Sticky flows on the circle and their noises.

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Abstract

This paper gives a construction of sticky flows on the circle. Sticky flows give examples of stochastic flows of kernels that interpolates between Arratia's coalescing flow and the deterministic diffusion flow. They are associated with systems of sticky independent Brownian particles on the circle, for some fixed parameter of stickyness.

It is proved that the noise generated by Brownian sticky flows is black. A new proof of the fact that the noise of Arratia's coalescing flow is black is given.

Introduction

The purpose of this note is to give examples of stochastic flows of kernels as defined in [4], which naturally interpolate between the Arratia's coalescing flow associated with systems of coalescing independent Brownian particles on the circle and the deterministic diffusion flow associated with independent Brownian particles (actually, the results are given in the slightly more general framework of symmetric Levy processes for which points are not polar). The construction is performed using Dirichlet form theory and the extension of De Finetti's theorem given in [4]. The sticky flows of kernels are associated with systems of sticky independent Levy particles on the circle, for some fixed parameter of stickyness. Some elementary asymptotic properties of the flow are also given.

In section 2, it is proved that, in the case the one-point motion is symmetric stable process of order $\alpha \in]1,2]$, the noises generated by the sticky flows are black. In section 3, the noises generated by Arratia's coalescing flows (in the case the one-point motion is symmetric stable process of order $\alpha \in]1,2]$) are black.

1 Construction of sticky flows.

1.1 Compatible family of Dirichlet forms.

Let $(\mathcal{E}_n)_{n\geq 1}$ be a family of Dirichlet forms[1], respectively defined on $L^2(M^n, m_n)$, where M is a metric space and $(m_n)_{n\geq 1}$ is a family of probability measures on M^n. We will denote by D_n the domain of the Dirichlet form $\mathcal{E}_n$. For all $n \geq 1$, S_n denotes the group of permutations of $\{1, \dots, n\}$.

Definition 1 *We will say that the family of probability measures $(m_n)_{n\geq 1}$ is consistent and exchangeable if for all $n \geq 1$, m_n is the law of $(X_1, \dots, X_n)$, where $(X_i)_{i\geq 1}$ is an exchangeable sequence of M-valued random variables. By Kolmogorov's theorem, this holds if and only if*

(i) *for all $n \geq 1$, $\sigma \in S_n$ and $f \in L^1(m_n)$, $\int f_\sigma dm_n = \int f dm_n$.*

(ii) *for all $n \geq 1$ and $f \in L^1(m_n)$, $\int (f \otimes 1) dm_{n+1} = \int f dm_n$.*

Let $(\mathsf{P}_t^{(n)})_{n\geq 1}$ be the family of Markovian semigroups associated with this family of Dirichlet forms. We assume that $(m_n)_{n\geq 1}$ is consistent and exchangeable.

Definition 2 *We will say that the family of Dirichlet forms $(\mathcal{E}_n)_{n\geq 1}$ is consistent if the family of Markovian semigroups $(\mathsf{P}_t^{(n)})_{n\geq 1}$ is consistent, that is if the following assertions are satisfied*

(i) *for all $f \in L^2(m_n)$ and $\sigma \in S_n$, $\mathsf{P}_t^{(n)} f_\sigma = (\mathsf{P}_t^{(n)} f)_\sigma$, where $f_\sigma(x_1, \dots, x_n) = f(x_{\sigma_1}, \dots, x_{\sigma_n})$;*

(ii) *for all $f \in L^2(m_n)$, $f \otimes 1 \in L^2(m_{n+1})$ and $\mathsf{P}_t^{(n+1)}(f \otimes 1) = (\mathsf{P}_t^{(n)} f) \otimes 1$.*

Then we can define for all $n \geq 1$ the conditional expectations $\pi_n : L^2(m_{n+1}) \to L^2(m_n)$ such that for all $f \in L^2(m_{n+1})$, $\pi_n(f) \otimes 1$ is the orthogonal projection in $L^2(m_{n+1})$ of f onto $\{g \otimes 1,\ g \in L^2(m_n)\}$.

[1] We refer the reader not familiar with Dirichlet forms and symmetric Markov processes to [3]

Proposition 3 *The family of Dirichlet forms $(\mathcal{E}_n)_{n\geq 1}$ is consistent if and only if*

(i) *For all $n \geq 1$, $\sigma \in S_n$ and $(f,g) \in D_n^2$, we have $(f_\sigma, g_\sigma) \in D_n^2$ and $\mathcal{E}_n(f_\sigma, g_\sigma) = \mathcal{E}_n(f,g)$.*

(ii) *For all $n \geq 1$, $f \in D_{n+1}$ and $g \in D_n$, we have $g \otimes 1 \in D_{n+1}$, $\pi_n(f) \in D_n$ and*

$$\mathcal{E}_{n+1}(f, g\otimes 1) = \mathcal{E}_n(\pi_n(f), g), \quad (1)$$
$$\mathcal{E}_{n+1}(f) = \mathcal{E}_n(\pi_n(f)) + \mathcal{E}_{n+1}(f - \pi_n(f)\otimes 1). \quad (2)$$

This proposition is an immediate corollary of the following theorem.

Theorem 4 *Let $\mathcal{E}_1$ and $\mathcal{E}$ be two Dirichlet forms respectively defined on $L^2(E_1, \mathcal{F}_1, m_1)$ and on $L^2(E_1 \times E_2, \mathcal{F}_1 \otimes \mathcal{F}_2, m)$, with domain D_1 and D, where m_1 and m are probability measures on E_1 and on $E_1 \times E_2$ satisfying $\int (g\otimes 1) dm = \int g dm_1$ for all $g \in L^1(m_1)$. Let P_t^1 and P_t be the associated Markovian semigroups. Then,* **(i)** *and* **(ii)** *are equivalent, where*

(i) *For all $g \in L^2(m_1)$, $\mathsf{P}_t(g\otimes 1) = (\mathsf{P}_t^1 g)\otimes 1$.*

(ii) *For all $g \in D_1$ and $f \in D$, we have $\pi(f) \in D_1$, $g \otimes 1 \in D$ and*

$$\mathcal{E}(f, g\otimes 1) = \mathcal{E}_1(\pi(f), g), \quad (3)$$
$$\mathcal{E}(f) = \mathcal{E}_1(\pi(f)) + \mathcal{E}(f - \pi(f)\otimes 1). \quad (4)$$

where $\pi(f)\otimes 1$ is the orthogonal projection of f in $L^2(m)$ onto $\{g \otimes 1,\ g \in L^2(m_1)\}$.

Proof. Suppose first **(i)**.

For all $f \in L^2(m)$, $\mathcal{E}(f)$ is the increasing limit as $t \to 0+$ of

$$\frac{1}{t}\left(\|f\|^2_{L^2(m)} - \|\mathsf{P}_{t/2} f\|^2_{L^2(m)}\right)$$

and $f \in D$ if and only if $\mathcal{E}(f) < \infty$. Using this relation and (i), we show that for all $g \in L^2(m_1)$, $\mathcal{E}(g\otimes 1) = \mathcal{E}_1(g)$. Thus $g \in D_1$ if and only if $g \otimes 1 \in D$.

Let $f \in L^2(m)$ and $u = f - \pi(f)\otimes 1$. Then u is orthogonal in $L^2(m)$ to $V = \{g\otimes 1,\ g \in L^2(m_1)\}$. Then for all positive t, $\mathsf{P}_t u$ is orthogonal to V since

$$\langle \mathsf{P}_t u, g\otimes 1\rangle_{L^2(m)} = \langle u, \mathsf{P}_t(g\otimes 1)\rangle_{L^2(m)} = \langle u, (\mathsf{P}_t^1 g)\otimes 1\rangle_{L^2(m)} = 0.$$

This implies that $\|\mathsf{P}_{t/2}f\|^2_{L^2(m)} = \|\mathsf{P}_{t/2}u\|^2_{L^2(m)} + \|\mathsf{P}^1_{t/2}\pi(f)\|^2_{L^2(m_1)}$. Moreover, $\|f\|^2_{L^2(m)} = \|u\|^2_{L^2(m)} + \|\pi(f)\|^2_{L^2(m_1)}$ and we prove $\mathcal{E}(f) = \mathcal{E}(u) + \mathcal{E}_1(\pi(f))$. It follows that $\mathcal{E}_1(\pi(f)) \leq \mathcal{E}(f)$, which implies that $\pi(f) \in D_1$ when $f \in D$.

To prove **(ii)**, it remains to prove (3). This follows from the computation

$$\begin{aligned}
\mathcal{E}(f, g\otimes 1) &= \lim_{t\to 0}\frac{1}{t}(\langle f, g\otimes 1\rangle_{L^2(m)} - \langle f, \mathsf{P}_t(g\otimes 1)\rangle_{L^2(m)}) \\
&= \lim_{t\to 0}\frac{1}{t}(\langle \pi(f), g\rangle_{L^2(m_1)} - \langle \pi(f), \mathsf{P}^1_t g\rangle_{L^2(m_1)}) \\
&= \mathcal{E}_1(\pi(f), g).
\end{aligned}$$

We now assume **(ii)** is satisfied. Then **(i)** is satisfied if for all $\alpha > 0$ and $g \in L^2(m_1)$, $G_\alpha(g\otimes 1) = (G^1_\alpha g)\otimes 1$, where G_α and G^1_α are respectively the resolvents of P_t and of P^1_t. Let $\mathcal{E}_\alpha$ and $\mathcal{E}^1_\alpha$ be respectively the forms $\mathcal{E} + \alpha\langle\cdot,\cdot\rangle_{L^2(m)}$ and $\mathcal{E}_1 + \alpha\langle\cdot,\cdot\rangle_{L^2(m_1)}$. Let $g \in L^2(m_1)$ and $\alpha > 0$, then $G_\alpha(g\otimes 1)$ is the unique element of $L^2(m)$ such that for all $f \in L^2(m)$ we have $\mathcal{E}_\alpha(f, G_\alpha(g\otimes 1)) = \langle f, g\otimes 1\rangle_{L^2(m)}$. This implies that

$$\mathcal{E}_\alpha(f, G_\alpha(g\otimes 1)) = \langle \pi(f), g\rangle_{L^2(m_1)}.$$

Using equation (3) and the definition of $G^1_\alpha g$, we also have

$$\mathcal{E}_\alpha(f, (G^1_\alpha g)\otimes 1) = \mathcal{E}^1_\alpha(\pi(f), G^1_\alpha g) = \langle \pi(f), g\rangle_{L^2(m_1)}.$$

This proves that $G_\alpha(g\otimes 1) = (G^1_\alpha g)\otimes 1$ and **(i)** follows. □

1.2 Consistent families of probability measures and exchangeable random partitions.

For generalities on exchangeable random partitions, we refer to Pitman's St Flour course [7]. For all $n \geq 1$, we let $\mathcal{P}_n$ denote the set of all partitions of $[n] = \{1, \ldots, n\}$. The number of elements of a partition π is denoted $|\pi|$. A random partition Π_n of $[n]$ is called *exchangeable* if its distribution is invariant under the obvious action of S_n on $\mathcal{P}_n$. Equivalently, for each partition $\{A_1, \ldots, A_k\}$ of $[n]$,

$$\mathsf{P}[\Pi_n = \{A_1, \ldots, A_k\}] = p(|A_1|, \ldots, |A_k|)$$

for some symmetric function p, called the *exchangeable partition probability function* (EPPF) of Π_n.

A sequence of exchangeable random partitions $(\Pi_n)_{n\geq 1}$ is called *consistent in distribution* if for all $1 \leq m \leq n$, the restriction, denoted by $\Pi_{m,n}$, of Π_n to $[m]$ has the same distribution as Π_m. The associated sequence of EPPFs are also called *consistent*. Note that being given a consistent sequence of EPPFs (p_n), it is possible to construct a sequence of exchangeable random partitions, called an *infinite exchangeable random partition* $\Pi_\infty = (\Pi_n)_{n\geq 1}$ such that the EPPF of Π_n is p_n and with $\Pi_{m,n} = \Pi_m$ for all $1 \leq m \leq n$.

Using Kingman's representation theorem, given an infinite exchangeable random partition Π_∞, it is possible to construct a random sequence in $[0,1]$, $(P_i)_{i\geq 1}$, with $\sum_{i\geq 1} P_i \leq 1$ and the law of Π_∞ given $(P_i)_{i\geq 1}$ is the same as if Π_∞ were given by random sampling from a random distribution with ranked atoms $(P_i)_{i\geq 1}$. The random partition Π_∞ is called *proper* when $\sum_i P_i = 1$.

For all partition $\pi = \{A_1, \ldots, A_k\}$ of $[n]$, let E_π denote the set of all $x \in M^n$ such that for all $1 \leq l \leq k$ and $(i,j) \in A_l^2$, we have $x_i = x_j$. Then E_π is isomorphic to M^k. Let λ_π be the probability measure on E_π given by $(\varphi_\pi)_*(\lambda^{\otimes k})$, where λ is a probability measure on M and $\varphi_\pi : M^k \to E_\pi$ with $(\varphi_\pi(y_1, \ldots, y_k))_i = y_j$ for all i, j such that $i \in A_j$.

Let us be given a proper infinite random partition Π_∞. Let $m_n = \sum_{\pi\in\mathcal{P}_n} p_\pi \lambda_\pi$, where $p_n = (p_\pi)_{\pi\in\mathcal{P}_n}$ is the distribution of Π_n, the restriction of Π_∞ to $[n]$. Then $(m_n)_{n\geq 1}$ is consistent and exchangeable.

Kingman's representation theorem then implies that $m_n = \mathsf{E}[\mu^{\otimes n}]$, where μ is a random measure on M. This random measure can be described with $(X_i)_{i\geq 1}$ a sequence of independent M-valued random variables of law λ and an independent sequence, $(P_i)_{i\geq 1}$, of $[0,1]$-valued random variables with $\sum_{i\geq 1} P_i = 1$, and μ is defined by the relation $\mu = \sum_{i\geq 1} P_i \delta_{X_i}$.

In this paper we will be interested in the case where $(P_i)_{i\geq 1}$ is distributed like a Dirichlet process of parameter $\frac{1-\tau}{\tau}$, where $\tau \in [0,1[$ is a fixed parameter. More precisely, the sequence $(P_i)_{i\geq 1}$ is distributed as the jumps of a process $\left(\frac{\Gamma_{u\theta}}{\Gamma_\theta}\right)_{0\leq u\leq 1}$, where $(\Gamma_s)_{s\geq 0}$ is a standard Γ-process, i.e., the subordinator whose marginal laws are given by gamma distribution of parameter s. In this case the family $(m_n)_{n\geq 1}$ can be constructed by the relation $m_1 = \lambda$ and $m_{n+1}(d\bar{x}_n, dx_{n+1}) = m_n(d\bar{x}_n)\pi_n(\bar{x}_n, dx_{n+1})$ where $\bar{x}_n = (x_1, \ldots, x_n)$ and

$$\pi_n(\bar{x}_n, dx_{n+1}) = \frac{(1-\tau)\lambda(dx_{n+1}) + \tau\sum_{i=1}^n \delta_{x_i}(dx_{n+1})}{(1-\tau) + n\tau}. \tag{5}$$

Then the EPPF p satisfies

$$\begin{cases} p(n_1,\dots,n_k,1) & = \left(\frac{1-\tau}{(1-\tau)+n\tau}\right) p(n_1,\dots,n_k), \\ p(n_1,\dots,n_{k-1},n_k+1) & = \left(\frac{n_k\tau}{(1-\tau)+n\tau}\right) p(n_1,\dots,n_k). \end{cases} \tag{6}$$

1.3 Construction of consistent families of Dirichlet forms on $\mathbb{S}^1$.

We now let M denote the unit circle $\mathbb{S}^1$ and λ denote the Lebesgue measure on $\mathbb{S}^1$. For a fixed parameter $\tau \in [0,1[$, assume we are given the consistent family of probability measures $(m_n)_{n\geq 1}$ defined in the previous section. Since these measures depend on τ, we will denote them m_n^τ. We will also denote the corresponding EPPF p^τ.

Let P_t be the Markovian semigroup of a symmetric Levy process of exponent ψ on $\mathbb{S}^1$, for which points are not polar, i.e., such that

$$\sum_{k\in\mathbb{Z}} (\alpha + \psi(k))^{-1} < \infty \tag{7}$$

for $\alpha > 0$. We denote by $\mathcal{E}$ the associated Dirichlet form. This Dirichlet form is defined on $L^2(\lambda)$.

For every integer $k \geq 1$, $\mathcal{E}^{\odot k}$ denotes the Dirichlet form associated with k independent Levy processes, i.e., with the Markovian semigroup $\mathsf{P}_t^{\otimes k}$.

For every integer $n \geq 1$, we define the Dirichlet form on $C^1((\mathbb{S}^1)^n) \subset L^2(m_n^\tau)$ by the formula

$$\mathcal{E}_n^\tau = \sum_{\pi\in\mathcal{P}_n} p_\pi^\tau \mathcal{E}_\pi, \tag{8}$$

where $\mathcal{E}_\pi = \varphi_\pi(\mathcal{E}^{\odot k})$ (with $k = |\pi|$). More precisely, for $f \in C^1(E_\pi)$, $\mathcal{E}_\pi(f) = \mathcal{E}^{\odot k}(f \circ \varphi_\pi)$.

The Dirichlet form $\mathcal{E}_n^\tau$ being defined by the superposition of closable Dirichlet forms, $\mathcal{E}_n^\tau$ is closable in $L^2(m_n)$ (see proposition 3.1.1 p.214 in [2]). Moreover, these Dirichlet forms are regular by construction. We denote by $\mathsf{P}_t^{(n),\tau}$ the associated Markovian semigroup.

Remark 5 *Let A_n^τ denote the generator of $\mathcal{E}_n^\tau$ and let $A^{(n)}$ denote the generator of $\mathsf{P}_t^{\otimes n}$. (When P_t is the heat semigroup, $A^{(n)}$ is the Laplacian on*

$H_2((\mathbb{S}^1)^n)$). Then for $f \in C^2((\mathbb{S}^1)^n)$

$$A_n^\tau f \ m_n^\tau = \sum_{\pi \in \mathcal{P}_n} p_\pi^\tau (A^{(|\pi|)}(f \circ \varphi_\pi) \circ \varphi_\pi^{-1}) \ \lambda_\pi. \tag{9}$$

Proposition 6 *The family of Dirichlet forms $(\mathcal{E}_n^\tau)$ verifies the following recurrence property :*

$$\mathcal{E}_{n+1}^\tau(g) = \frac{1}{(1-\tau)+n\tau}\left((1-\tau)\mathcal{E}_n^\tau \odot \mathcal{E}(g) + \tau \sum_{i=1}^n \mathcal{E}_n(g^i)\right), \tag{10}$$

where $g^i(x_1,\dots,x_n)$ denotes $g(x_1,\dots,x_n,x_i)$ and $\mathcal{E}_n \odot \mathcal{E}$ is the Dirichlet form defined on $C^1((\mathbb{S}^1)^n) \subset L^2(m_n^\tau \otimes \lambda)$ associated with the Markovian semigroup $\mathsf{P}_t^{(n),\tau} \otimes \mathsf{P}_t$.

Proof. We have

$$\mathcal{E}_{n+1}^\tau(g) = \sum_{\pi \in \mathcal{P}_n} p_{\pi \cup \{n+1\}}^\tau \mathcal{E}_{\pi \cup \{n+1\}}(g) + \sum_{\pi \in \mathcal{P}_n} \sum_{j=1}^{|\pi|} p_{\pi_j}^\tau \mathcal{E}_{\pi_j}(g), \tag{11}$$

where π_j is the partition of $[n+1]$ obtained by adding $n+1$ to the jth set A_j of $\pi = \{A_1,\dots,A_{|\pi|}\}$. We conclude using the definition of the EPPF p^τ given by (6), the fact that $\mathcal{E}_{\pi\cup\{n+1\}}(g) = \mathcal{E}_\pi \odot \mathcal{E}(g)$ and that for all $i \in A_j$, $\mathcal{E}_{\pi_j}(g) = \mathcal{E}_\pi(g^i)$. □

Theorem 7 **(i)** *The family of Dirichlet forms $(\mathcal{E}_n^\tau)_{n\geq 1}$ defined above is consistent.*

(ii) *The family of Markovian semigroups $(\mathsf{P}_t^{(n),\tau})_{n\geq 1}$ associated with this family of Dirichlet forms are strong Feller semigroups.*

Proof. **(i)**: Fix $n \geq 1$. We denote D_n^τ the domain of $\mathcal{E}_n^\tau$. Then $C^\infty((\mathbb{S}^1)^n)$ is dense in D_n^τ with respect to the scalar product $\mathcal{E}_n^\tau + \langle\cdot,\cdot\rangle_{L^2(m_n^\tau)}$. From the definition of $(m_n^\tau)_{n\geq 1}$, the projection operator π_n^τ given in (5) maps $C^\infty((\mathbb{S}^1)^{n+1})$ onto $C^\infty((\mathbb{S}^1)^n)$.

Let $g \in D_n^\tau$, then using (10) it is easy to check that $\mathcal{E}_{n+1}^\tau(g \otimes 1) = \mathcal{E}_n^\tau(g)$, which implies $g \otimes 1 \in D_{n+1}^\tau$.

For all $f \in C^\infty((\mathbb{S}^1)^{k+1})$ and $g \in C^\infty((\mathbb{S}^1)^k)$,

$$\mathcal{E}^{\odot(k+1)}(f, g\otimes 1) = \int \mathcal{E}^{\odot k}(f_x, g)\lambda(dx), \tag{12}$$

where $f_x(x_1,\dots,x_k)=f(x_1,\dots,x_k,x)$. This holds since

$$\mathcal{E}^{\odot(k+1)}(f,g\otimes 1)=\lim_{t\to 0}\frac{1}{t}\left(\langle f,g\otimes 1\rangle_{L^2(\lambda^{\otimes(k+1)})}-\langle f,(\mathsf{P}_t^{\otimes k}g)\otimes 1\rangle_{L^2(\lambda^{\otimes(k+1)})}\right).$$

This implies, using (8), that for all $f\in C^\infty((\mathbb{S}^1)^{n+1})$ and $g\in C^\infty((\mathbb{S}^1)^n)$,

$$\mathcal{E}_n^\tau\odot\mathcal{E}(f,g\otimes 1)=\int\mathcal{E}_n^\tau(f_x,g)\lambda(dx).$$

Using this relation and (10), we show that for all $f\in C^\infty((\mathbb{S}^1)^{n+1})$ and $g\in C^\infty((\mathbb{S}^1)^n)$,

$$\mathcal{E}_{n+1}^\tau(f,g\otimes 1)=\mathcal{E}_n^\tau(\pi_n^\tau(f),g). \tag{13}$$

This easily implies that for all $f\in C^\infty((\mathbb{S}^1)^{n+1})$

$$\mathcal{E}_{n+1}^\tau(f)=\mathcal{E}_n^\tau(\pi_n^\tau(f))+\mathcal{E}_{n+1}^\tau(f-\pi_n^\tau(f)\otimes 1), \tag{14}$$

and that

$$\mathcal{E}_n^\tau(\pi_n^\tau(f))\le\mathcal{E}_{n+1}^\tau(f). \tag{15}$$

Since we also have $\|\pi_n^\tau(f)\|_{L^2(m_n^\tau)}\le\|f\|_{L^2(m_{n+1}^\tau)}$, (15) extends to all $f\in D_{n+1}^\tau$. This implies that π_n^τ maps D_{n+1}^τ onto D_n^τ. It is now easy to check that (13) and (14) extend to all $f\in D_{n+1}^\tau$ and $g\in D_n^\tau$. Applying theorem 3 proves that the family of Dirichlet forms $(\mathcal{E}_n^\tau,\ n\ge 1)$ is consistent.

(ii): We now define an everywhere defined version $\tilde{\mathsf{P}}_t^{(n),\tau}$ of $\mathsf{P}_t^{(n),\tau}$.

Fix $n\ge 1$. Let $\hat{X}$ be the stationary Markov process associated with $\mathsf{P}_t^{(n+1),\tau}$. The stationary law is m_{n+1}^τ. Let X_t and Y_t be the processes defined by

$$\begin{aligned}
X_t^i &= \hat{X}_t^i && \text{for } i\le n,\\
Y_t^i &= \hat{X}_t^i && \text{for } i\le n-1,\\
Y_t^n &= \hat{X}_t^{n+1} && \text{for } t\le T,\\
Y_t^n &= \hat{X}_t^n && \text{for } t> T,
\end{aligned}$$

where $T=\inf\{s,\ \hat{X}_s^n=\hat{X}_s^{n+1}\}$. Note that for $t\ge T$, $X_t=Y_t$. Moreover, the consistency of the family $(\mathsf{P}_t^{(n),\tau})_{n\ge 1}$ and the strong Markov property implies that (X_t) and (Y_t) are both stationary Markov processes associated with $\mathsf{P}_t^{(n),\tau}$.

In the following, $x = (\hat{x}_1, \dots, \hat{x}_n)$ and $y = (\hat{x}_1, \dots, \hat{x}_{n-1}, \hat{x}_{n+1})$. For any bounded function f on $(\mathbb{S}^1)^n$ and $t > 0$, and any bounded function h on $(\mathbb{S}^1)^{n+1}$, we have using the consistency of the family $(\mathsf{P}_t^{(n),\tau})_{n\geq 1}$ that

$$\begin{aligned}
&\left| \int_{(\mathbb{S}^1)^{n+1}} (\mathsf{P}_t^{(n),\tau} f(x) - \mathsf{P}_t^{(n),\tau} f(y)) h(\hat{x}) \; m_{n+1}^\tau(d\hat{x}) \right| \\
&\qquad = \; |\mathsf{E}[(f(X_t) - f(Y_t)) h(\hat{X}_0)]| \\
&\qquad \leq \; 2\|f\|_\infty \mathsf{E}[1_{t\leq T} |h(\hat{X}_0)|] \\
&\qquad \leq \; 2\|f\|_\infty \int |h(\hat{x})| \varepsilon_t(d(\hat{x}_n, \hat{x}_{n+1})) \; m_{n+1}^\tau(d\hat{x}), \qquad (16)
\end{aligned}$$

where $\varepsilon_t(r)$ is the probability that two independent Levy processes of exponent ψ, respectively started at x and y with $r = d(x, y)$, have not met before time t. Therefore, since points are not polar for the Levy process, for all $t > 0$, $\lim_{r\to 0} \varepsilon_t(r) = 0$.

Since the estimate (16) holds for all bounded function h, we have

$$|\mathsf{P}_t^{(n),\tau} f(x) - \mathsf{P}_t^{(n),\tau} f(y)| \leq 2\|f\|_\infty \varepsilon_t(d(\hat{x}_n, \hat{x}_{n+1})), \qquad m_{n+1}^\tau(d\hat{x}) - a.e. \quad (17)$$

Using this coupling n times, we obtain

$$|\mathsf{P}_t^{(n),\tau} f(x) - \mathsf{P}_t^{(n),\tau} f(y)| \leq 2\|f\|_\infty \sum_{i=1}^n \varepsilon_t(d(x_i, y_i)), \qquad m_{2n}^\tau(dx, dy) - a.e. \quad (18)$$

Therefore there exists a continuous function $\tilde{\mathsf{P}}_t^{(n),\tau} f$ such that $m_n^\tau(dx)$-a.e., $\tilde{\mathsf{P}}_t^{(n),\tau} f = \mathsf{P}_t^{(n),\tau} f$. It can be obtained by convolution with a sequence of continuous functions ρ_N with support in $B(0, 1/N)$ approximating δ_0, noting that the convolution products are equicontinuous.

Thus, this defines an everywhere defined version of $\mathsf{P}_t^{(n),\tau}$, which is a strong Feller semigroup. □

Remark 8 *Let f be a Lipschitz function on $(\mathbb{S}^1)^d$. We denote by $Lip(f)$ the Lipschitz constant of f. Using the same coupling as in the previous proof, it can be proved that*

$$|\mathsf{P}_t^{(n),\tau} f(x) - \mathsf{P}_t^{(n),\tau} f(y)| \leq Lip(f) \times \sum_{i=1}^n \mathsf{E}_{(x_i,y_i)}^{\otimes 2}[d(x_t, y_t) 1_{t\leq T_\Delta}], \qquad (19)$$

where under $\mathsf{P}_{(x,y)}^{\otimes 2}$, x_t and y_t are two independent $\mathbb{S}^1$-valued Levy processes started at x and y, and T_Δ is the first time x_t and y_t meet.

1.4 Sticky flows.

Let us now apply theorem 2.4.1 in [4] to the family of semigroups constructed in the previous section. This theorem (a generalization of De Finetti's theorem) states that starting from a consistent family of Feller semigroups it is possible to construct a stochastic flow of kernels $(K_{s,t},\ s \leq t)$ such that $\mathsf{E}[K_{0,t}^{\otimes n}] = \mathsf{P}_t^{(n),\tau}$. Thus to the family of Dirichlet forms $(\mathcal{E}_n^\tau)_{n\geq 1}$ is associated a stochastic flow of kernels which we will call the *sticky flow of parameter* τ *and exponent* ψ.

In [4], it is proved that every stochastic flow of kernels K admits a measurable modification K', i.e., for all $s \leq t$ and x, $K_{s,t}(x) = K'_{s,t}(x)$ a.s. and $(x,\omega) \mapsto K_{s,t}(x,\omega)$ is measurable. Thus, the sticky flows we will consider in the following will be assumed to be measurable.

Proposition 9 *Let $(K_{s,t},\ s \leq t)$ be a sticky flow of parameter τ and exponent ψ. Let μ denote an independent random probability measure $\sum_{i\geq 1} P_i \delta_{X_i}$, where (P_i) is a Dirichlet process of parameter $\frac{1-\tau}{\tau}$ and (X_i) is a sequence of independent random variables of law λ and independent of (P_i). Then,*

(a) *For every $t \in \mathbb{R}$, $\lambda K_{-T,t}$ converges weakly as $T \to \infty$ towards a probability measure on $\mathbb{S}^1$, denoted μ_t. Moreover, μ_t has the same law as μ.*

(b) *The sticky flow induces a Feller process ν_t on the space $\mathcal{M}_1^+(\mathbb{S}^1)$ of probability measures on $\mathbb{S}^1$ by the relation $\nu_t = \nu_0 K_{0,t}$. The stationary distribution of this Feller process is given by the law of μ.*

(c) *For all $x \in \mathbb{S}^1$ and $s \leq t$, a.s. $K_{s,t}(x)$ is an atomic measure.*

Proof. (a) and (b) : The fact that $\lambda K_{-T,t}$ converges weakly follows from the fact that for every continuous function f, $\lambda K_{-T,t} f = \int K_{-T,t} f(x) \lambda(dx)$ is a martingale in T, which therefore converges a.s.

The Feller property of the semigroup associated with ν_t can be easily proved from the Feller properties of the semigroups $\mathsf{P}_t^{(n),\tau}$ using the dense algebra of polynomial functions on $\mathcal{M}_1^+(\mathbb{S}^1)$ of the form

$$\hat{g}(\nu) = \int g(x_1, \ldots, x_n)\ \nu(dx_1) \cdots \nu(dx_n), \qquad g \in C((\mathbb{S}^1)^n), \quad n \in \mathbb{N}.$$

It is clear that the law of μ_t is a stationary distribution for this Feller process on $\mathcal{M}_1^+(\mathbb{S}^1)$. Thus it remains to show that the law of μ is the unique stationary distribution.

Note that for all $n \geq 1$, $\mathsf{E}[(\nu_t)^{\otimes n}] = \nu_0^{\otimes n} \mathsf{P}_t^{(n),\tau}$. Since $\mathsf{P}_t^{(n),\tau}$ is irreducible, $\mathsf{E}[(\nu_t)^{\otimes n}]$ converges towards $m_n^\tau = \mathsf{E}[\mu^{\otimes n}]$. This implies that if ν is a random probability measure on $\mathbb{S}^1$ whose law is a stationary distribution, then for all $n \geq 1$ we have $\mathsf{E}[\nu^{\otimes n}] = \mathsf{E}[\mu^{\otimes n}]$. This implies, since $\mathcal{M}_1^+(\mathbb{S}^1)$ is compact, that ν and μ have the same law.

(c) Note that the set of atomic measures is a Borel set in $\mathcal{M}_1^+(\mathbb{S}^1)$. Since μ is atomic and stationary, $\mu K_{0,t}$ is atomic and distributed like μ. Since $\mu K_{0,t} = \int K_{0,t}(x)\mu(dx)$, we have $\mu(dx)$-a.e. $K_{0,t}(x)$ is atomic a.s. Thus, since $K_{0,t}$ and μ are independent, $\lambda \otimes \mathsf{P}$-a.s. $K_{0,t}(x)$ is atomic. Using the rotation invariance, it implies that for every x, we have that a.s., $K_{0,t}(x)$ is atomic. □

2 The noise of a sticky flow is black.

Let us first recall Tsirelson's definition of a noise (see [8, 9]) : A *noise* consists of a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a family $(\mathcal{F}_{s,t})_{s\leq t}$ of sub-σ-fields (also called a factorization) of $\mathcal{F}$ and a L^2-continuous one-parameter group $(T_h)_{h\in\mathbb{R}}$ of transformations of Ω preserving P such that

(i) for all $s \leq t \leq u$, $\mathcal{F}_{s,u} = \sigma(\mathcal{F}_{s,t} \cup \mathcal{F}_{t,u})$.

(ii) for all $t_0 < \cdots < t_n$, $(\mathcal{F}_{t_{i-1},t_i})_{1\leq i\leq n}$ is a family of independent σ-fields.

(iii) for all $s \leq t$ and $h \in \mathbb{R}$, $T_h(\mathcal{F}_{s,t}) = \mathcal{F}_{s+h,t+h}$.

In [4], a noise $N = (\Omega, \mathcal{F}, (\mathcal{F}_{s,t}), \mathsf{P}, (T_h))$ is associated to the law of a stochastic flow of kernels on a locally compact separable metric space M, with $\Omega = \prod_{s<t} E$ (E is the space of kernels on M), $\mathcal{F} = \otimes_{s<t}\mathcal{B}(E)$ ($\mathcal{B}(E)$ is the Borel σ-field on E), $K_{s,t}^0(\omega) = \omega(s,t)$, P the law of the stochastic flow of kernels, $\mathcal{F}_{s,t}$ the σ-field $\sigma(K_{u,v}^0,\ s < u < v < t)$ completed by all P-negligible sets of $\mathcal{F}$ and T_h defined by $T_h(\omega)(s,t) = \omega(s+h, t+h)$. This noise is called *the noise of the stochastic flow of kernels* K^0. In the following, K denotes a measurable modification of K^0.

A process $X = (X_{s,t})_{s\leq t}$ is called a *centered linear representation* of a noise $N = (\Omega, \mathcal{F}, (\mathcal{F}_{s,t}), \mathsf{P}, (T_h))$ if for all $s \leq t$, $X_{s,t} \in L_0^2(\mathcal{F}_{s,t})$ (L_0^2 is the set of all L^2-functions with mean 0) and for all $s \leq t \leq u$, a.s. $X_{s,u} = X_{s,t} + X_{t,u}$.

Such a process is also called a decomposable process. We denote by H_0^{lin} the space of centered linear representations of N. The *linear part* of N, denoted $N^{\text{lin}} = (\Omega, \mathcal{F}, (\mathcal{F}_{s,t}^{\text{lin}}), \mathsf{P}, (T_h))$, is a subnoise of N (i.e., $\mathcal{F}_{s,t}^{\text{lin}} \subset \mathcal{F}_{s,t}$) where $\mathcal{F}_{s,t}^{\text{lin}} = \sigma(X_{u,v},\ X \in H_0^{\text{lin}},\ s \le u \le v \le t)$. A noise is called *black* if N^{lin} is a trivial noise, or equivalently if $H_0^{\text{lin}} = \{0\}$.

A sticky flow of parameter τ and exponent ψ with $\psi(k) = \sigma^2|k|^\alpha$ is called a *stable sticky flow of parameter τ and index α*. When $\alpha = 2$ it is called a *Brownian sticky flow of parameter τ*.

Note that condition (7) implies that $\alpha \in]1,2]$. The one-point motion X_t^1 of a stable sticky flow of index α can be described with a stable process x_t of index α by the relation $X_t^1 = x_t\ [2\pi]$. We denote by P_x the law of a stable process starting from x. We recall the scaling property : If X_t is a stable process of law P_x, then the law of $(\lambda^{-1/\alpha} x_{\lambda t},\ t \ge 0)$ is $\mathsf{P}_{\lambda^{-1/\alpha} x}$. Note also that for all $\rho < \alpha$, we have $\mathsf{E}_0[|x_t|^\rho] = t^{\rho/\alpha}\mathsf{E}_0[|x_1|^\rho] < \infty$.

The purpose of this section is to prove

Theorem 10 *The noise of a stable sticky flow is black.*

Remark 11 *We conjecture the noise of any sticky flow is black.*

A noise N is called *continuous* (see [9]) (or the factorization $(\mathcal{F}_{s,t})$ is called continuous) if for all $s < t$, $\cup_{\varepsilon > 0}\mathcal{F}_{s+\varepsilon, t-\varepsilon}$ generates $\mathcal{F}_{s,t}$ and $\cup_{n=1}^\infty \mathcal{F}_{-n,n}$ generates $\mathcal{F}$.

Note that a noise N is continuous as soon as $L^2(\Omega, \mathcal{F}, \mathsf{P})$ is separable (see remark 3e3 in [9] and lemma 2.1 in [8]). Indeed, the continuity of N follows from the continuity of the filtrations $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$ and $(\mathcal{F}_{s,\infty})_{s\in\mathbb{R}}$. Due to the separability, the sets of discontnuities of these filtrations are at most countable and are invariant under the action of the shift operators $(T_h)_{h\in\mathbb{R}}$. Thus these sets are empty and the filtrations are continuous. This implies that the noise of a stochastic flow of kernels is continuous (the separability condition is satisfied).

From now on, $N_{\tau,\alpha}$ denotes the noise of a stable sticky flow of parameter τ and index α.

Let $\mathcal{H}_1 = \{Z \in L^2(\Omega, \mathcal{F}, \mathsf{P}) :\ ((Z_{s,t} = \mathsf{E}[Z|\mathcal{F}_{s,t}]))_{s\le t} \in H_0^{\text{lin}}\}$ be the first chaos of N. Let $H_1(s,t) = \{Z_{s,t} :\ Z \in H_0^{\text{lin}}\} = \mathcal{H}_1 \cap L^2(\Omega, \mathcal{F}_{s,t}, \mathsf{P})$. For every $Z \in L_0^2(\Omega)$, $H_{s,t}(Z)$ denotes the orthogonal projection of Z on $H_1(s,t)$.

We set $H = H_{0,1}$. Since $N_{\tau,\alpha}$ is continuous,

$$H(Z) = \lim_{n\to\infty} \sum_{k=1}^{2^n} \mathsf{E}[Z|\mathcal{F}_{(k-1)2^{-n},k2^{-n}}] \tag{20}$$

(see theorem 6.a.4 in [9]). Note that $N_{\tau,\alpha}$ is black if $H(Z) = 0$ for all $Z \in L_0^2(\mathsf{P})$, or equivalently for all $Z \in L_0^2(\mathcal{F}_{0,1})$.

In the following $\mathsf{P}_t^{(d)}$ denotes the Feller semigroup associated with the d-point motion of a sticky flow of parameter τ and exponent ψ.

Proposition 12 *Let f be a Lipschitz function on $(\mathbb{S}^1)^d$ and h be a nonnegative continuous function on $\mathbb{S}^1$, then*

$$H\left(\langle K_{0,1}^{\otimes d} f, h\rangle_{L^2(m_d^\tau)} - \langle \mathsf{P}_1^{(d)} f, h\rangle_{L^2(m_d^\tau)}\right) = 0. \tag{21}$$

Proof in the Brownian case.

Set $Z = \langle K_{0,1}^{\otimes d} f, h\rangle_{L^2(m_d^\tau)} - \langle \mathsf{P}_1^{(d)} f, h\rangle_{L^2(m_d^\tau)}$. The result holds if

$$\sum_{k=1}^{2^n} \mathsf{E}[Z|\mathcal{F}_{(k-1)2^{-n},k2^{-n}}]$$

converges towards 0 in $L^2(\mathsf{P})$ as $n \to \infty$. Since the terms in the sum are independent, this holds if

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \mathsf{E}\left[\left(\mathsf{E}[Z|\mathcal{F}_{(k-1)2^{-n},k2^{-n}}]\right)^2\right] = 0. \tag{22}$$

For all $0 \le s < t \le 1$,

$$\mathsf{E}[\langle K_{0,1}^{\otimes d} f, h\rangle_{L^2(m_d^\tau)} | \mathcal{F}_{s,t}] = \langle \mathsf{P}_s^{(d)} K_{s,t}^{\otimes d} \mathsf{P}_{1-t}^{(d)} f, h\rangle_{L^2(m_d^\tau)}.$$

Set $\mu_s(dx) = \int_{x_0 \in (\mathbb{S}^1)^d} h(x_0) \mathsf{P}_s^{(d)}(x_0, dx) m_d^\tau(dx_0)$, $g_t = \mathsf{P}_{1-t}^{(d)} f$ and $\varepsilon = t - s$. Then

$$\mathsf{E}\left[(\mathsf{E}[Z|\mathcal{F}_{s,t}])^2\right] = \mu_s^{\otimes 2}\left(\mathsf{P}_\varepsilon^{(2d)}(g_t \otimes g_t) - \mathsf{P}_\varepsilon^{(d)\otimes 2}(g_t \otimes g_t)\right),$$

where $\mathsf{P}_t^{(2d)}$ is the semigroup of the $2d$-point motion of the sticky flow and $\mathsf{P}_t^{(d)\otimes 2}$ is the semigroup of two independent d-point motions. The measure μ_s is absolutely continuous with respect to m_d^τ and we have $\frac{d\mu_s}{dm_d^\tau} \le \|h\|_\infty < \infty$ (indeed, for $f \ge 0$, $\mu_s f = \langle \mathsf{P}_s^{(d)} f, h\rangle_{L^2(m_d^\tau)} = \langle f, \mathsf{P}_s^{(d)} h\rangle_{L^2(m_d^\tau)} \le \|h\|_\infty m_d^\tau f$).

Lemma 13 *There exists a contant C such that for all $t \in [0,1]$ we have that $Lip(g_t) \leq C Lip(f)$.*

Proof. For $t \geq 0$, x and y in $\mathbb{S}^1$,

$$\mathsf{E}^{\otimes 2}_{(x,y)}[d(X_t, Y_t) 1_{t \leq T_\Delta}] \leq \mathsf{E}[|d(x,y) + B_t| 1_{t \leq T}],$$

where B is a Brownian motion starting from 0 with $\mathsf{E}[B_t^2] = 2\sigma^2 t$ and T is the first time $d(x,y) + B_t = 0$. Since $M_t = d(x,y) + B_t$ is a martingale, we have

$$\mathsf{E}[|d(x,y) + B_t| 1_{t \leq T}] = \mathsf{E}[M_{t \wedge T}] = d(x,y).$$

We conclude using remark 8. □

It is easy to check, using the consistency relations of the family $(\mathsf{P}^{(n)}_t)$ that for all $(x,y) \in (\mathbb{S}^1)^d \times (\mathbb{S}^1)^d$

$$\begin{aligned}\left(\mathsf{P}^{(2d)}_\varepsilon(g_t \otimes g_t) - \mathsf{P}^{(d)\otimes 2}_\varepsilon(g_t \otimes g_t)\right)(x,y) \\ = \frac{1}{2}(\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)})[(g_t(X_\varepsilon) - g_t(Y_\varepsilon))^2],\end{aligned} \quad (23)$$

where under $\mathsf{P}^{(2d)}_{(x,y)}$ (resp. under $\mathsf{P}^{(d)\otimes 2}_{(x,y)}$) (X_t, Y_t) is the $2d$-point motion started at (x,y) (resp. X_t and Y_t are independent d-point motions respectively started at x and y).

Lemma 14 *There exists $C_1 = C_1(d, \psi)$ a positive constant such that for all Lipschitz function g on $(\mathbb{S}^1)^d$, $t > 0$, x and y in $(\mathbb{S}^1)^d$, we have*

$$(\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)})[(g(X_t) - g(Y_t))^2] \leq C_1 (Lip(g))^2 \times t. \quad (24)$$

Proof. We have

$$\begin{aligned}&(\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)})[(g(X_t) - g(Y_t))^2] \\ &= (\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)})[((g(X_t) - g(x)) + (g(y) - g(Y_t)))^2] \\ &+ (\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)})[(g(x) - g(y))^2] \\ &+ 2(\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)})[(g(x) - g(y))((g(X_t) - g(x)) + (g(y) - g(Y_t)))].\end{aligned}$$

It is easy to see that the second and the third terms vanish. Thus

$$
\begin{aligned}
(\mathsf{E}^{(d)\otimes 2}_{(x,y)} - \mathsf{E}^{(2d)}_{(x,y)}) & \left[(g(X_t) - g(Y_t))^2\right] \\
& \leq \mathsf{E}^{(d)\otimes 2}_{(x,y)}[((g(X_t) - g(x)) + (g(y) - g(Y_t)))^2] \\
& \leq 2\mathsf{E}^{(d)\otimes 2}_{(x,y)} \left[(g(X_t) - g(x))^2 + (g(y) - g(Y_t))^2\right] \quad (25) \\
& \leq 2(\mathrm{Lip}(g))^2 \mathsf{E}^{(d)\otimes 2}_{(x,y)} \left[d(X_t, x)^2 + d(Y_t, y)^2\right]. \quad (26) \\
& \leq 2(\mathrm{Lip}(g))^2 \times (2d\sigma^2 t). \quad (27)
\end{aligned}
$$

This proves the lemma. □

Set $T = \inf\{t,\ \{X_t\} \cap \{Y_t\} \neq \emptyset\} = \inf_{1 \leq i,j \leq d} T_{i,j}$, where $T_{i,j} = \inf\{t,\ X_t^i = Y_t^j\}$.

Lemma 15 *There exists a constant $C_2 = C_2(d)$ such that for all positive ε,*

$$\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T < \varepsilon] < C_2\sqrt{\varepsilon}. \quad (28)$$

Proof. We have $\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T < \varepsilon] \leq \sum_{1 \leq i,j \leq d} \mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T_{i,j} < \varepsilon]$. Let us remark that for all i, j

$$\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T_{i,j} < \varepsilon] = \frac{1}{2\pi^2} \int_{0<x<y<2\pi} \mathsf{P}_{y-x}[T \leq 2\varepsilon] dx dy,$$

with $T = \inf\{t, B_t \in \{0, 2\pi\}\}$ and where, under P_z, B_t is a Brownian motion starting at z (we use the fact that the law of $(Y^j_{t/2} - X^i_{t/2},\ t < 2T)$ under $\mathsf{P}^{(d)\otimes 2}_{(x,y)}$ is the same as the law of $(B_t,\ t < T)$ under P_z). Since $\mathsf{P}_z[T \leq 2\varepsilon] \leq \mathsf{P}_z[T_0 \leq 2\varepsilon] + \mathsf{P}_{2\pi - z}[T_0 \leq 2\varepsilon]$, where $T_0 = \inf\{t,\ B_t = 0\}$, we have

$$\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T_{i,j} < \varepsilon] \leq \frac{2}{\pi} \int_0^{2\pi} \mathsf{P}_z[T_0 < 2\varepsilon]\ dz.$$

Using the reflection principle, we get $\mathsf{P}_z[T_0 < 2\varepsilon] \leq 2\mathsf{P}_0[B_{2\varepsilon} > z]$. Thus

$$
\begin{aligned}
\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T_{i,j} < \varepsilon] & \leq \frac{4}{\pi} \int_0^{2\pi} \mathsf{P}_0[B_{2\varepsilon} > z]\ dz \\
& \leq \frac{4}{\pi} \sqrt{2\varepsilon}\, \mathsf{E}_0[|B_1|].
\end{aligned}
$$

This proves the lemma. □

Lemma 16 *There exists a constant* $C_3 = C_3(d, Lip(f), \|h\|_\infty)$ *such that for all* $0 < s < t < 1$ *and* $\varepsilon = t - s$,

$$\mu_s^{\otimes 2}\left(\mathsf{P}_\varepsilon^{(2d)}(g_t \otimes g_t) - \mathsf{P}_\varepsilon^{(d)\otimes 2}(g_t \otimes g_t)\right) \leq C_3 \times \varepsilon^{3/2}. \tag{29}$$

Proof. Using first the fact that $((X_s, Y_s),\ s \leq T)$ has the same law under $\mathsf{P}_{(x,y)}^{(2d)}$ and under $\mathsf{P}_{(x,y)}^{(d)\otimes 2}$, then the strong Markov property at time T and finally lemma 14, we have for all Lipschitz function g

$$\begin{aligned}
&(\mathsf{E}_{(x,y)}^{(d)\otimes 2} - \mathsf{E}_{(x,y)}^{(2d)})[(g(X_\varepsilon) - g(Y_\varepsilon))^2] \\
&\quad = (\mathsf{E}_{(x,y)}^{(d)\otimes 2} - \mathsf{E}_{(x,y)}^{(2d)})[1_{T<\varepsilon}(g(X_\varepsilon) - g(Y_\varepsilon))^2] \\
&\quad = \mathsf{E}_{(x,y)}^{(d)\otimes 2}\left[1_{T<\varepsilon}(\mathsf{E}_{(X_T,Y_T)}^{(d)\otimes 2} - \mathsf{E}_{(X_T,Y_T)}^{(2d)})[(g(X_{\varepsilon-T}) - g(Y_{\varepsilon-T}))^2]\right] \\
&\quad \leq (C_1(\mathrm{Lip}(g))^2) \times \varepsilon \times \mathsf{P}_{(x,y)}^{(d)\otimes 2}[T < \varepsilon].
\end{aligned}$$

Using the fact that for all $t \in [0,1]$, $\mathrm{Lip}(g_t) \leq C\mathrm{Lip}(f)$ and equation (23), we get

$$\begin{aligned}
&\mu_s^{\otimes 2}\left(\mathsf{P}_\varepsilon^{(2d)}(g_t \otimes g_t) - \mathsf{P}_\varepsilon^{(d)\otimes 2}(g_t \otimes g_t)\right) \\
&\qquad \leq (C_1(C\mathrm{Lip}(f))^2) \times \varepsilon \times \int \mu_s^{\otimes 2}(dx, dy)\mathsf{P}_{(x,y)}^{(d)\otimes 2}[T < \varepsilon] \\
&\qquad \leq (C_1(C\mathrm{Lip}(f))^2\|h\|_\infty^2) \times \varepsilon \times \mathsf{P}_{(m_d^\tau)^{\otimes 2}}^{(d)\otimes 2}[T < \varepsilon]. \qquad \square
\end{aligned}$$

Lemma 16 and lemma 15 implies that

$$\mathsf{E}[(\mathsf{E}[Z|\mathcal{F}_{(k-1)2^{-n},k2^{-n}}])^2] \leq C_3 \times 2^{-3n/2}.$$

Thus

$$\sum_{k=1}^{2^n} \mathsf{E}[(\mathsf{E}[Z|\mathcal{F}_{(k-1)2^{-n},k2^{-n}}])^2] \leq C_3 \times 2^{-n/2}, \tag{30}$$

which converges towards 0 as $n \to \infty$. Therefore $H(Z) = 0$. This proves proposition 12. $\square$

Proof in the case $1 < \alpha < 2$.

The proof is essentially the same. We replace lemma 13 by

Lemma 17 *Let f be a Lipschitz function f. Then there exists a constant $C = C(Lip(f), d)$ such that for all $t \in [0,1]$, we have*

(i) *$\mathsf{P}_t^{(d)} f$ is Hölder of exponent $\alpha - 1$, and*

(ii) *for all x and y in $(\mathbb{S}^1)^d$,*

$$|\mathsf{P}_t^{(d)} f(y) - \mathsf{P}_t^{(d)} g(y)| \leq C \times d(x,y)^{\alpha - 1}.$$

Before proving this lemma we recall this result

Lemma 18 *There exists $c > 0$ such that as $t \to \infty$,*

$$\mathsf{P}_0[T_1 \geq s] \sim c s^{-1+1/\alpha}, \tag{31}$$

where P_0 is the law of a stable process x_t of index α starting from 0 and T_1 is the first time it hits 1.

Proof. For $q > 0$,

$$\int_0^\infty q e^{-qs} \mathsf{P}_0[T_1 > s] ds = 1 - E_1[e^{-qT_0}].$$

Corollary 18 and Theorem 19 in the section II.5 in [1] imply

$$1 - E_1[e^{-qT_0}] = \left(\int_0^\infty \frac{1 - \cos(t)}{q + t^\alpha} dt \right) \times \left(\int_0^\infty \frac{dt}{q + t^\alpha} \right)^{-1}.$$

We have

$$\lim_{q \to 0} \int_0^\infty \frac{1 - \cos(t)}{q + t^\alpha} dt = \int_0^\infty \frac{1 - \cos(t)}{t^\alpha} dt < \infty$$

and

$$\int_0^\infty \frac{dt}{q + t^\alpha} = q^{-1+1/\alpha} \int_0^\infty \frac{dt}{1 + t^\alpha}.$$

Thus there exists a constant $c > 0$ such that as $q \to 0+$, $1 - E_1[e^{-qT_0}] \sim c q^{1-1/\alpha}$. We conclude using the Tauberian theorem. □

Proof of lemma 17. We still use remark 8. Note that

$$\mathsf{E}^{\otimes 2}_{(x,y)}[d(x_t, y_t) 1_{t \leq T_\Delta}] \leq \mathsf{E}_{d(x,y)}[|z_t| 1_{t \leq T_0}],$$

where z_t is a stable process of index α (with $\psi(k) = \exp(-2c|k|^\alpha)$) and T_0 (respectively, T_r) is the first time it hits 0 (respectively, r). Set $r = d(x,y)$ and $s = r^{-\alpha}t$. We have

$$\mathsf{E}_r[|z_t|1_{t\le T_0}] = (\mathsf{E}_r[|z_t|] - \mathsf{E}_0[|z_t|]) + (\mathsf{E}_0[|z_t|] - \mathsf{E}_r[|z_t|1_{t\ge T_0}]). \quad (32)$$

Note that

$$|\mathsf{E}_r[|z_t|] - \mathsf{E}_0[|z_t|]| = |\mathsf{E}_0[|r+z_t| - |z_t|]| \le r.$$

We have $\mathsf{E}_0[|z_t|] = t^{1/\alpha}\mathsf{E}_0[|z_1|]$. Using the strong Markov property at T_0 and the scaling property we have

$$\begin{aligned}\mathsf{E}_r[|z_t|1_{t\ge T_0}] &= \mathsf{E}_r[1_{t\ge T_0}(t-T_0)^{1/\alpha}] \times \mathsf{E}_0[|z_1|]\\ &= \mathsf{E}_0[1_{t\ge T_r}(t-T_r)^{1/\alpha}] \times \mathsf{E}_0[|z_1|].\end{aligned}$$

Thus,

$$\mathsf{E}_0[|z_t|] - \mathsf{E}_r[|z_t|1_{t\ge T_0}] = t^{1/\alpha}(1 - \mathsf{E}_0[1_{t\ge T_r}(1-T_r/t)^{1/\alpha}]) \times \mathsf{E}_0[|z_1|]. \quad (33)$$

Since under P_0, T_r is distributed like $r^\alpha T_1$, we have

$$\mathsf{E}_0[1_{t\ge T_r}(1-T_r/t)^{1/\alpha}] = \mathsf{E}_0[1_{s\ge T_1}(1-T_1/s)^{1/\alpha}].$$

We then have

$$\begin{aligned}1 - \mathsf{E}_0[1_{t\ge T_r}(1-T_r/t)^{1/\alpha}] &= 1 - \mathsf{E}_0[1_{s\ge T_1}(1-T_1/s)^{1/\alpha}]\\ &= \mathsf{P}_0[T_1 > s] + \mathsf{E}_0[1_{s\ge T_1}(1-(1-T_1/s)^{1/\alpha})].\end{aligned}$$

We remark that

$$\begin{aligned}\mathsf{E}_0[1_{s\ge T_1}(1-(1-T_1/s)^{1/\alpha})] &= \alpha\int_0^1 \mathsf{P}_0[vs \le T_1 \le s](1-v)^{-1+1/\alpha}dv\\ &\le \alpha\int_0^1 \mathsf{P}_0[vs \le T_1](1-v)^{-1+1/\alpha}dv.\end{aligned}$$

Since as $s \to \infty$, $\mathsf{P}_0[T_1 \ge s] \sim cs^{-1+1/\alpha}$, there exists a constant c' such that

$$\mathsf{P}_0[T_1 \ge s] \le c's^{-1+1/\alpha}, \qquad s \ge 0.$$

Thus

$$\begin{aligned}\mathsf{E}_0[1_{s\ge T_1}(1-(1-T_1/s)^{1/\alpha})] &\le \alpha c's^{-1+1/\alpha}\int_0^1 v^{-1+1/\alpha}(1-v)^{-1+1/\alpha}dv\\ &= Cs^{-1+1/\alpha}\end{aligned}$$

where C is a finite constant. Finally the second term in (32) is dominated by $(c'+C)\mathsf{E}_0[|z_1|]t^{1/\alpha}s^{-1+1/\alpha}$. Since $s=r^{-\alpha}t$, $t\leq 1$ and $\alpha\leq 2$,

$$t^{1/\alpha}s^{-1+1/\alpha}=r^{\alpha-1}t^{\frac{2-\alpha}{\alpha}}\leq r^{\alpha-1}.$$

This proves the lemma (for $r\leq 1$, $r\leq r^{\alpha-1}$). □

We replace lemma 14 by

Lemma 19 *There exists $C_1=C_1(d,\psi)$ a positive constant such that for all Hölder function g of order $\alpha-1$ on $(\mathbb{S}^1)^d$, $t>0$, x and y in $(\mathbb{S}^1)^d$, we have*

$$(\mathsf{E}^{(d)\otimes 2}_{(x,y)}-\mathsf{E}^{(2d)}_{(x,y)})[(g(X_t)-g(Y_t))^2]\leq C_1(H\ddot{o}l(g))^2\times t^{2\frac{\alpha-1}{\alpha}}, \tag{34}$$

where $H\ddot{o}l(g)=\sup_{x,y}|g(y)-g(x)|/d(x,y)^{\alpha-1}$.

Proof. We follow the proof of lemma 14 up to (25) and we dominate the term (25) by

$$2(\text{Höl}(g))^2\mathsf{E}^{(d)^{\otimes 2}}_{(x,y}[d(X_t,x)^{2(\alpha-1)}+d(X_t,y)^{2(\alpha-1)}].$$

We have for $p\in\mathbb{S}^1$

$$\begin{aligned}\mathsf{E}^{(d)}_x[d(X_t,x)^{2(\alpha-1)}] &\leq d^{2(\alpha-1)}\mathsf{E}^{(1)}_p[d(X_t,p)^{2(\alpha-1)}]\\ &\leq d^{2(\alpha-1)}\mathsf{E}_0[|x_t|^{2(\alpha-1)}].\end{aligned}$$

Note that since $2(\alpha-1)<\alpha$,

$$\mathsf{E}_0[|x_t|^{2(\alpha-1)}]=t^{2(\alpha-1)/\alpha}\mathsf{E}_0[|x_1|^{2(\alpha-1)}]<\infty.$$

This proves the lemma. □

We replace lemma 15 by

Lemma 20 *There exists a constant $C_2=C_2(d)$ such that for all positive ε,*

$$\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T<\varepsilon]<C_2\varepsilon^{1/\alpha}. \tag{35}$$

Proof. We have

$$\mathsf{P}^{(d)\otimes 2}_{(m_d^\tau)^{\otimes 2}}[T<\varepsilon]\leq c_d\mathsf{P}^{\otimes 2}_{m^{\otimes 2}}[T_\Delta<\varepsilon]$$

where T_Δ is the first time $x_t = y_t$. We also have for x and y in $\mathbb{S}^1$ and $r = |x - y| \in [0, \pi]$ (below, under P_0, x_t is a stable process of index α starting at 0)

$$\begin{aligned} \mathsf{P}^{\otimes 2}_{(x,y)}[T_\Delta < \varepsilon] &= \mathsf{P}_0[\exists t < \varepsilon,\ x_t = r[2\pi]] \\ &\leq \mathsf{P}_0[S_\varepsilon \geq r], \end{aligned}$$

where $S_\varepsilon = \sup_{t<\varepsilon} |x_t|$. The scaling property implies that

$$\mathsf{P}_0[S_\varepsilon \geq r] = \mathsf{P}_0[S_1 \geq \varepsilon^{-1/\alpha} r].$$

Thus for some constant C,

$$\begin{aligned} \mathsf{P}^{(d)\otimes 2}_{(m^\tau_d)^{\otimes 2}}[T < \varepsilon] &\leq C \int_0^\pi \mathsf{P}_0[S_1 \geq \varepsilon^{-1/\alpha} r] dr \\ &\leq C\varepsilon^{1/\alpha} \int_0^{\pi\varepsilon^{-1/\alpha}} \mathsf{P}_0[S_1 \geq r] dr. \end{aligned}$$

Since there exists a constant $k > 0$ such that $\mathsf{P}_0[S_1 \geq r] \sim kr^{-\alpha}$ as $r \to \infty$ (see proposition VIII.4 in [1]), $\int_0^\infty \mathsf{P}_0[S_1 \geq r]dr$ is finite. This proves the lemma. □

We replace lemma 16 by

Lemma 21 *There exists a constant $C_3 = C_3(d, Lip(f), \|h\|_\infty)$ such that for all $0 < s < t < 1$ and $\varepsilon = t - s$,*

$$\mu_s^{\otimes 2}\left(\mathsf{P}^{(2d)}_\varepsilon(g_t \otimes g_t) - \mathsf{P}^{(d)\otimes 2}_\varepsilon(g_t \otimes g_t)\right) \leq C_3 \times \varepsilon^{2-1/\alpha}. \tag{36}$$

Proof. This is the same proof but now we use lemmas 19 and 20 instead oflemmas 14 and 15. □

We end the proof of proposition 12 by replacing the upperestimate (30) by $C_3 2^{-n}$ by an upperestimate by $C_3 2^{-n(1-1/\alpha)}$ which also converges towards 0. □

Proof of the theorem.

We are going to show that $H(Z) = 0$ for $Z \in L^2_0(\mathcal{F}_{0,1})$. Let V denote the vector space spanned by constants and functions of the form

$$Z = \prod_{i=1}^n \langle K^{\otimes d}_{t_{i-1},t_i} f^i, h^i \rangle_{L^2(m^\tau_d)}, \tag{37}$$

for all $n \geq 1$, $d \geq 1$, $(f_i)_i$ a family of Lipschitz functions on $(\mathbb{S}^1)^d$, (h_i) a family of nonnegative continuous functions on $(\mathbb{S}^1)^d$ and $0 = t_0 < t_1 < \cdots < t_n = 1$. Let V_0 be the set of functions in V such that $\mathsf{E}[Z] = 0$. Then V is dense in $L^2(\mathcal{F}_{0,1})$ and V_0 is dense in $L^2_0(\mathcal{F}_{0,1})$. Thus to prove that $H(Z) = 0$ for all $Z \in L^2_0(\mathcal{F}_{0,1})$, it is enough to prove $H(Z) = 0$ for all $Z \in V_0$. Let Z be in the form (37), then for all i, there exists a constant c_i such that

$$\mathsf{E}[Z|\mathcal{F}_{t_{i-1},t_i}] = c_i \langle K^{\otimes d}_{t_{i-1},t_i} f^i, h^i \rangle_{L^2(m^\tau_d)}.$$

Thus, if we take Z such that for all i, $\mathsf{E}[\langle K^{\otimes d}_{t_{i-1},t_i} f^i, h^i\rangle_{L^2(m^\tau_d)}] = 0$, $\mathsf{E}[Z] = 0$ (V_0 is spanned by functions in this form) and

$$\begin{aligned} H(Z) &= \sum_{i=1}^n H_{t_{i-1},t_i}(Z) \\ &= \sum_{i=1}^n c_i H_{t_{i-1},t_i}(\langle K^{\otimes d}_{t_{i-1},t_i} f^i, h^i\rangle_{L^2(m^\tau_d)}). \end{aligned}$$

Proposition 12 shows that $H(Z) = 0$. This proves the theorem. □

3 Arratia's coalescing flows.

Let P_t be the Feller semigroup of a Levy process on $\mathbb{S}^1$ for which points are not polar. Using theorem 4.3.1. in [4], starting from the consistent family of Feller semigroups $(\mathsf{P}^{\otimes n}_t)$, we construct a consistent family of Feller semigroups $(\mathsf{P}^{\otimes n,c}_t)$, such that for every $x \in (\mathbb{S}^1)^n$, if $X^{(n),c}$ denotes the associated n-point motion starting from x and $T_{i,j} = \inf\{t, X^{(n),c}_i(t) = X^{(n),c}_j(t)\}$, then for $t \geq T_{i,j}$, $X^{(n),c}_i(t) = X^{(n),c}_j(t)$ and $(X^{(n),c}_t,\ t \leq \inf_{1\leq i,j\leq n} T_{i,j}\}$ is distributed like n independent Levy processes respectively starting from $x_1, \ldots, x_n$ and stopped at the first time two of them hit.

To this family of Feller semigroups is associated a coalescing flow. We call it a *Arratia's coalescing flow of exponent* ψ. When $\psi(k) = \sigma^2|k|^\alpha$ with $\alpha \in]1, 2]$, we call it a *Arratia's coalescing flow of index* α.

Theorem 22 *The noise generated by a Arratia's coalescing flow of index α is black.*

Remark 23 *In the case the one-point motion is a Brownian motion, this result was proved by Tsirelson (cf [9]). The proof we give here is based on* *a criterium he gave to prove that a noise is black (zero quadratic variation), but involves different techniques.*

Proof. We follow the proof of theorem 10. For $d \geq 1$, set $m_d = m_d^{\frac{1}{2}}$. We let φ be an Arratia's coalescing flow of index α.

Like in the proof of theorem 10, we only need to prove that if f is a Lipschitz function on $(\mathbb{S}^1)^d$ and h is a nonnegative continuous function on $\mathbb{S}^1$, $H(Z) = 0$, where

$$Z = \langle f \circ \varphi_{0,1}^{\otimes d}, h\rangle_{L^2(m_d)} - \langle \mathsf{P}_1^{(d),c} f, h\rangle_{L^2(m_d)}.$$

In the following, in order to simplify the notation, we denote $\mathsf{P}_t^{\otimes d,c}$ by $\mathsf{P}_t^{(d)}$. For all $0 \leq s < t \leq 1$,

$$\mathsf{E}[\langle f \circ \varphi_{0,1}^{\otimes d} f, h\rangle_{L^2(m_d)} | \mathcal{F}_{s,t}] = \langle \mathsf{P}_s^{(d)} K_{s,t}^{\otimes d} \mathsf{P}_{1-t}^{(d)} f, h\rangle_{L^2(m_d^\tau)},$$

where $K_{s,t}(x)$ denotes $\delta_{\varphi_{s,t}(x)}$.

Set $\mu_s(dx) = \int_{x_0 \in (\mathbb{S}^1)^d} h(x_0) \mathsf{P}_s^{(d)}(x_0, dx) m_d(dx_0)$, $g_t = \mathsf{P}_{1-t}^{(d)} f$ and $\varepsilon = t - s$. Then

$$\mathsf{E}\left[(\mathsf{E}[Z|\mathcal{F}_{s,t}])^2\right] = \mu_s^{\otimes 2}\left(\mathsf{P}_\varepsilon^{(2d)}(g_t \otimes g_t) - \mathsf{P}_\varepsilon^{(d)\otimes 2}(g_t \otimes g_t)\right).$$

Lemma 24 *There exists a positive constant $D_1 = D_1(d)$ such that for all $\pi \in \mathcal{P}_d$, $\lambda_\pi \mathsf{P}_s^{(d)}$ is absolutely continuous with respect to m_d, with*

$$\frac{d\lambda_\pi \mathsf{P}_s^{(d)}}{dm_d} \leq D_1.$$

Proof. For $\pi \in \mathcal{P}_d$, let $j_\pi : (\mathbb{S}^1)^d \to E_\pi$ defined by $(j_\pi(x_1, \ldots, x_d))_k = (y_1, \ldots, y_k)$ with $y_i = x_j$, where j is the smaller element of the set of π containing i.

The coalescing d point Levy process $X_t^{(d)}$ can be constructed starting with d independent Levy processes X_t^i by applying progressively in time the following rule : when two paths meet at time t, the path of higher index is replaced after time t by the path of lower index. Then for each time t, there exists a random partition $\pi_t \in \mathcal{P}_d$ such that $X_t^{(d)} = j_{\pi_t}(X_t^1, \ldots, X_t^d)$.

Thus, for all positive function f,

$$\begin{aligned}
\mathsf{P}_t^{(d)} f(x) &= \mathsf{E}_{x_1,\ldots,x_d}^{\otimes d}[f \circ j_{\pi_t}(X_t^1, \ldots, X_t^d)] \\
&\leq \sum_{\pi \in \mathcal{P}_d} \mathsf{E}_{x_1,\ldots,x_d}^{\otimes d}[f \circ j_\pi(X_t^1, \ldots, X_t^d)] \\
&\leq \sum_{\pi \in \mathcal{P}_d} \mathsf{P}_t^{\otimes d}(f \circ j_\pi)(x).
\end{aligned}$$

This implies that for all $\pi \in \mathcal{P}_d$,

$$\begin{aligned} \lambda^{\otimes d}\mathsf{P}_t^{(d)} f &\leq \sum_{\pi\in\mathcal{P}_d} \lambda^{\otimes d}\mathsf{P}_t^{\otimes d}(f\circ j_\pi)(x). \\ &\leq \sum_{\pi\in\mathcal{P}_d} \lambda_\pi(f\circ j_\pi) \quad = \quad \sum_{\pi\in\mathcal{P}_d} \lambda_\pi f. \end{aligned}$$

Note also that for $\pi \in \mathcal{P}_d$, $\lambda_\pi \mathsf{P}_t^{(d)} f = \lambda^{\otimes|\pi|}\mathsf{P}_t^{(|\pi|)} f \circ \varphi_\pi$ (φ_π was defined in section 1.2). The lemma easily follows. □

This lemma implies the existence of a constant D such that the measure μ_s is absolutely continuous with respect to m_d with $\frac{d\mu_s}{dm_d} \leq D\|h\|_\infty$.

Then, we just follow the proof of theorem 10. □

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References

[1] BERTOIN, J. (1996). *Lévy processes.* Cambridge university press.

[2] BOULEAU, N. and HIRSCH, F. (1991). *Dirichlet forms and analysis on Wiener spaces.* De Gruyter, Berlin.

[3] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). *Dirichlet forms and symmetric Markov processes.* De Gruyter, Berlin.

[4] LE JAN, Y. and RAIMOND, O. Flows, coalescence and noise, math.PR/0203221, To appear in The Annals of Probability.

[5] LE JAN, Y. and RAIMOND, O. (2002) Sticky flows on the circle.

[6] LE JAN, Y. and RAIMOND, O. (2002) The noise of a Brownian sticky flow is black.

[7] PITMAN, J. (2002). *Combinatorial Stochastic Processes.* Saint Flour lecture notes, Juillet 2002.

[8] TSIRELSON, B. (1998). Unitary Brownian motions are linearizable, math.PR/9806112

[9] TSIRELSON, B. (2002). *Scaling limit, noise, stability.* Saint Flour lecture notes, Juillet 2002.