

**INTEREST RATE EXPLOSIONS IN HEATH,
JARROW, MORTON BOND MODELS**

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INTEREST RATE EXPLOSIONS IN HJM MODELS

Many questions in mathematical finance are concerned with effects of trading several assets—selling some assets to finance purchases of other assets.

To address these questions, it is necessary to set up an asset model that provides some consistency (to be explained shortly) for asset prices within the model.

A bond market (think of U.S. treasury bonds) has a large number of assets because each bond issued has a unique maturity date.

A bond's value at maturity is fixed in advance, but before that date its price is at the mercy of bond buyers and bond sellers.

The bond price fluctuates minute by minute, much like a stock price. The bonds are the basic assets, and we attempt to model their prices.

Bond Price Equations

We will use a superscript T to denote specific maturity dates, and for each time $t \leq T$ the price of the T -maturity bond is denoted by

$$P_t^T$$

In an HJM *one-factor model* the stochastic evolution of each bond price is determined by two (typically stochastic) parametric functions .

A standard Brownian motion B_t models the noise, and all bonds share this single trading noise, hence the term “one-factor” model.

Multi-factor versions of these models are easily defined and HJM described them in their second paper on this subject:

Heath, Jarrow, and Morton (1992) “Bond Pricing and the Term structure of Interest Rates: A New Methodology for Contingent Claims Valuation” *Econometrica* 60: 77–106.

Each bond satisfies its own SDE; that is, its drift and diffusion coefficient are unique to each process P_t^T .

The trading noise dB_t is common to all these assets.

Bond Dynamics: Parameters μ and Σ

Each bond price satisfies the SDE

$$(1) \quad dP_t^T = \mu_t^T dt + \Sigma_t^T dB_t$$

The parameters can not be arbitrary. It is customary to normalize prices so that $P_T^T \equiv 1$ and for this it is necessary that $\Sigma_t^T \rightarrow 0$ as $t \rightarrow T$.

Restrictions on the Drift Parameter

To avoid financial imbalances in the model (eliminate arbitrage) all the following ratios must have a common value.

$$\frac{\mu_t^T - r_t P_t^T}{\Sigma_t^T}$$

This fixed adapted function $\lambda_t(\omega)$ is called the *market price of risk*. In the ratio above, r_t denotes the short term interest rate function.

So we have the restriction that for all T ,

$$\mu_t^T = r_t P_t^T + \lambda_t \Sigma_t^T$$

Consequently, Equation (1) now has the following form:

$$dP^T = \{rP^T + \lambda\sigma^T\}dt + \Sigma^T dB_t$$

which is best written as

$$= rP^T dt + \Sigma^T \{dB_t + \lambda dt\}$$

Under a mild restriction on λ_t , we may change the measure (switch to an equivalent one) using Girsanov's theorem. The new measure, called the *risk-neutral measure* has

$$B_t + \int_0^t \lambda_s ds$$

as a standard Brownian motion, and one obtains this important equation as the risk-neutral dynamics of bond prices:

$$(2) \quad dP^T = rP^T dt + \Sigma^T dW_t$$

This dynamical equation is extremely useful for all sorts of investment calculations. This follows from the fact that each price, discounted by the “money market account”, is a martingale.

More complicated investments formed with these assets share the same property.

HJM's Contribution—a Methodology to Choose Σ_t^T and Obtain Solutions of Equation (2)

Although it is unreasonable to think that a bond price is a smooth function of time, it is reasonable to assume that for fixed t the prices P_t^T are smooth in the maturity date T .

This expresses the idea of a *forward interest rate*. Such a rate f_t^T is defined as

$$f_t^T =: -\frac{\partial \log P_t^T}{\partial T}$$

Since a forward rate is a function of the bond price, Equation (2) and Ito's formula determines the SDE for the forward rate.

It has an especially nice form with respect to the risk-neutral measure, as Heath, Jarrow, and Morton discovered:

Risk - Neutral Dynamics of Forward Interest Rates

$$(3) \quad df = \sigma_t^T \int_t^T \sigma_t^u du \cdot dt + \sigma_t^T dW$$

In order to form a bond model, it is enough to make a choice for σ^T and solve Equation (3).

One can then recover values for r_t and Σ_t^T . That is, one can construct the risk-neutral form of the bond model.

In particular, we have $r_t = f(t, t)$ and the risk-neutral dynamics of bonds are expressed as

$$(4) \quad dP^T = r_t P^T dt + \sigma_t^{*T} P^T dW_t$$

where

$$\sigma_t^{*T} = - \int_t^T \sigma(t, u) du$$

The Log Normal HJM Model

A. Morton considered the choice of σ which is a linear function of the forward rate:

$$\sigma_t^T = c \cdot f_t^T$$

Here c is a constant, and the modeling Equation (3) simplifies to

$$(5) \quad df_t^T = c^2 f_t^T \int_t^T f_t^u du \cdot dt + c f_t^T dB_t$$

Morton observed that, unfortunately, the quadratic drift term causes local solutions to this equation to explode. For this reason, the model was abandoned since it allowed zero bond prices and infinite interest rates!

However, my student, Kyounghee Kim, and I have found a way to revive this model. Our approach is to condition the risk-neutral measure so that no explosion is possible before a specified time, T_0 .

Conditioning via Change of Measure

We explain our idea by temporarily assuming that $\sigma_t^{*T_0}$ is regular enough so that we have an equivalent measure where

$$d\tilde{W}_t := dW_t - \sigma_t^{*T_0} dt$$

defines a standard Brownian motion (Girsanov applied to W_t).

The forward rate dynamics look quite interesting using this measure:

Old Dynamics

$$df = -\sigma_t^T \sigma_t^{*T} dt + \sigma_t^T dW$$

Conversion to New Dynamics

$$\begin{aligned} &= -\sigma_t^T \sigma_t^{*T} dt + \sigma_t^T \sigma_t^{*T_0} dt + \sigma_t^T \{dW - \sigma_t^{*T_0} dt\} \\ &= \sigma_t^T \{\sigma_t^{*T_0} - \sigma_t^{*T}\} dt + \sigma_t^T d\tilde{W}_t \end{aligned}$$

But, σ^* is an integral of $-\sigma$ so we have

New Dynamics

$$(6) \quad df^T = -\sigma_t^T \int_T^{T_0} \sigma_t^u du \cdot dt + \sigma_t^T dW$$

The significance here is that σ is positive in the models of interest ; Equation (6) has negative drift in contrast to the risk-neutral dynamics.

New Dynamics and the Log Normal Model

Our approach is to forget (temporarily) how the New Dynamical equation arose and instead solve it and develop properties of its solutions.

Equation (6) specializes to

$$(6) \quad df_t^T = -c^2 f_t^T \int_T^{T_0} f_t^u du \cdot dt + f_t^T dW$$

This SDE problem has a particularly nice form if we integrate the maturity parameter over an interval $[\tau, T_0]$.

For fixed $\tau < T_0$ let Y_t^τ denote the process

$$(7) \quad Y_t^\tau = \int_\tau^{T_0} f_t^u du$$

By integrating term by term (6) we obtain a simple, autonomous SDE for Y_t^τ ; it has an explicit solution.

$$(8) \quad dY_t^\tau = -\frac{c^2}{2} (Y_t^\tau)^2 dt + cY_t^\tau d\tilde{W}_t$$

THEOREM 1:

For each non-negative initial condition Y_0^τ , Equation (8) has a positive (non-exploding) solution given by

$$Y_t^\tau = \frac{M_t}{\frac{1}{Y_0^\tau} + \frac{c^2}{2} \int_0^t M_s ds}.$$

M_t denotes the exponential martingale

$$M_t = \exp\left\{c\tilde{W}_t - \frac{c^2}{2}t\right\}$$

Remarks

1. The only distinguishing feature between various solutions is their dependence on the initial condition. Equation (8) is a single SDE as opposed to the typical situation where one has a system of SDE's which define forward rates.

2. The solution involves exponential Brownian motion and its time integral. The time integral appears in option pricing problems where an average of asset prices over time defines the option payoff (Asian options).

In this model, the time integral of geometric Brownian motion appears as a nominal interest rate.

CONSTRUCTION OF THE MODEL

We make the following assumption concerning the initial values Y_t^τ

Y_0^τ is a nonnegative, decreasing C^1 function of τ

This is reasonable because we will reverse our previous steps to define forward rates in terms of Y . As a consequence of definition 1. below, we have

$$Y_0^\tau = \log \left(\frac{P_0^\tau}{P_0^{T_0}} \right).$$

Since a longer bond has a lower price, we are in effect assuming that the initial yield curve is positive and C^1 .

Forward Rates and Dynamics Relative to the Forward Measure

We define a family of forward rate functions.

Definition 1.

$$f_t^\tau := -\frac{\partial Y}{\partial \tau}$$

Note that we then have $Y = \int_\tau^{T_0} f_t^u du$ and also $Y = \log \left(\frac{P_t^\tau}{P_t^{T_0}} \right)$.

One sees directly from the solution of Equation (8) that the partial derivative is given by

$$f_t^\tau = \frac{-M_t \frac{\partial Y}{\partial \tau}(0, \tau)}{\left(1 + \frac{c^2}{2} Y(0, \tau) \int_0^t M_s ds\right)^2}$$

Furthermore, we define the forward rate volatility, σ_t^τ , as

$$\sigma_t^\tau = c f_t^\tau.$$

Proposition 1. f_t^τ satisfies the SDE:

$$(9) \quad df_t^\tau = -c^2 f_t^\tau Y_t^\tau dt + c f_t^\tau d\tilde{W}_t.$$

Consequently, our model satisfies the “New Dynamics” presented in a heuristic manner earlier.

But, the question remains, have we accomplished anything? Are there associated bond prices and if so, do these asset prices allow arbitrage or not?

This is answered by the following result.

Definition 2. The price of a τ -maturity bond is

$$P_t^\tau := \exp\{-Y_t^t + Y_t^\tau\}.$$

Proposition 2:

For each $\tau \leq T_0$ the discounted asset

$$\frac{P_t^\tau}{P_t^{T_0}}$$

is a local martingale.

Proof. Apply Ito's lemma to the process

$$\frac{P_t^\tau}{P_t^{T_0}} = \exp\{Y_t^\tau\}.$$

The stochastic equation for the differential has zero drift. Hence, it is a local martingale.

THEOREM 2.

The bond prices P_t^τ for $0 \leq t \leq \tau \leq T_0$ form a no-arbitrage price system.