Products of Beta matrices and sticky flows

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In [2], a family of stochastic flows of kernels on S^1 called "sticky flows" is described. Sticky flows are defined by their "moments" which are consistent systems of transition kernels on S^1. In this note, a discrete version of sticky flows is presented in the case the sticky flows are associated with a system of Brownian particles on S^1. This discrete model is defined by products of Beta matrices on the discrete torus $\mathbb{Z}/N\mathbb{Z}$ and will be called a Beta flow. Similarly to the continuous case, the moments of the Beta flow are consistent systems of transition matrices on $\mathbb{Z}/N\mathbb{Z}$. A convergence of the Beta matrices to sticky kernels is shown at the level of the moments.

1 Beta matrices and Polya scheme

Let a be a positive parameter and N be an even positive integer. We define a random transition matrix K on the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ as follows :

$$K(i,j) = X_i \mathbf{1}_{j=i+1} + (1 - X_i)\mathbf{1}_{j=i-1}$$

where $X_1, \ldots, X_N$ are independent Beta$(\frac{a}{N}, \frac{a}{N})$ random variables.
Let $(K_n)_n$ be a i.i.d sequence of such random transition matrices and let $\{Z(t), t \geq 0\}$ be an independent Poisson process on $\mathbb{R}$ with intensity N^2. The family of matrices $(K_{N,s,t})_{s \leq t}$ defined by: $K_{N,s,t} = K_{Z(s)+1}K_{Z(s)+2} \cdots K_{Z(t)}$ for every $s \leq t$, is a stochastic flow of kernels on $\mathbb{Z}/N\mathbb{Z}$. It will be called the Beta flow on $\mathbb{Z}/N\mathbb{Z}$.

1.1 Description of the n-point motion

Following the generalization of De Finetti theorem given in [1], it is shown that the law of such a stochastic flow of transition kernels is given by a consistent system of n-point Markovian semigroups $\{\Pi_t^{(n)} = E(K_{0,t}^{\otimes n}),\ n \in \mathbb{N}^*\}$. In our case, these Markovian semigroups $(\Pi_t^{(n)})_t$ are associated with a jump Markov process on $(\mathbb{T}_N)^n$ with holding times N^2 and transition matrices $P_N^{(n)} = E(K^{\otimes n})$. Let us compute the positive transition probabilities: for $x = (x_1, \dots, x_n) \in (\mathbb{T}_N)^n$ and $\varepsilon \in \{-1, 1\}^n$,

$$P_N^{(n)}(x, x+\varepsilon) = E\left(\prod_{j=1}^{n}(Z_{x_j}\mathbf{1}_{\varepsilon_j=1} + (1-Z_{x_j})\mathbf{1}_{\varepsilon_j=-1})\right) = \prod_{l\in\mathbb{T}_N} E(Z_l^{s_l^+(x,\varepsilon)}(1-Z_l)^{s_l^-(x,\varepsilon)}).$$

where $s_l^{\pm}(x,\varepsilon) = \mathrm{Card}(\{i \in \{1, \dots, n\}, x_i = l \text{ and } \varepsilon_i = \pm 1\})$.
Let $s_l(x)$ denote the number of coordinates of x equal to l.

$$P_N^{(n)}(x, x+\varepsilon) = \prod_{l\in\mathbb{T}_N} \frac{\prod_{i=0}^{s_l^+(x,\varepsilon)-1}(\frac{a}{N}+i)\prod_{i=0}^{s_l^-(x,\varepsilon)-1}(\frac{a}{N}+i)}{\prod_{i=0}^{s_l(x)-1}(\frac{2a}{N}+i)}.$$

Let us note that

$$\begin{aligned} P^{(1)}(x, x+\varepsilon) &= \frac{1}{2} \text{ for } n = 1, \\ P^{(n)}(x, x+\varepsilon) &= P^{(n-1)}(\underline{x}, \underline{x}+\underline{\varepsilon})\frac{(\frac{a}{N}+s_{x_n}^+(\underline{x},\underline{\varepsilon}))^{\mathbf{1}_{\varepsilon_n=1}}(\frac{a}{N}+s_{x_n}^-(\underline{x},\underline{\varepsilon}))^{\mathbf{1}_{\varepsilon_n=-1}}}{\frac{2a}{N}+s_{x_n}(\underline{x})} \text{ for } n \geq 2 \end{aligned}$$

where $\underline{x} = (x_1, \dots, x_{n-1})$.
Thus the transition mechanism can be described as follows: the first point moves from a site i to the site $i+1$ or to the site $i-1$ with equal probability. The motion of the $(k-1)$ first points being known, the k-th point jumps by $+1$ with probability $\frac{u+\frac{a}{N}}{u+v+\frac{2a}{N}}$ if among the $k-1$ first points, $u+v$ were located in the same site, u jumping by $+1$ and v by -1. This is a combination of independent Polya urns attached at each site.

1.2 Invariant measures and Dirichlet processes

We shall focus our attention on the irreducible component

$$T_N^{(n)} = \{x \in (\mathbb{Z}/N\mathbb{Z})^n,\ x_i - x_1 \in 2\mathbb{Z}, \text{ for all } 1 \leq i \leq n\}.$$

of the n-point process.

Proposition 1 *For every $n \in \mathbb{N}^*$, the n-point process on $T_N^{(n)}$ has a reversible measure $m_N^{(n)}$ given by*

$$m_N^{(n)}(x) = \frac{\prod_{l\in\mathbb{Z}/N\mathbb{Z}}(\prod_{i=0}^{s_l(\underline{x})-1}(\frac{2a}{N}+i))}{2\prod_{i=0}^{n-1}(a+i)} \text{ for all } x \in T_N^{(n)} \quad (1)$$

Proof. The ratio $\frac{P^{(n)}(x,x+\varepsilon)}{P^{(n)}(x+\varepsilon,x)}$ is equal to

$$\prod_{l\in\mathbb{T}_N}\left(\frac{\prod_{i=0}^{s_l^+(x,\varepsilon)-1}(\frac{a}{N}+i)\prod_{i=0}^{s_l^-(x,\varepsilon)-1}(\frac{a}{N}+i)}{\prod_{i=0}^{s_l^+(x+\varepsilon,-\varepsilon)-1}(\frac{a}{N}+i)\prod_{i=0}^{s_l^-(x+\varepsilon,-\varepsilon)-1}(\frac{a}{N}+i)}\right)\prod_{l\in\mathbb{T}_N}\left(\frac{\prod_{i=0}^{s_l(x+\varepsilon)-1}(\frac{2a}{N}+i)}{\prod_{i=0}^{s_l(x)-1}(\frac{2a}{N}+i)}\right)$$

As $s_l^+(x+\varepsilon,-\varepsilon)=s_l^-(x,\varepsilon)$, the first quotient in the parentheses is equal to one. Thus the probability measure proportional to the measure α_n on $T_N^{(n)}$, defined by

$$\alpha_n(x)=\prod_{l\in\mathbb{T}_N}(\prod_{i=0}^{s_l(x)-1}(\frac{2a}{N}+i))\text{ for all }x\in T_N^{(n)},$$

is a reversible measure for the n-point motion. It remains to note that the total mass of α_n is $2\prod_{i=0}^{n-1}(a+i)$ (this equality can be proved by iteration on n). □

Remark. This result can be extended to a more general situation of a non reversible chain on a finite graph. It will be described in a forthcoming paper.

Note that $m_N^{(1)}$ is the uniform law on $\mathbb{Z}/N\mathbb{Z}$ and that $m_N^{(n)}$ verifies the following iterative relation:

$$m_N^{(n+1)}(\underline{x},x_{n+1})=\frac{\frac{2a}{N}+s_{x_{n+1}}(\underline{x})}{a+n}m_N^{(n)}(\underline{x})\quad\forall(\underline{x},x_{n+1})\in T_N^{(n+1)}\text{ and }n\in\mathbb{N}^*\tag{2}$$

It follows from the expression (1) that $x\mapsto m_N^{(n)}(x)$ is constant on each class of the equivalent relation: $x\sim y$ if and only if $\forall\ i,j\in\{1,\dots,n\}$, $x_i=x_j\Rightarrow y_i=y_j$. Thus $m_N^{(n)}$ can be expressed as a mixture of uniform laws on these equivalent classes. In order to give a precise decomposition of $m_N^{(n)}$, let us introduce some notations associated with a partition π of $[n]=\{1,\dots,n\}$.
Let $\mathcal{P}_n$ be the set of all partitions of $[n]$. Let $|\pi|$ denote the number of non-empty blocks of π. Let C_π be the set of points $x\in(S^1)^n$ such that $x_i=x_j$ if and only if i and j are in the same block of π. The sets $C_\pi\cap T_N^{(n)}$ are the equivalent classes for the equivalent relation described above. Let E_π denote the set of points $x\in(S^1)^n$ such that, if i and j are in the same block of π, then $x_i=x_j$. The uniform measure on $E_\pi\cap T_N^{(n)}$ denoted by $\lambda_{N,\pi}$ can be expressed as the image of the uniform measure on $T_N^{(|\pi|)}$ by the one-to-one map ϕ_π defined as follows: for all $x\in(S^1)^{|\pi|}$, $\phi_\pi(x)=(y_1,\dots,y_n)$ where $y_i=x_l$ if i belongs to the l-th block of π.

Proposition 2 *The invariant measure $m_N^{(n)}$ of the n-point motion on $T_N^{(n)}$ has the following decomposition:*

$$m_N^{(n)}=\sum_{\pi\in\mathcal{P}_n}p_\pi^{(a)}\lambda_{N,\pi}$$

where $p_\pi^{(a)}=a^k\frac{\prod_{i=1}^k(n_i-1)!}{\prod_{i=0}^{n-1}(a+i)}$ if π is a partition of $[n]$ with k nonempty blocks of length $n_1,\dots,n_k$.

Remark. $(p_\pi^{(a)})_{\pi\in\mathcal{P}_n}$ is the exchangeable partition function of an exchangeable sequence of random variables governed by the Blackwell-MacQueen Urn scheme [3].

Proof. The proof of the proposition can be established by induction on n using the iterative definition (2) of $m_N^{(n)}$ and the following iterative definition of $(p_\pi^{(a)})_{\pi\in\mathcal{P}_n}$: let $\pi = (B_1, \dots, B_k)$ be a partition of $[n]$ with k nonempty blocks. Then

$$p_{\tilde{\pi}}^{(a)} = \begin{cases} \frac{a}{a+n} p_\pi^{(a)} & \text{if } \tilde{\pi} = (B_1, \dots, B_k, \{n+1\}) \\ \frac{a+n_k}{a+n} p_\pi^{(a)} & \text{if } \tilde{\pi} = (B_1, \dots, B_k \cup \{n+1\}) \end{cases}$$

□

Let us note that the jump chain is two-periodic. The set of states $T_N^{(n)}$ of the chain can be divided in two sets: $C_{1,N}^{(n)}$ the set of points $x \in T_N^{(n)}$ with odd coordinates and $C_{2,N}^{(n)}$ the set of points $x \in T_N^{(n)}$ with even coordinates. If $x = (x_1, \dots, x_n) \in C_{2,N}^{(n)}$ then

$$2m_N^{(n)}(x) = E(\prod_{i=1}^{N/2} Y_{2i}^{s_{2i}(x)}) = E(\prod_{i=1}^{n} Y_{x_i})$$

where $(Y_2, Y_4, \dots, Y_N)$ is a random vector with symmetric Dirichlet law $\mathcal{D}_{N/2}(\frac{2a}{N}, \dots, \frac{2a}{N})$[1]. Thus for all $n \in \mathbb{N}^*$, the restriction[2] of $m_N^{(n)}$ to $C_{2,N}^{(n)}$ is equal to $E(\mu^{\otimes n})$ where μ is a Dirichlet process[3] on $2\mathbb{Z}/N\mathbb{Z}$ with parameter a. Equivalently, for all $n \in \mathbb{N}^*$, the restriction of $m_N^{(n)}$ to $C_{1,N}^{(n)}$ is equal to $E(\nu^{\otimes n})$ where ν is a Dirichlet process on $(2\mathbb{Z}+1)/N\mathbb{Z}$ with parameter a.

2 Consistent system of sticky kernels on S^1

We now consider the Beta flow as defined on the lattice $\frac{1}{N}(\mathbb{Z}/N\mathbb{Z})$ of S^1. It is a discrete model of the sticky flow defined by Y. Le Jan and O. Raimond in [2] for the parameter $\tau = \frac{1}{a+1}$ and the exponent $\psi : \lambda \mapsto \frac{1}{2}|\lambda|^2$. Let us summarize the properties of this sticky flow. For a detailed presentation of the sticky flows in a more general setting, we refer the reader to [2].

For $n \in \mathbb{N}^*$, let us consider the measure $m^{(n)}$ defined on $(S^1)^n$ by $m^{(n)} = \sum_{\pi\in\mathcal{P}_n} p_\pi^{(a)} \lambda_\pi$ where λ_π is the image of the Lebesgue measure $\lambda^{\otimes|\pi|}$ on $(S^1)^{|\pi|}$ by the map ϕ_π. As this defines an exchangeable and consistent system of measures, it follows from Kingman's representation theorem that there exists a random measure on S^1 such that $m^{(n)} = E(\mu^{\otimes n})$ for every $n \in \mathbb{N}^*$. In our case, μ is the Dirichlet process of parameter a on S^1.

Using this family of measures, a consistent system of Feller semigroups denoted $(P_t^{(n)})$ can be defined via their Dirichlet forms:

[1]The Dirichlet law $\mathcal{D}_r(a_1, \cdots, a_r)$ where $a_1 > 0, \dots, a_r > 0$ is a law on the simplex $S_r = \{x \in (0,1)^r,\ x_1 + \cdots + x_r = 1\}$ defined by $\frac{\Gamma(a_1+\cdots+a_r)}{\Gamma(a_1)\cdots\Gamma(a_r)} x_1^{a_1-1} \dots x_r^{a_r-1} 1_{x_1+\cdots+x_r=1} dx_1 \cdots dx_{r-1}$. Thus its moment of order $(k_1, \dots, k_r)$ is $\frac{\Gamma(a_1+k_1)\cdots\Gamma(a_r+k_r)}{\Gamma(a_1)\cdots\Gamma(a_r)} \frac{\Gamma(a_1+\cdots+a_r)}{\Gamma(a_1+\cdots+a_r+k_1+\cdots+k_r)}$

[2]The restriction of a probability measure m to a measurable set B is the measure $\frac{m(\,\cdot\,\cap B)}{m(B)}$.

[3]A Dirichlet process of parameter $a > 0$ on a compact metric space M is a random measure μ on M such that for every measurable finite partition $(B_1, \dots, B_k)$ of M, the random vector $(\mu(B_1), \mu(B_2) \dots, \mu(B_k))$ has the Dirichlet law $\mathcal{D}_k(a\lambda(B_1), \dots, a\lambda(B_k))$ where λ denotes the uniform distribution on M.

Proposition 3 (Y. Le Jan and O. Raimond, [2]) *For $k \in \mathbb{N}^*$, let $\mathcal{E}^{\odot k}$ be the Dirichlet form defined on $L^2((S^1)^k, \lambda^{\otimes k})$, associated with k independent Brownian motions on S^1. For every $n \in \mathbb{N}^*$, let $\mathcal{E}^{(n)}$ denote the Dirichlet form on $C^1((S^1)^n)$ defined as follows:*

$$\forall f, g \in C^1((S^1)^n),\ \mathcal{E}^{(n)}(f,g) = \sum_{\pi \in \mathcal{P}_n} p_\pi \mathcal{E}^{\odot|\pi|}(f \circ \phi_\pi,\ g \circ \phi_\pi).$$

- *The semigroups $(P_t^{(n)})$, $n \in \mathbb{N}^*$ associated with the Dirichlet forms $\mathcal{E}^{(n)}$, $n \in \mathbb{N}^*$ define a consistent system of strong Feller semigroups.*
- *For every $n \in \mathbb{N}^*$, the generator of $\mathcal{E}^{(n)}$ denoted by $A^{(n)}$ has the following expression:*

$$A^{(n)}(f) = \frac{1}{2} \sum_{\pi \in \mathcal{P}_n} (\Delta^{(|\pi|)}(f \circ \phi_\pi)) \circ \phi_\pi^{-1} 1_{C_\pi} \quad \forall f \in C^2((S^1)^n) \tag{3}$$

Since $\{(P_t^{(n)})_t,\ n \in \mathbb{N}^*\}$ is a compatible family of Feller semigroups on S^1, it follows from theorem 1.1.4 in [1] that it is possible to construct a stochastic flow of kernels $(K_{s,t})$ such that $E(K_{0,t}^{\otimes n}) = P_t^{(n)}$ for every $t \in \mathbb{R}$ and $n \in \mathbb{N}^*$. This stochastic flow was named a sticky flow of parameter $\tau = \frac{1}{a+1}$ and exponent[4] $\psi : \lambda \mapsto \frac{1}{2}|\lambda|^2$.

3 Convergence theorem

We will now establish the weak convergence of the moments of the Beta kernels to the moments of the sticky kernels:

Theorem 4 *For $n \in \mathbb{N}^*$, let $(P_{N,t}^{(n)})_t$ denote the semigroup of the n-point process defined on $\bar{T}_N^{(n)}$ by the Beta flow. Let $(P_t^{(n)})_t$ denote the semigroup of the n-point process defined on $(S^1)^n$ by the sticky flow of parameter a and exponent $\psi : \lambda \mapsto \frac{1}{2}|\lambda|^2$.*
For every $n \in \mathbb{N}^*$, if f and g are continuous functions on $(S^1)^n$, then $\int g P_{N,t}^{(n)}(f) dm_N^{(n)}$ converges to $\int g P_t^{(n)}(f) dm^{(n)}$ as N tends to $+\infty$.

Before going into details, let us explain the scheme of the proof. The main step of the proof of this theorem, is to show the following convergence of the resolvents:

Proposition 5 *Let $n \in \mathbb{N}^*$. Let $(V_{N,\lambda}^{(n)})_{\lambda>0}$ denote the resolvent associated with the n-point motion of the Beta flow and let $(V_\lambda^{(n)})_{\lambda>0}$ denote the resolvent of the n-point motion of the sticky flow of parameter a and exponent $\psi : \lambda \mapsto \frac{1}{2}|\lambda|^2$.*
For all continuous functions f and g on $(S^1)^n$, $\int V_{N,\lambda}^{(n)}(f) g dm_N^{(n)}$ converges to $\int V_\lambda^{(n)}(f) g dm^{(n)}$ as N tends to $+\infty$.

As the discrete and the continuous n-point processes are both reversible, an argument using spectral measures allows to deduce the weak convergence of the semigroups from the weak convergence of the resolvents.
The convergence of the resolvents is based on the convergence of the invariant measure $m_N^{(n)}$ and of the generator of the n-point motion of the Beta flow together with a Lipschitz property of the discrete resolvent $V_{N,\lambda}^{(n)}$.

Before proving proposition 5, let us give precise statements of these three points.

[4]More generally, sticky flows can be constructed using a Levy process with no polar points instead of a Brownian motion; ψ refers to the exponent of the chosen Levy process.

Convergence of the invariant measures

Lemma 6 *For every partition π of $[n]$, let f_π be a function defined on $(S^1)^n$, Lipschitz on E_π and vanishing outside E_π. Let $f = \sum_{\pi \in \mathcal{P}_n} f_\pi$. Then*

$$|\int f dm_N^{(n)} - \int f dm^{(n)}| \leq \frac{C_n}{N} \sum_{\pi \in \mathcal{P}_n} (||f_\pi||_{Lip} + ||f_\pi||_\infty).$$

Proof. Let $E_{N,\pi}$ denote the intersection of E_π with $\bar{T}_N^{(n)}$ and let $\lambda_{N,\pi}$ be the uniform distribution on $\bar{T}_N^{(n)}$. It suffices to prove that for all partitions π, π' of $[n]$,

$$|\int_{E_{N,\pi'}} f_\pi d\lambda_{N,\pi'} - \int_{E_{\pi'}} f_\pi d\lambda_{\pi'}| \leq \frac{C_n}{N}(||f_\pi||_{Lip} + ||f_\pi||_\infty).$$

Let us first consider the case π is not thinner than π', that is there is a nonempty block B of π intersecting at least two blocks of π'. Let $\tilde{\pi}$ be the partition of $[n]$ obtained by merging the blocks of π that intersect the same block of π'. Then $E_{\pi'} \cap E_\pi$ is a subset of $E_{\tilde{\pi}}$. As $\tilde{\pi}$ is a coarser partition than π', $\lambda_{\pi'}(E_{\tilde{\pi}}) = 0$ and $|E_{N,\tilde{\pi}}| \leq \frac{2}{N}|E_{N,\pi'}|$. Thus

$$|\int_{E_{N,\pi'}} f_\pi d\lambda_{N,\pi'} - \int_{E_{\pi'}} f_\pi d\lambda_{\pi'}| = \frac{1}{|E_{N,\pi'}|} |\sum_{x \in E_{N,\pi'}} f_\pi(x)| \leq \frac{2}{N}||f_\pi||_\infty.$$

Let us now consider the case π is equal or thinner than π'. Then $E_{\pi'} \subset E_\pi$. If π' has k nonempty blocks then

$$|\int_{E_{N,\pi'}} f_\pi d\lambda_{N,\pi'} - \int_{E_{\pi'}} f_\pi d\lambda_{\pi'}| = |\frac{2^{k-1}}{N^k} \sum_{x \in T_N^{(k)}} f_\pi(\phi_{\pi'}(\frac{x}{N})) - \int_{(S^1)^k} f_{\pi'}(\phi_{\pi'}(x))dx|.$$

The function $f_{\pi'} \circ \phi_{\pi'}$ is a Lipschitz function with Lipschitz coefficient smaller than $n||f_{\pi'}||_{Lip}$. Thus it remains to establish the following result: for every $k \in \mathbb{N}^*$, there exists a constant C_k such that for every Lipschitz function g on $(S^1)^k$,

$$|\frac{2^{k-1}}{N^k} \sum_{x \in T_N^{(k)}} g(\frac{x}{N}) - \int_{(S^1)^k} g(x)dx| \leq \frac{C_k}{N}||g||_{Lip}.$$

The proof can done by induction on k. □

Convergence of the generators

Lemma 7 *For every $n \in \mathbb{N}^*$, let $A_N^{(n)}$ denote the generator of the n-point motion of the Beta flow on $\bar{T}_N^{(n)}$. For every C^2 function f on $(S^1)^n$,*

$$\sup_{x \in T_N^{(n)}} |A_N^{(n)}(f)(\frac{x}{N}) - A^{(n)}(f)(\frac{x}{N})|$$

converges to 0 as N tends to $+\infty$.

Proof. Let $n \in \mathbb{N}^*$ and f be a C^2 function defined on $(S^1)^n$. Let us recall the expression of $A^{(n)}(f)$:

$$A^{(n)}(f) = \frac{1}{2} \sum_{\pi \in \mathcal{P}_n} \Delta_\pi(f) \mathbf{1}_{C_\pi} \text{ where } \Delta_\pi(f)(x) = \Delta^{(|\pi|)}(f \circ \phi_\pi)(\phi_\pi^{-1}(x)).$$

Thus it suffices to prove that for every partition π of $[n]$, $\sup_{x \in C_\pi \cap T_N^{(n)}} \mid 2A_N^{(n)}(f)(\frac{x}{N}) - \Delta_\pi f(\frac{x}{N}) \mid$ converges to 0 as N tends to $+\infty$.
The expression linking the generator $A_N^{(n)}$ and the transition matrice $P_N^{(n)}$ is the following: for a function g defined on $(S^1)^n$ and $x \in T_N^{(n)}$,

$$A_N^{(n)} g(\frac{x}{N}) = N^2 \left(\sum_{\varepsilon \in \{\pm 1\}^n} P_N^{(n)}(x, x+\varepsilon) g(\frac{x+\varepsilon}{N}) - g(\frac{x}{N}) \right).$$

As $s_l^+(x,\varepsilon) = s_l^-(x,-\varepsilon)$, $P_N^{(n)}(x,x+\varepsilon) = P_N^{(n)}(x,x-\varepsilon)$. Thus,

$$2A_N^{(n)} g(\frac{x}{N}) = \sum_{\varepsilon \in \{\pm 1\}^n} P_N^{(n)}(x, x+\varepsilon) L_{N,n}(g)(x,\varepsilon) \ \forall x \in T_N^{(n)}.$$

where $L_{N,n}(g)(x,\varepsilon) = N^2(g(\frac{x+\varepsilon}{N}) + g(\frac{x-\varepsilon}{N}) - 2g(\frac{x}{N}))$.
For a C^2 function g on $(S^1)^r$ and a point $x \in (S^1)^r$, set $L_r(g)(x,\varepsilon) = \sum_{i=1}^r \sum_{j=1}^r \varepsilon_i \varepsilon_j \partial^2_{i,j} g(x)$. It follows from a Taylor expansion with integral remainder that $|L_{N,r}(g)(k,\varepsilon) - L_r(g)(\frac{k}{N},\varepsilon)|$ converges to zero uniformly on $k \in T_N^{(r)}$ and $\varepsilon \in \{\pm 1\}^r$ as N tends to $+\infty$.
Let π be a partition of $[n]$ having d nonempty blocks. Let us define a discrete version of the Δ_π. First, let $\Delta_N^{(d)}$ denote the discrete Laplacian on $\bar{T}_N^{(d)}$:

$$\Delta_N^{(d)} g(x) = \frac{N^2}{2^d} \sum_{\varepsilon \in \{\pm 1\}^d} (g(x + \frac{\varepsilon}{N}) + g(x - \frac{\varepsilon}{N}) - 2g(x)) \text{ for a function } g \text{ defined on } \bar{T}_N^{(d)}.$$

Then set $\Delta_{N,\pi} g(\cdot) = \Delta_N^{(d)}(g \circ \phi_\pi)(\phi_\pi^{-1}(\cdot))$.
It follows that $\Delta_{N,\pi} f(\frac{k}{N}) - \Delta_\pi f(\frac{k}{N})$ converges to zero uniformly on $k \in T_N^n$.
Let us note that the restriction of ϕ_π to $\{\pm 1\}^d$ is a one-to-one map onto $E_\pi \cap \{\pm 1\}^n$, whence $\Delta_{N,\pi} f(\frac{k}{N}) = \frac{1}{2^d} \sum_{\varepsilon \in \{\pm 1\}^n \cap E_\pi} L_{N,n}(f)(k,\varepsilon)$. Consequently, the expression of $2A_N^{(n)} f(\frac{k}{N}) - \Delta_\pi f(\frac{k}{N})$ can be split into the three following terms:

$$\begin{aligned}
I_N^1(k) &= \sum_{\varepsilon \in E_\pi \cap \{\pm 1\}^n} (P_N^{(n)}(k, k+\varepsilon) - \frac{1}{2^d}) L_{N,n}(f)(\frac{k}{N}, \varepsilon) \\
I_N^2(k) &= \Delta_{N,\pi} f(\frac{k}{N}) - \Delta_\pi f(\frac{k}{N}) \\
I_N^3(k) &= \sum_{\varepsilon \in E_\pi^c \cap \{\pm 1\}^n} P_N^{(n)}(k, k+\varepsilon) L_{N,n}(f)(\frac{k}{N}, \varepsilon)
\end{aligned}$$

It remains to study the asymptotic behaviour of $P_N^{(n)}(k, k+\varepsilon)$. Let $(B_1, \ldots, B_d)$ denote the nonempty blocks of π and let $k \in C_\pi$. Then

$$P_N^{(n)}(k, k+\varepsilon) = \prod_{l=1}^d \frac{\gamma_{a/N}(\sum_{i \in B_l} 1_{\varepsilon_i = 1}) \gamma_{a/N}(\sum_{i \in B_l} 1_{\varepsilon_i = -1})}{\gamma_{2a/N}(|B_l|)} \ \forall \varepsilon \in \{\pm 1\}^n$$

where $\gamma_b(u)$ denotes the product $\prod_{i=0}^{u-1}(b+i)$ for $b>0$ and $u\in\mathbb{N}$ with the convention $\prod_{i=0}^{-1}=1$. A computation shows that

- if $u,v\in\mathbb{N}^*$ then

$$\frac{\gamma_{b/N}(u)\gamma_{b/N}(v)}{\gamma_{2b/N}(u+v)}\leq\frac{b}{N}\frac{u!v!}{(u+v-1)!}\text{ for }N\geq b.$$

- if $u\in\mathbb{N}^*$ and $v=0$ then

$$|\frac{\gamma_{b/N}(u)\gamma_{b/N}(0)}{\gamma_{2b/N}(u)}-\frac{1}{2}|=\frac{1}{2}(1-\prod_{i=1}^{u-1}(1-\frac{b}{2b+Ni}))\leq\frac{1}{2}(1-(1-\frac{b}{2b+N})^{u-1}.$$

Thus $\sup_{k\in C_\pi,\ \varepsilon\in\{\pm1\}^n\cap E_\pi^c}P_N^{(n)}(k,k+\varepsilon)$ and $\sup_{k\in C_\pi,\ \varepsilon\in\{\pm1\}^n\cap E_\pi}|P_N^{(n)}(k,k+\varepsilon)-\frac{1}{2^d}|$ converge to 0 as N tends to $+\infty$. □

Lipschitz property of the resolvents associated with the Beta flow

Lemma 8 *If $f:\bar{T}_N^{(n)}\to\mathbb{R}$ is a Lipschitz function then $V_{N,\lambda}^{(n)}(f)$ is a Lipschitz function with a Lipschitz coefficient bounded by $\frac{1}{\lambda}||f||_{Lip}$.*

Proof. We use a coupling argument. Let $\underline{x}=(x_1,\ldots,x_{n+1})$ be a point of $\bar{T}_N^{(n+1)}$ such that $x_1\neq x_2$. Let $X_t=(X_t^{(1)},\ldots,X_t^{(n+1)})$ be a Markov chain on $\bar{T}_N^{(n+1)}$ with transition matrix $P_N^{(n+1)}$ and with initial point $\underline{x}$. Set $\tau=\inf\{s>0,\ X_s^{(1)}=X_s^{(2)}\}$. Since $(X_t^{(1)},X_t^{(2)})$ is a positive recurrent Markov chain on $\bar{T}_N^{(2)}$, τ is almost surely finite. Let us define two processes $(Y_t)_t$ and $(Z_t)_t$ on $\bar{T}_N^{(n)}$:

- $Y_t^{(1)}=X_t^{(1)}$ and $Y_t^{(i)}=X_t^{(i+1)}$ for $i\in\{2,\ldots,n\}$,
- $Z_t^{(1)}=X_t^{(2)}\mathbf{1}_{t\leq\tau}+X_t^{(1)}\mathbf{1}_{t>\tau}$ and $Z_t^{(i)}=X_t^{(i+1)}$ for $i\in\{2,\ldots,n\}$.

Let $(P_{N,t}^{(n)})_t$ denote the n-point Markovian semigroup of the Beta flow on $\frac{1}{N}(\mathbb{Z}/N\mathbb{Z})$. As $\{(P_{N,t}^{(n)})_t,\ n\in\mathbb{N}^*\}$ defines a consistent family of Markovian semigroups, $(Y_t)_t$ and $(X_t^{(2)},X_t^{(3)},\ldots,X_t^{(n+1)})_t$ are both Markov processes with semigroup $(P_{N,t}^{(n)})_t$. The strong Markov property implies that $(Z_t)_t$ is also a Markov process with semigroup $(P_{N,t}^{(n)})_t$. As $Y_t=Z_t$ if $t\geq\tau$,

$$\begin{aligned}&|V_{N,\lambda}^{(n)}(f)(x_1,x_3,\ldots,x_{n+1})-V_{N,\lambda}^{(n)}(f)(x_2,x_3,\ldots,x_{n+1})|\\&\quad=|\int_0^{+\infty}E(f(Y_{t\wedge\tau})-f(Z_{t\wedge\tau}))e^{-\lambda t}dt|\leq||f||_{Lip}\int_0^{+\infty}e^{-\lambda t}E(d(X_{t\wedge\tau}^{(1)},X_{t\wedge\tau}^{(2)}))dt.\end{aligned}$$

Let us show that $E(d(X_{t\wedge\tau}^{(1)},X_{t\wedge\tau}^{(2)}))\leq d(x_1,x_2)$. Without loss of generality, one may assume that $w=x_1-x_2\in\{0,\ldots,N-1\}$. Let $(\hat{X}_t^{(1)},\hat{X}_t^{(2)})$ be the Markov chain on $(\frac{1}{N}\mathbb{Z})^2$ starting from (x_1,x_2) with transition matrix $\hat{P}$ defined by: $\hat{P}(\frac{i+kN}{N},\frac{j+lN}{N})=P_N^{(2)}(i,j)$ for all $(i,j)\in T_N^{(2)}$. Set $W_t=\hat{X}_t^{(1)}-\hat{X}_t^{(2)}$. Since w is even, for every $t\geq0$, $W_{t\wedge\tau}$ remains nonnegative. Thus $E(d(\hat{X}^{(1)}_{t\wedge\tau},\hat{X}^{(2)}_{t\wedge\tau}))\leq E(W_{t\wedge\tau})=w$ (since $(W_t)_t$ is a martingale). We have obtained the following inequality:

$$|V^{(n)}_\lambda(f)(x_1,x_3,\dots,x_{n+1})-V^{(n)}_\lambda(f)(x_2,x_3,\dots,x_{n+1})|\leq\frac{1}{\lambda}||f||_{Lip}d(x_1,x_2).$$

As the semigroup $(P^{(n)}_{N,t})_t$ is invariant by the action of a permutation,

$$\forall x,y\in\bar{T}^{(n)}_N,\ |V^{(n)}_{N,\lambda}(f)(x)-V^{(n)}_{N,\lambda}(f)(y)|\leq\frac{1}{\lambda}||f||_{Lip}\sum_{i=1}^n d(x_i,y_i).$$

□

Proof of the proposition 5.

A density argument reduces the problem to showing that for all C^1 functions f and g on $(S^1)^n$, $\int V^{(n)}_{N,\lambda}(f)gdm^{(n)}_N$ converges to $\int V^{(n)}_\lambda(f)gdm^{(n)}$ as N tends to $+\infty$.
Let us introduce an extension of $V^{(n)}_{N,\lambda}(f)$ to $(S^1)^n$:

Lemma 9 *A Lipschitz function g on $\bar{T}^{(n)}_N$ can be extended to a function $\tilde{g}$ such that :*

- $\|\tilde{g}\|_\infty=\|g\|_\infty$
- $\|\tilde{g}\|_{Lip}\leq C_n\|g\|_{Lip}$ *where C_n is a constant only depending on n.*
- *$\tilde{g}$ is differentiable on $(S^1)^n-\mathcal{R}$ where $\mathcal{R}$ is the subset of points having at least one coordinate in $\frac{1}{N}(\mathbb{Z}/N\mathbb{Z})$ and $||\partial_i\tilde{g}(x)||\leq C_n||g||_{Lip}$ for all $i\in\{1,\dots,n\}$ and $x\in(S^1)^n-\mathcal{R}$.*

Proof. Firstly, a function g on $\bar{T}^{(n)}_N$ is extended to a function $\bar{g}$ on the lattice $(\frac{1}{N}(\mathbb{Z}/N\mathbb{Z}))^n$ as follows: for $x\in(\frac{1}{N}(\mathbb{Z}/N\mathbb{Z}))^n-\bar{T}^{(n)}_N$, set $\bar{g}(x)=\frac{1}{|V_x|}\sum_{y\in V_x}g(y)$ where V_x is the set of the nearest points of x in $(\frac{1}{N}(\mathbb{Z}/N\mathbb{Z}))^n$ in the sense of the distance $d_n(x,y)=\sum_{i=1}^n d(x_i,y_i)$. This extension has the following properties:

- $||\bar{g}||_\infty=||g||_\infty$,
- there is a constant $C_n>0$ such that for every Lipschitz function $g:\bar{T}^{(n)}_N\to\mathbb{R}$, $||\bar{g}||_{Lip}\leq C_n||g||_{Lip}$.

Lastly, a function f defined on $(\frac{1}{N}(\mathbb{Z}/N\mathbb{Z}))^n$ is extended to a function $\hat{f}$ on $(S^1)^n$ as follows. A point $x=(x_1,\dots,x_n)$ in an elementary cube $\prod_{i=1}^n]\frac{k_i}{N},\frac{k_i+1}{N}[$ is the barycentre of the vertices of this cube $\{\frac{k+\eta}{N},\ \eta\in\{0,1\}^n\}$ with the weights

$$\alpha_n(k+\eta,x)=N^n\prod_{i=1}^n(x_i-\frac{k_i}{N})^{\eta_i}(\frac{k_i+1}{N}-x_i)^{1-\eta_i}$$

respectively. Then we set $\hat{f}(x)$ as the convex combination of the points $f(\frac{k+\eta}{N})$ with the weights $\alpha_n(k+\eta,x)$ for every $\eta\in\{0,1\}^n$: $\hat{f}(x)=\sum_{\eta\in\{0,1\}^n}\alpha_n(k+\eta,x)f(\frac{k+\eta}{N})$. Let us list some properties of this extension:

- $||\hat{f}||_\infty = ||f||_\infty$ and f is differentiable in $(S^1)^n - \mathcal{R}$.
- If f is a Lipschitz function then $||\hat{f}||_{Lip} \le ||f||_{Lip}$ and for every $i \in \{1, \dots, n\}$, $|\partial_i f(x)| \le ||f||_{Lip}$ if $x \in (S^1)^n - \mathcal{R}$.

□

On applying lemmas 6 and 8, we obtain that the difference between $\int V_{N,\lambda}^{(n)}(f) g dm_N^{(n)}$ and $\int \widetilde{V_{N,\lambda}^{(n)}}(f) g dm^{(n)}$ converges to zero as N tends to $+\infty$.
The following lemma allows to complete the proof:

Lemma 10 *If f is a C^1 function on $(S^1)^n$ then for every $g \in L^2(m^{(n)})$, $\int g \widetilde{V_{N,\lambda}^{(n)}}(f) dm^{(n)}$ converges to $\int g V_\lambda^{(n)}(f) dm^{(n)}$.*

Proof. As the weak convergence in the Dirichlet space $(\mathcal{H}^{(n)}, \mathcal{E}_\lambda^{(n)})$ implies the weak convergence in $L^2(m^{(n)})$, it suffices to prove that for all $g \in \mathcal{H}^{(n)}$, $\mathcal{E}_\lambda^{(n)}(\widetilde{V_{\lambda,N}^{(n)}}(f), g)$ tends to $\mathcal{E}_\lambda^{(n)}(V_\lambda(f), g)$.
As the set of C^3 functions is dense in the Dirichlet space of each $\mathcal{E}_\pi$, it is dense in $\mathcal{H}^{(n)} = \cap_{\pi \in \mathcal{P}_n} \mathcal{H}_\pi$. It follows from lemmas 8 and 9 that, for $\lambda > 0$,

$$\mathcal{E}_\lambda^{(n)}(\widetilde{V_{\lambda,N}^{(n)}}(f)) \le C_{n,\lambda}(||f||_{Lip}^2 + ||f||_\infty^2)$$

where $C_{n,\lambda}$ is a constant depending only on n and λ. This reduces the problem to proving the convergence for a C^3 function g.
Let g be a C^3 function. The difference $I_N = \mathcal{E}_\lambda^{(n)}(\widetilde{V_{N,\lambda}^{(n)}}(f), g) - \mathcal{E}_\lambda^{(n)}(V_\lambda^{(n)}(f), g)$ is the sum of two terms:

$$\begin{aligned} I_N^{(1)} &= \mathcal{E}_\lambda^{(n)}(\widetilde{V_{N,\lambda}^{(n)}}(f), g) - \mathcal{E}_{N,\lambda}^{(n)}(V_{N,\lambda}^{(n)}(f), g) \\ I_N^{(2)} &= \int f(x) g(x) dm_N^{(n)}(x) - \int f(x) g(x) dm^{(n)}(x) \end{aligned}$$

By lemma 6, $I_N^{(2)}$ goes to zero as N tends to $+\infty$.
Let us split up $I_N^{(1)}$:

$$I_N^{(1)} = \int \widetilde{V_{N,\lambda}^{(n)}}(f)(\lambda - A^{(n)})(g) dm^{(n)} - \int \widetilde{V_{N,\lambda}^{(n)}}(f)(\lambda - A^{(n)})(g) dm_N^{(n)} + \int V_{N,\lambda}^{(n)}(f)(A_N^{(n)} - A^{(n)})(g) dm_N^{(n)}$$

By lemma 8, $V_{N,\lambda}^{(n)}(f)$ is Lipschitz with Lipschitz coefficient bounded by $\frac{1}{\lambda}||f||_{Lip}$. On the other hand, it follows from the expression of the generator (3) that $A^{(n)}(g)$ is the sum of Lipschitz functions on E_π that vanish out of E_π. Thus the lemma 6 can be applied. The last integral is bounded by

$$\frac{||f||_\infty}{\lambda} \sup_{x \in \bar{T}_N^{(n)}} |\, (A_N^{(n)} - A^{(n)})(g)(x) \,| \,.$$

Thus it converges to 0 by lemma 7. □

Proof of the theorem 4.

Let f be a continuous function on $(S^1)^n$. As the two n-point processes are reversible Markov processes, their generators $A_N^{(n)}$ and $A^{(n)}$ are self-adjoint operators on the Hilbert spaces $L^2(m_N^{(n)})$ and $L^2(m^{(n)})$ with nonpositive spectra. Let ν_N^f and ν^f denote the spectral measures of $A_N^{(n)}$ and $A^{(n)}$ respectively associated with the function f :

$$\langle f,\ \psi(A_N^{(n)})f\rangle = \int_{\mathbb{R}_-} \psi d\nu_N^f \text{ and } \langle f,\ \psi(A^{(n)})f\rangle = \int_{\mathbb{R}_-} \psi d\nu^f$$

for every continuous function ψ on $\mathbb{R}_-$.
The relation between the resolvent and the generator, given by $V_{N,\lambda}^{(n)} = (\lambda - A_N^{(n)})^{-1}$ in the discrete case, and the convergence theorem imply that for every $t > 0$, $\int \frac{1}{t-x} d\nu_N^f(x)$ converges to $\int \frac{1}{t-x} d\nu^f(x)$. By the Stone-Weierstrass theorem, the following set of functions defined on $\mathbb{R}_-$, $\{x \mapsto \frac{1}{t-x},\ t > 0\}$, is dense in the continuous functions on $\mathbb{R}_-$ that tends to 0 at $-\infty$. Thus $\int f(x) P_{N,t}^{(n)}(f)(x) dm_N^{(n)}(x) = \int_{-\infty}^{0} e^{tx} d\nu_N^f(x)$ converges to $\int f(x) P_t^{(n)}(f)(x) dm^{(n)}(x) = \int_{-\infty}^{0} e^{tx} d\nu^f(x)$ for all $t > 0$. The polarization identity lets us recover the announced convergence.

Remark. The convergence result in theorem 4 also holds with respect to the uniform measures on $\bar{T}_N^{(n)}$ and $(S^1)^n$: let f and g be continuous functions on $(S^1)^n$.
Then $\frac{1}{|\bar{T}_N^{(n)}|} \sum_{x \in \bar{T}_N^{(n)}} g(x) P_{N,t}^{(n)}(f)(x)$ converges to $\int g(x) P_t^{(n)}(f)(x) dx$ as N tends to $+\infty$.

Proof. Let g and f be continuous functions on $(S^1)^n$.
For $\varepsilon > 0$, set $V_\varepsilon = \{x \in (S^1)^n,\ \exists\, i \neq j,\ |x_i - x_j| < \varepsilon\}$ and consider a continuous function g_ε on $(S^1)^n$ such that $g_\varepsilon = g$ outside V_ε and $||g_\varepsilon||_\infty \le ||g||_\infty$.
Outside V_0, the measures $m_N^{(n)}$ and $m^{(n)}$ coincide with the uniform measures $\lambda_N^{(n)}$ and $\lambda^{(n)}$ on $\bar{T}_N^{(n)}$ and $(S^1)^n$ respectively. Thus $\int g_\varepsilon P_{N,t}^{(n)}(f) d\lambda_N^{(n)}$ converges to $\int g_\varepsilon P_t^{(n)}(f) d\lambda^{(n)}$. As $\lambda^{(n)}(\partial V_\varepsilon) = 0$, $\lambda_N^{(n)}(V_\epsilon)$ converges to $\lambda^{(n)}(V_\epsilon)$. Thus,

$$\begin{aligned}
\Big|\int g P_{N,t}^{(n)}(f) d\lambda_N^{(n)} - \int g P_t^{(n)}(f) d\lambda^{(n)}\Big| &\le \int |g - g_\varepsilon| P_{N,t}^{(n)}(f) d\lambda_N^{(n)} \\
&+ \Big|\int g_\varepsilon P_{N,t}^{(n)}(f) d\lambda_N^{(n)} - \int g_\varepsilon P_t^{(n)}(f) d\lambda^{(n)}\Big| + \int |g - g_\varepsilon| P_t^{(n)}(f) d\lambda^{(n)} \\
&\le 6||g||_\infty ||f||_\infty \lambda^{(n)}(V_\epsilon)
\end{aligned}$$

As $\lambda^{(n)}(V_\epsilon)$ converges to $\lambda^{(n)}(V_0) = 0$ as ε tends to 0, $|\int g P_{N,t}^{(n)}(f) d\lambda_N^{(n)} - \int g P_t^{(n)}(f) d\lambda^{(n)}|$ converges to 0 as N tends to $+\infty$. □

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