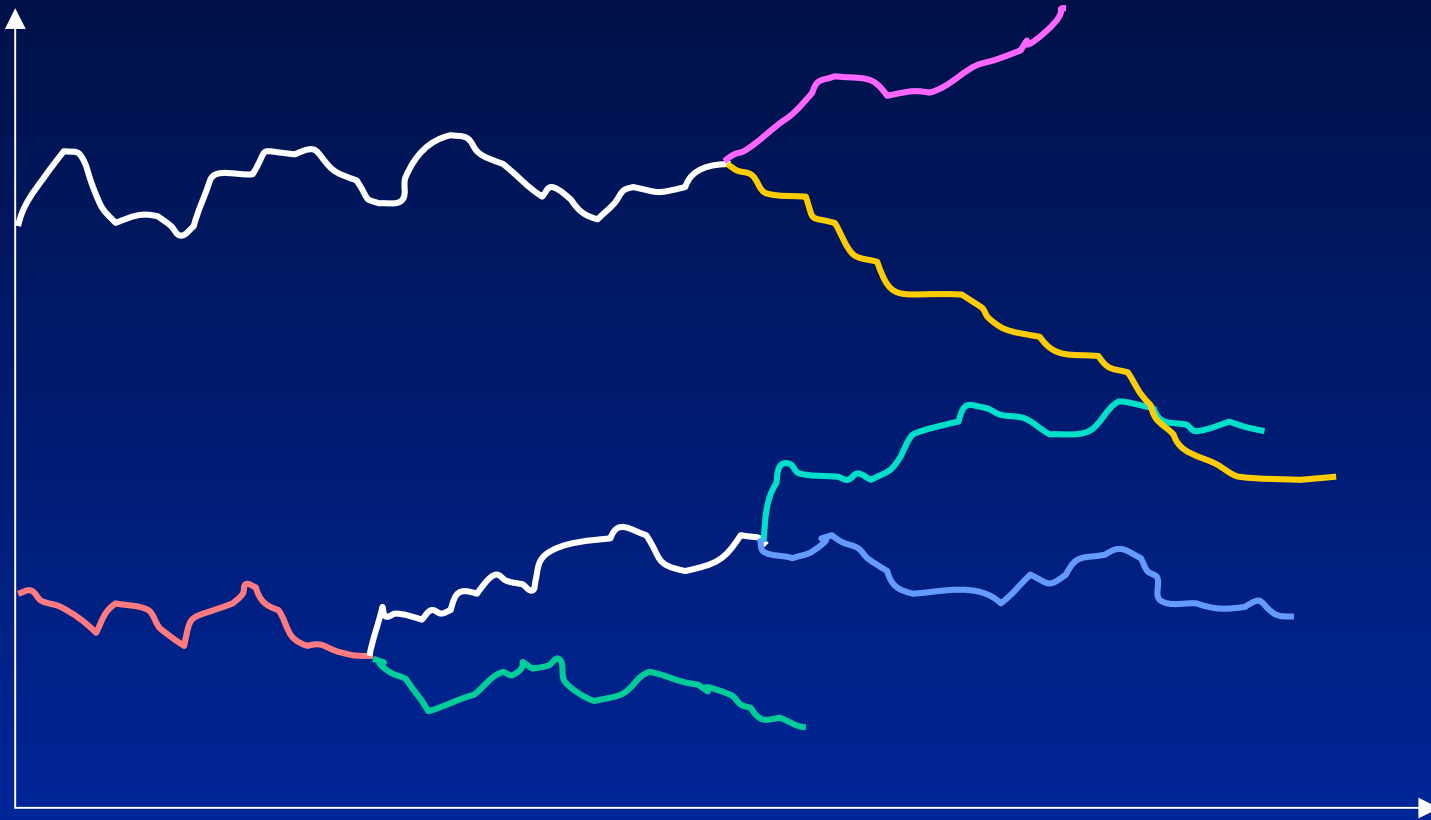
Spatial motion assumption

Independent Brownian motions



Branching mechanism

Exponential lifetime and binary branching independently



Proportional rescaling limit

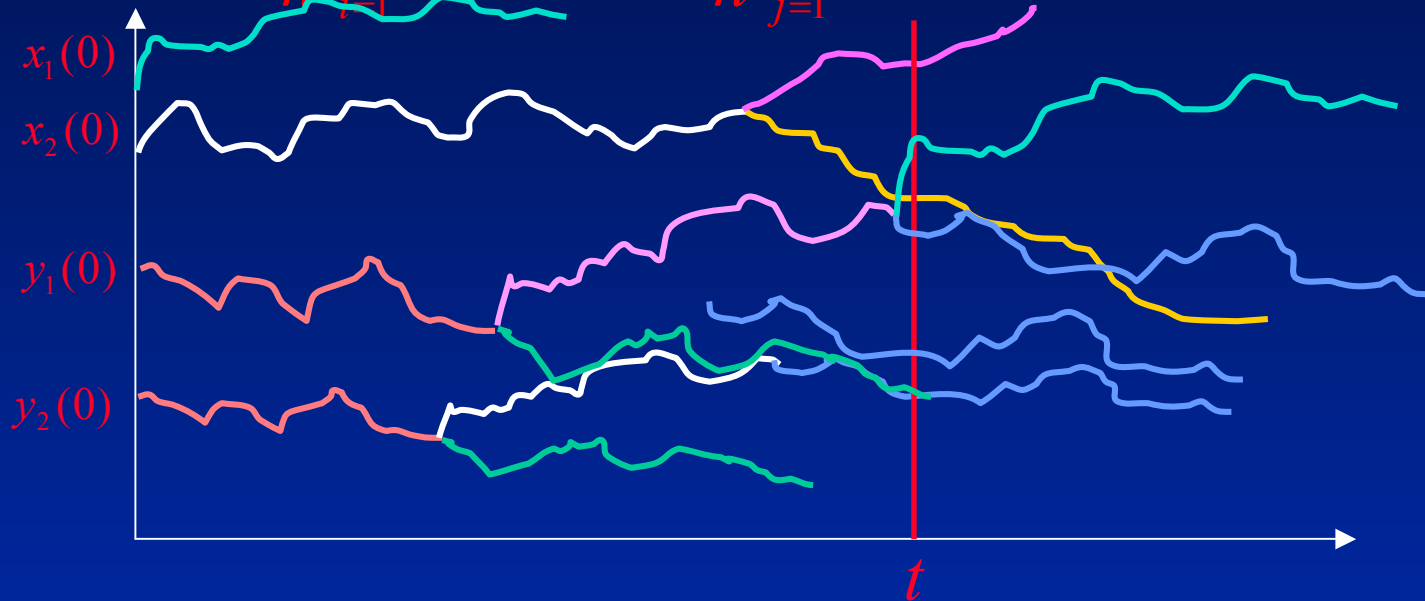
convergence of mean lifetime, particle mass and initial distribution



Empirical measure-valued processes

$$\mu_0 = \frac{1}{n} \sum_{i=1}^m \delta_{x_i(0)} \quad \nu_0 = \frac{1}{n} \sum_{j=1}^n \delta_{y_j(0)} \quad \delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

$$\mu_t = \frac{1}{n} \sum_{i=1}^{m_t} \delta_{x_i(t)} \quad \nu_t = \frac{1}{n} \sum_{j=1}^{n_t} \delta_{y_j(t)} \quad t \geq 0$$



Multiplicative property and infinite divisibility

If we denote the particles' empirical measures by

$$\mu_t = \frac{1}{n} \sum_{i=1}^{m_t} \delta_{x_i(t)} \quad \nu_t = \frac{1}{n} \sum_{j=1}^{n_t} \delta_{y_j(t)} \quad t \geq 0$$

$$\langle \phi, \mu \rangle \square \int_{\mathbf{R}} \phi(x) \mu(dx)$$

The following multiplicative property holds:

$$\begin{aligned} & \mathbf{E}_{\mu_0 + \nu_0} (\exp \{-\langle \phi, \mu_t + \nu_t \rangle\}) \\ &= \mathbf{E}_{\mu_0} (\exp \{-\langle \phi, \mu_t \rangle\}) \mathbf{E}_{\nu_0} (\exp \{-\langle \phi, \nu_t \rangle\}) \end{aligned}$$

MuP is equivalent to infinite divisibility

$$\begin{aligned} & \mathbf{E}_{\mu_0 + \nu_0} (\exp \{-\langle \phi, \mu_t + \nu_t \rangle\}) \\ &= \mathbf{E}_{\mu_0} (\exp \{-\langle \phi, \mu_t \rangle\}) \mathbf{E}_{\nu_0} (\exp \{-\langle \phi, \nu_t \rangle\}) \end{aligned}$$

Multiplicative property (MuP) is the fact that a measure-valued Markov process has MP if two such processes start at μ_0 and ν_0 , respectively, then their sum is equal in law to the same process starting at $\mu_0 + \nu_0$. This is just the infinite divisibility

Log-Laplace functional and evolution equation

Based on the MuP, we have following log-Laplace Functional equation:

$$\mathbf{E}_{\mu}(\exp\{-\langle X_t, \phi \rangle\}) = \exp\{-\langle \mu, u(t) \rangle\}$$

Where $u(t)$ is the solution of following nonlinear evolution equation

$$\frac{\partial u(t)}{\partial t} = \frac{1}{2} \Delta u(t) - \frac{1}{2} u(t)^2$$
$$u(0) = \phi$$

See M. Jirina58; S. Watanabe68; D. Dawson75;
M.L. Silverstein68.

Structural properties of Super-Brownian motion

Suppose that the initial measure is the Lebesgue measure denoted by $\lambda(dx) = dx$

According to Dawson-Hochberg and Roelly-Coppoletta

$$\mathbf{P}_\lambda(X_t(dx) \perp dx) = 1 \quad d \geq 2 \quad t > 0$$

$$\mathbf{P}_\lambda(X_t(dx) \ll dx) = 1 \quad d = 1 \quad t \geq 0$$

SPDE for Super-Brownian motion

For $d = 1$ according to Konno-Shiga88, the density process $X_t(x)$ is continuous in t and x and satisfies following SPDE

$$X_t(x) - X_{t_0}(x) = \int_{t_0}^t \sqrt{X_s(x)} \dot{W}(s, x) ds \\ + \int_{t_0}^t \frac{1}{2} \Delta X_s(x) ds \quad t_0 > 0$$

New Model, spatial motion assumption

For $d = 1$, between branchings, the motion of each particle is driven by following SDE:

$$dx_i(t) = \int_{\mathbf{R}} g(y - x_i(t)) \dot{W}(t, y) dy dt + \varepsilon dB_i(t) \quad \varepsilon \in \mathbf{R}, i \in N$$

Where $\dot{W}(y, t)$ is the space-time white noise.

New Model, assumptions

$$dx_i(t) = \int_{\mathbf{R}} g(y - x_i(t)) \dot{W}(t, y) dy dt + \varepsilon dB_i(t) \quad \varepsilon \in \mathbf{R}, i \in N$$

Assumption SS: $g(\cdot)$ is square integrable and has continuous second derivative.

According as $\varepsilon = 0$ or $\varepsilon \neq 0$, the condition will be referred as degenerate case or no-degenerate case, respectively.

New Model, branching mechanism

When a particle dies, it produces j particles with probability

p_j which satisfies

$$\sum_{j=0}^{\infty} j p_j = 1, \quad m_2 \square \sum_{j=0}^{\infty} j^2 p_j < \infty$$

Dependency of the particles in the new model

The quadratic variational process for any two particles is

$$\langle x_i(t), x_j(t) \rangle = \int_0^t \rho(x_i(s) - x_j(s)) ds + \varepsilon^2 t \delta_{ij}$$

$$\rho(x) \square \int_{\mathbf{R}} g(x-y)g(y)dy \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Loss of the multiplicative property

The log-Laplace functional does not hold for the new or interacting model. (See Wang98 for a counter example)

How to construct this class of interacting superprocesses?

Construction : pregenerator

By Ito's formula, we can formally find out the pregenerator of the limiting interacting superprocess of the branching particle systems as follows.

$$\mathbf{L}F(\mu) \square \mathbf{A}F(\mu) + \mathbf{B}F(\mu)$$

where

$$\begin{aligned} \mathbf{A}F(\mu) \square & \frac{1}{2} \int_{\mathbf{R}} (\rho(0) + \varepsilon^2) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbf{R}} \int_{\mathbf{R}} \rho(x-y) \frac{d}{dx} \frac{d}{dy} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy) \\ F(\mu) \square & f(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \end{aligned}$$

Construction : existence of solution of MP

$$\mathbf{B} \quad F(\mu) \square \sigma \int_{\mathbf{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx)$$

where

$$\frac{\delta F(\mu)}{\delta \mu(x)} \square \lim_{\varepsilon \downarrow 0} \frac{F(\mu + \varepsilon \delta_x) - F(\mu)}{\varepsilon}$$

Then, the existence of the MP can be obtained by tightness argument from finite branching particle system.

Uniqueness is a difficult problem.

Construction : uniqueness of solution of MP

By a theorem of Stroock-Varadhan, if there exists a bounded measurable function $\Lambda_{F,t}$ which is independent of X_t and is only function of F,t,μ such that

$$\mathbf{E}_{\mu}(F(X_t)) = \Lambda_{F,t}(\mu)$$

holds. Then, the MP is well-posed.

How to find the function $\Lambda_{F,t}$?

Change the form of the generator

The motivation to find out the dual process comes from the following observation of the generator. For monomial function

$$F_f(\mu) \square \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} f(x_1, \cdots, x_n) \mu^{\otimes n}(dx_1 \cdots dx_n) = F_\mu(f)$$

We have

$$\mathbf{L}F_f(\mu) = \mathbf{L}^*F_\mu(f) + \frac{1}{2}\sigma n(n-1)F_\mu(f)$$

$$\mathbf{L}^*F_\mu(f) \square F_\mu(G^n f) + \frac{1}{2}\sigma \sum_{j,k=1, j \neq k}^n (F_\mu(\Phi_{j,k}f) - F_\mu(f))$$

Change the form of the generator

where

$$G^n f(x_1, \dots, x_n) \square \frac{1}{2} \sum_{i=1}^n \varepsilon^2 \frac{\partial^2}{\partial x_i^2} f(x_1, \dots, x_n) \\ + \frac{1}{2} \sum_{i,j=1}^n \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_n)$$

$$[\Phi_{j,k} f](x_1, \dots, x_{n-2}, x)$$

$$\square \begin{cases} f(\dots, x_{j-1}, x, x_{j+1}, \dots, x_{k-1}, x, x_{k+1}, \dots, x_{n-2}) & j < k \\ f(\dots, x_{k-1}, x, x_{k+1}, \dots, x_{j-1}, x, x_{j+1}, \dots, x_{n-2}) & k < j \end{cases}$$

Dual generator

L^* has the structure of generator of function-valued Markov process $Y(t)$ described as follows:

(1) Random jump-mechanism:

$$f(x_1, \dots, x_k) \rightarrow \Phi_{ij} f(x_1, \dots, x_{k-2}, x)$$

(2) Deterministic spatial motion between jumps:

$$f(x_1, \dots, x_k) \rightarrow T_t^k f(x_1, \dots, x_k)$$

Duality

Define Stroock-Varadhan function $\Lambda_{F,t}$ as follows:

$$\Lambda_{F,t} \square \hat{\mathbf{E}}[\langle Y(t), \mu^{N(Y(t))} \rangle \exp\{\frac{1}{2} \sigma \int_0^t N(Y(u))(N(Y(u)) - 1) du\}]$$

where $N(Y(t)) = n$ if $Y(t)$ is a n -dimensional function.

Then, by Feynman-Kac formula we get the duality and the uniqueness follows.

Duality for singular coefficient

If $g(\cdot)$ is a singular function and $\rho(\cdot)$ is a local Lipschitz Function, the dual process is not directly available. However, in this case, we still can find the limiting duality by a Limit duality method. (See Wang95 and 98)

Existence of the density process

According to Wang97, in SS non-degenerate case, (i.e. if $g(\cdot)$ is a square integrable, has continuous second derivative, and $\varepsilon \neq 0$), then

$$\mathbf{P}_\mu(\mu_t \ll dx, t > 0) = 1$$

The idea to prove this result is estimating the moments of the dual process since the log-Laplace functional does not exist in the interacting case.

Purely-atomic measure state

According to Wang97, in SS degenerate case, (i.e. if $g(\cdot)$ is a square-integrable, has continuous second derivative, and $\varepsilon = 0$), then

$$\mathbf{P}_\mu(\mu_t \in N_a, t > 0) = 1$$

$N_a \square \{ \text{Space of purely-atomic measures} \}$

The proof of this result is based on following facts:

(1) The generator \mathbf{B} drives the superprocess immediately entering into the space of purely-atomic measures.

Degenerate and coalescence

(2) In the degenerate case, by considering the distance process of any two particles and using Feller's criterion of accessibility, we can get the coalescence property: ***Any two particles either never separate or never meet according as they have same initial states or not.***

Feller's criterion of accessibility

In SS degenerate case, define

$$\eta_t \square x_i(t) - x_j(t)$$

According to Feller's criterion of accessibility

$$\mathbf{P} \left(\inf_{t>0} |\eta_t| = 0 \right) = \begin{cases} 1 \\ 0 \end{cases} \Leftrightarrow \int_0^1 \frac{y}{\rho(0) - \rho(y)} dy \begin{cases} < \infty \\ = \infty \end{cases}$$

Inaccessibility of zero

Since $\rho(\cdot)$ is non-negative definite, by Bochner-Khinchin Theorem, there exists a $F(\cdot)$ such that

$$\begin{aligned} 0 \leq 1 - \frac{\rho(y)}{\rho(0)} &= \int_{\mathbf{R}} (1 - \cos(xy)) dF(x) \\ &\leq \int_{\mathbf{R}} \frac{1}{2} (xy)^2 dF(x) = \frac{1}{2} y^2 |\rho''(0)| \end{aligned}$$

Hence

$$0 \leq \sup_y \frac{\rho(0) - \rho(y)}{y^2} \leq \frac{1}{2} |\rho''(0)|$$

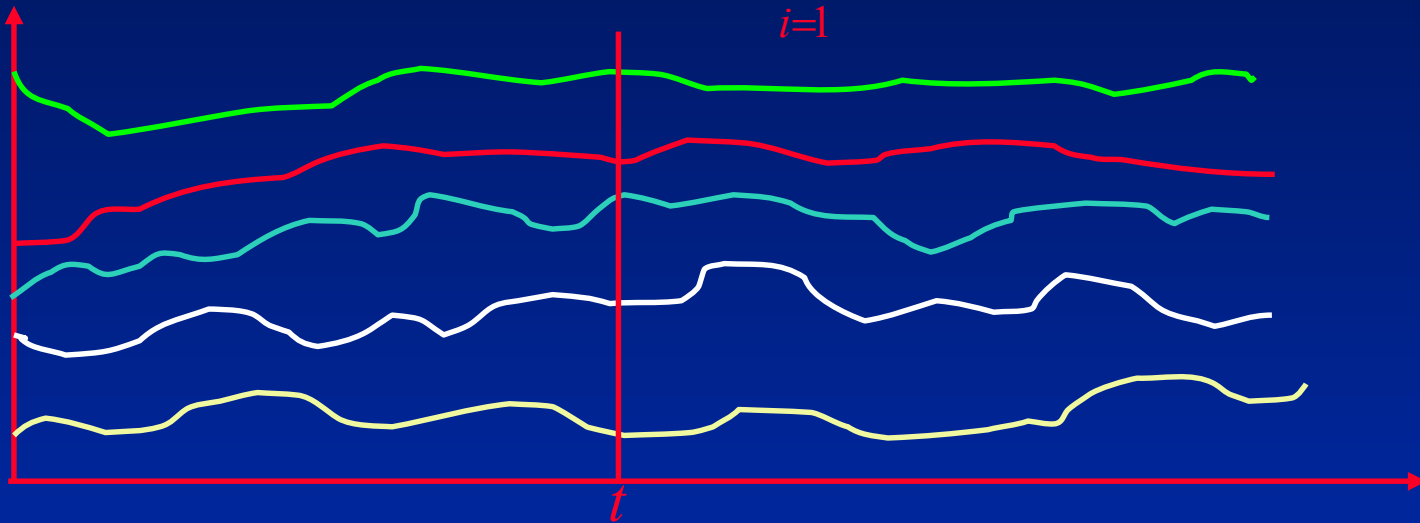
state 0 is inaccessible.

Commutativity of semigroups

(3) In the SS degenerate case, the semigroup generated by **A** commutes with the semigroup generated by **B**.

Intuitive explanation:

$$\mu_t = \sum_{i=1}^{m_t} a_i(t) \delta_{x_i(t)}$$



Derivations of SPDEs for density processes

SPDE for Super-Brownian motion(space-time martingale

Representation theorem, Konno-Shiga 1988)

$$\begin{aligned} X_t(x) - X_{t_0}(x) &= \int_{t_0}^t \sqrt{X_s(x)} \dot{W}(s, x) ds \\ &+ \int_{t_0}^t \frac{1}{2} \Delta X_s(x) ds \end{aligned}$$

SPDE for the interacting superprocess(decomposition

Theorem, Dawson-Vaillancourt-Wang 2000)

$$\begin{aligned} \ell_t(x) - \ell_{t_0}(x) &= \int_{t_0}^t \sqrt{\sigma \ell_s(x)} \dot{U}(s, x) ds \\ &+ \frac{1}{2} \rho_\varepsilon \int_{t_0}^t \Delta \ell_s(x) ds + \int_{t_0}^t \int_{\mathbf{R}} \{\nabla [g(y-x) \ell_s(x)]\} \dot{V}(s, y) dy ds \end{aligned}$$

Degenerate SPDE for purely-atomic superprocess

Dawson-Li-Wang2002 has studied a degenerate SPDE for purely-atomic measure valued superprocesses and proved the existence and uniqueness of its strong solution.

$$\begin{aligned} & \langle \phi, \mu_t \rangle - \langle \phi, \mu_{t_0} \rangle - \frac{1}{2} \rho(0) \int_{t_0}^t \langle \phi'', \mu_s \rangle ds \\ &= \sum_{i \in I(t_0)} \int_{t_0}^t \phi(x_i(s)) \sqrt{\sigma a_i(s)} dB_i(s) \quad t_0 > 0 \\ &+ \int_{t_0}^t \int_{\mathbf{R}} \langle g(y-x) \phi'(x), \mu_s(dx) \rangle \dot{W}(s, y) dy ds \end{aligned}$$

Simple facts

Under SS assumption, following equation has unique strong solution.

$$x(t) - x(0) = \int_0^t \int_{\mathbf{R}} g(y - x(0)) W(ds, dy)$$

According to Yamada-Watanabe, following equation has unique strong solution.

$$\xi(t) - \xi(0) = \int_0^t \sqrt{\sigma \xi(s)} dB(s)$$

One dimensional case

We can use stopping time technique and choice of test function to decompose following one-dimensional SPDE

$$\begin{aligned} & \xi_i(t)\phi(x_i(t)) - \xi_i(0)\phi(x_i(0)) = \\ & \int_0^t \int_{\mathbf{R}} \xi_i(s)\phi'(x_i(s))g(y - x_i(s))W(ds, dy) \\ & + \frac{1}{2}\rho(0) \int_0^t \xi_i(s)\phi''(x_i(s))ds + \int_0^t \phi(x_i(s))\sqrt{\sigma\xi_i(s)}dB_i(s) \Rightarrow \\ & x_i(t) - x_i(0) = \int_0^t \int_{\mathbf{R}} g(y - x_i(s))W(ds, dy), \\ & \xi_i(t) - \xi_i(0) = \int_0^t \sqrt{\sigma\xi_i(s)}dB_i(s) \end{aligned}$$

Multidimensional case

We can use stopping time technique and choice of test function to decompose the degenerate SPDE into a sequence of one-dimensional SPDEs

$$\begin{aligned} \xi_i(t)\phi(x_i(t)) - \xi_i(0)\phi(x_i(0)) = & \\ \int_0^t \int_{\mathbf{R}} \xi_i(s)\phi'(x_i(s))g(y - x_i(s))W(ds, dy) & \\ + \frac{1}{2} \rho(0) \int_0^t \xi_i(s)\phi''(x_i(s))ds & \\ + \int_0^t \phi(x_i(s))\sqrt{\sigma\xi_i(s)}dB_i(s) \quad i \in I(0) & \end{aligned}$$

A generalized new model and SDSM

Now we generalize the interacting model to a more general case: location dependent branching and general diffusion coefficient.

Since offspring distribution depends on spatial location, the motion affects branching. This model brings a new class of interacting. This new model has been studied extensively in Dawson-Li-Wang2001.(Construction, existence of density process, rescaling limit converging to Super-BM, catalytic and so on.)

New Model, spatial motion assumption

For $d = 1$, between branchings, the motion of each particle is driven by following SDE:

$$dx_i(t) = \int_{\mathbf{R}} g(y - x_i(t)) \dot{W}(t, y) dy dt + c(x_i) dB_i(t) \quad c \in C^2(\mathbf{R}), i \in N$$

New Model, branching mechanism

When a particle dies, it produces j particles with probability $p_j(x)$ which satisfies

$$\sum_{j=0}^{\infty} j p_j(x) = 1, \quad m_2(x) \square \sum_{j=0}^{\infty} j^2 p_j(x) < \infty$$

Dependency of the particles in the new model

The quadratic variational process for any two particles is

$$\langle x_i(t), x_j(t) \rangle = \int_0^t \rho(x_i(s) - x_j(s)) ds \\ + \int_0^t c^2(x_i(s)) ds \delta_{ij}$$

$$\rho(x) \square \int_{\mathbf{R}} g(x-y)g(y)dy \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Construction : pregenerator

By Ito's formula, we can formally find out the pregenerator of the limiting interacting superprocess of the branching particle systems as follows.

$$\mathbf{L}F(\mu) \square \mathbf{A}_c F(\mu) + \mathbf{B}_\sigma F(\mu)$$

where

$$\begin{aligned} \mathbf{A}_c F(\mu) \square & \frac{1}{2} \int_{\mathbf{R}} (\rho(0) + c(x)^2) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ & + \frac{1}{2} \int_{\mathbf{R}} \int_{\mathbf{R}} \rho(x-y) \frac{d}{dx} \frac{d}{dy} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy) \end{aligned}$$

Construction : existence of solution of MP

$$\mathbf{B} \sigma F(\mu) \square \int_{\mathbf{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx)$$

Then, the existence of the MP can be obtained by tightness argument from finite branching particle system. The Uniqueness can be obtained by duality method. However, state classification is a difficult problem.

State classification for this location dependent branching model

Since offspring distribution depends on spatial location, the motion affects branching. The method used in Wang97 does not work for this case. In Wang2002, to prove the purely-atomic conclusion for degenerate case, the following ideas are used:

- (1) Decomposition of generators
- (2) Trotter's product formula
- (3) Variable clock speed explanation

Generator decomposition technique

In order to explain SDSM immediately enters into purely-Atomic state from a absolutely continuous state, we define Following decomposition of generators.

$$\begin{aligned}\mathbf{B}_\sigma F(\mu) &= \mathbf{B}_d F(\mu) + \mathbf{B}_{\varepsilon/2} F(\mu) \\ \mathbf{B}_d F(\mu) &\square \frac{1}{2} \int_{\mathbf{R}} (\sigma(x) - \frac{\varepsilon}{2}) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx) \\ \mathbf{B}_{\varepsilon/2} F(\mu) &\square \frac{\varepsilon}{4} \int_{\mathbf{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx)\end{aligned}$$

Here we assume that $\sigma(x) \geq \varepsilon > 0$

Always purely-atomic measures

Let $\mathbf{L}_{0,d} F(\mu) \square \mathbf{A}_0 F(\mu) + \mathbf{B}_d F(\mu)$

$$\mathbf{A}_0 F(\mu) \square \frac{1}{2} \int_{\mathbf{R}} \rho(0) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx)$$

$$+ \frac{1}{2} \int_{\mathbf{R}} \int_{\mathbf{R}} \rho(x-y) \frac{d}{dx} \frac{d}{dy} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy)$$

If the initial measure is purely-atomic, the solution of the corresponding $\mathbf{L}_{0,d}$ MP is always a purely-atomic valued.

Variable clock speed

$$\text{Let } \mathbf{L}_{0,d} F(\mu) \square \mathbf{A}_0 F(\mu) + \mathbf{B}_d F(\mu)$$

$$\mathbf{L}_{0,\varepsilon} F(\mu) \square \mathbf{A}_0 F(\mu) + \mathbf{B}_{\varepsilon/2} F(\mu)$$

If the initial measure is a single atom $a \delta_x$, their solutions of the corresponding MPs are $a(t) \delta_{x(t)}$, $b(t) \delta_{x(t)}$, respectively. Then, the life time of $a(t)$ is shorter than that of $b(t)$ almost surely due to the clock speed.

Number of atoms dominated

Let U_t^d be the Feller semigroup of $\mathbf{L}_{0,d}F(\mu)$

V_t^ε be the Feller semigroup of $\mathbf{L}_{0,\varepsilon}F(\mu)$

$T_t^{\varepsilon/2}$ be the Feller semigroup of $\mathbf{B}_{\varepsilon/2}F(\mu)$

By Trotter's product formula, we have

$$\lim_{k \rightarrow 0} [U_{t/k}^d \square T_{t/k}^{\varepsilon/2}] F = U_t^\sigma F \Leftrightarrow \mathbf{L}_{0,\sigma} = \mathbf{L}_{0,d} + \mathbf{B}_{\varepsilon/2}$$

$$\lim_{k \rightarrow 0} [V_{t/k}^{\varepsilon/2} \square T_{t/k}^{\varepsilon/2}] F = V_t^\varepsilon F \Leftrightarrow \mathbf{L}_{0,\varepsilon} = \mathbf{L}_{0,\varepsilon/2} + \mathbf{B}_{\varepsilon/2}$$

According to Wang97 and the number of atoms dominated, the conclusion follows.

This is all for this presentation

Thanks