Stochastic heat and Burgers equations and their Singularities

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P&PDE Workshop,
Minneapolis, July 2003

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Abstract

The Arnol'd-Thom classification of caustics for the Burgers equation suggests that there should be an analogous one for the wavefronts of the corresponding heat equation.

We present a general theorem for Hamiltonian systems characterizing how the level surfaces of Hamilton's principal function meet the caustic surface in both the deterministic and stochastic cases.

Such a characterization allows one to give a fairly detailed description of the behaviour of the solution of the heat equation in the vicinity of the wavefront and caustic. It allows one to propose some reasons for the "blow-up" of the Burgers velocity field on the caustic.

In the case of small noise the shapes of the random wavefront and random caustic may easily be obtained, and to first order the caustic is merely displaced.

In the stochastic case we have the possibility of "rapid" changes in the caustic-wavefront intersection. This will engender stochastic turbulence in the Burgers velocity field and, due to its stochasticity, may be of an intermittent nature. There is no analogue of this in the deterministic case.

Throughout our studies much use has been made of computer algebra packages in building an understanding of the archetypal cases. Numerical simulations and numerical solutions of the partial differential equations involved have been immensely useful in clarifying conjectures and determining apt characterizations.

Stochastic Heat and Burgers Eqns

We have studied the inviscid limit of the stochastic viscous Burgers equation, for the velocity field $v^\mu(x,t)$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla_x) v^\mu = -\nabla c - \epsilon \nabla k \dot{W}_t + \frac{\mu^2}{2} \Delta v^\mu,$$

where $v^\mu(x,0) = \nabla S_0(x) + O(\mu^2)$ for some given S_0, $\dot{W}_t$ representing White Noise, by using the Hopf-Cole transformation,

$$v^\mu = -\mu^2 \nabla \ln u^\mu,$$

with u^μ satisfying the stochastic heat equation of Stratonovich type

$$du_t^\mu = \left[\frac{\mu^2}{2} \Delta u_t^\mu + \mu^{-2} c u_t^\mu\right] dt + \epsilon \mu^{-2} k u_t^\mu \circ dW_t,$$

with $u_0^\mu(x) = T_0(x) \exp\left(-S_0(x)/\mu^2\right)$, $S_0 \in C^1$ and T_0 a smooth positive function. c and k are functions of (x,t) but we illustrate time independent c herein.

Let $v^\mu = \nabla S^\mu$. Then S^μ satisfies the stochastic Hamilton-Jacobi equation

$$dS^\mu + \frac{1}{2}|\nabla S^\mu|^2\,dt + c\,dt + \epsilon k\,dW_t = \frac{1}{2}\mu^2 \Delta S^\mu\,dt.$$

and for $S^\mu \sim \sum_{j=0}^{\infty} \mu^{2j} S_j$ we have

$$\frac{\partial S_j}{\partial t} + \frac{1}{2} \sum_{\substack{i_1, i_2 \geq 0 \\ i_1+i_2=j}} \nabla S_{i_1} \cdot \nabla S_{i_2} = \frac{1}{2}\Delta S_{j-1},$$

with the convention $\frac{1}{2}\Delta S_{-1} = -c - \epsilon k \dot{W}_t$, $\dot{W}_t$ being white noise.

The above are the stochastic Hamilton-Jacobi continuity equations.

The corresponding stochastic mechanical flow Φ_s satisfies

$$d\dot{\Phi}_s = -\nabla c(\Phi_s)\,ds - \epsilon \nabla k(\Phi_s, s)\,dW_s,$$

with $\Phi_0(x) = x$, $\dot{\Phi}_0(x) = \nabla S(x, 0) = \nabla S_0(x)$.

Note that $S_0(x)$ and $S_0(x,t)$ differ.

Assuming $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism for $0 \le s < T(\omega)$, the caustic time, define

$$\tilde{S_0}(y,t) = \frac{1}{2}\int_0^t |\dot{\Phi}_s(y)|^2\, ds + S(y,0) \\ - \int_0^t c(\Phi_s y)\, ds - \epsilon \int_0^t k(\Phi_s y, s)\, dW_s$$

and let $S_0(x,t) = \tilde{S_0}(\Phi_t^{-1}x, t)$. For $c \in C^2(\mathbb{R}^d)$, $k \in C^2(\mathbb{R}^d \times \mathbb{R})$, $S_0(\cdot, 0) \in C^2(\mathbb{R}^d)$ we have:

$$\dot{\Phi}_t = \nabla S_0(\Phi_t, t)$$

for a.e. $\omega \in \Omega$, $0 \le t < T(\omega)$, with S_0 satisfying

$$dS_0(x,t) + \frac{1}{2}|\nabla S_0(x,t)|^2\, dt \\ = -c(x)\, dt - \epsilon k(x,t)\, dW_t.$$

For $\rho(x,t) = \left|\text{Det}\left(\frac{\partial}{\partial x}\Phi_t^{-1}(x)\right)\right|$ we have ρ satisfying the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla S_0) = 0.$$

for a.e. $\omega \in \Omega$, any $x \in \mathbb{R}^d$, $0 \le t < T(\omega)$.

Suppose now that $T_0 : \mathbb{R}^d \to \mathbb{R}$ is a smooth positive function. Define

$$T_0(y,t) = T_0(y)$$

and for $j = 1, 2, \ldots,$ let

$$T_j(y,t) \\ = \int_0^t \rho^{-\frac{1}{2}}(\cdot, s)\Delta_{\bullet}\left(\rho^{\frac{1}{2}}(\cdot, s) T_{j-1}(\Phi_s^{-1}\cdot, s)\right)\Big|_{\Phi_s y}\, ds.$$

If we now define ψ_j by

$$\psi_j(x,t) = T_j(\Phi_t^{-1}x, t)\rho^{\frac{1}{2}}(x,t).$$

we have the equalities

$$\frac{\partial \psi_j}{\partial t} + \nabla\psi_j \cdot \nabla S_0 = -\frac{1}{2}\psi_j \Delta S_0 + \Delta\psi_{j-1}$$

for $j = 0, 1, 2, \ldots,$ with $\psi_{-1} \equiv 0$.

We have now assembled the main tools we need to solve the equations

$$\frac{\partial S_j}{\partial t} + \frac{1}{2}\sum_{\substack{i_1, i_2 \geq 0 \\ i_1 + i_2 = j}} \nabla S_{i_1} \cdot \nabla S_{i_2} = \frac{1}{2}\Delta S_{j-1}, \quad j \geq 0.$$

For a.e. $\omega \in \Omega$ the solutions of the Hamilton-Jacobi continuity equations are given by

$$S_1(x,t) = -\ln \psi_0(x,t)$$

and for $j \geq 2$

$$\begin{aligned} &S_j(x,t) \\ &= 2^{1-j}\Bigg(-\frac{\psi_{j-1}}{\psi_0} \\ &+ \sum_{\substack{i_1,i_2\geq 1\\ i_1+i_2=j-1}} \frac{\psi_{i_1}\psi_{i_2}}{2\psi_0^2} - \sum_{\substack{i_1,i_2,i_3\geq 1\\ i_1+i_2+i_3=j-1}} \frac{\psi_{i_1}\psi_{i_2}\psi_{i_3}}{3\psi_0^3} \\ &+ \cdots + \frac{(-1)^{j-1}\psi_1^{j-1}}{(j-1)\psi_0^{j-1}}\Bigg)(x,t). \end{aligned}$$

An all important role is played by the Nelson diffusion process y_s^μ with drift given by $-\nabla\left(\sum_{j=0}^m \mu^{2j} S_j(y_s^\mu, t-s)\right)$

$$\begin{aligned} dy_s^\mu &= \mu\, dB(s) - \nabla \sum_{j=0}^{m} \mu^{2j} S_j(y_s^\mu, t-s)\, ds, \\ y_0^\mu &= x. \end{aligned}$$

We may represent u^μ and v^μ as below.

For each $m \geq 0$, with $v_j(x,t) = \nabla S_j(x,t)$,

$$\exp\left\{\mu^{-2}\sum_{j=0}^{m}\mu^{2j}v_j(x,t)\right\}u_t^{\mu}(x)$$

$$= \mathbb{E}\exp\left\{-\frac{\mu^{2m}}{2}\int_0^t \nabla\cdot v_m(y_s^{\mu},t-s)\,ds\right.$$

$$\left.+\frac{1}{2}\sum_{j=m+1}^{2m}\mu^{2(j-1)}\sum_{\substack{0\leq i_1,i_2\leq m\\ i_1+i_2=j}}\int_0^t v_{i_1}\cdot v_{i_2}(y_s^{\mu},t-s)\,ds\right\},$$

and the viscous Burgers velocity field is

$$v^{\mu}(x,t) = \sum_{j=0}^{m}\mu^{2j}v_j(x,t)$$

$$-\mu^2\nabla\ln\mathbb{E}\left\{\exp\left\{-\frac{\mu^{2m}}{2}\int_0^t\nabla\cdot v_m(y_s^{\mu},t-s)\,ds\right.\right.$$

$$\left.\left.+\frac{1}{2}\sum_{j=m+1}^{2m}\mu^{2(j-1)}\sum_{\substack{0\leq i_1,i_2\leq m\\ i_1+i_2=j}}\int_0^t v_{i_1}\cdot v_{i_2}(y_s^{\mu},t-s)\,ds\right\}\right\}.$$

Note that the second factor in each of the above is of the form $(1+O(\mu^{2m}))$.

Remark. *When $T_0 = 1$, ρ is simply given by*

$$\rho^{1/2}(x,t) = e^{-S_1(x,t)}.$$

In the case $T_0 \neq 1$, we note that initially ψ_0^2 is T_0^2 , and observe that up to the caustic time

$$\begin{aligned}
&\int_{\mathbb{R}^d} \psi_0^2(x,t)\,dx \\
&= \int_{\mathbb{R}^d} T_0^2(\Phi_t^{-1}(x))\rho(x,t)\,dx \\
&= \int_{\mathbb{R}^d} T_0^2(\Phi_t^{-1}(x))\left|\text{Det}\left(\nabla_x \Phi_t^{-1}(x)\right)\right|\,dx \\
&= \int_{\mathbb{R}^d} T_0^2(y)\,dy \\
&= \int_{\mathbb{R}^d} \psi_0^2(y,0)\,dy.
\end{aligned}$$

This shows that $\int_{\mathbb{R}_d} \psi_0^2\,dx$ is a conserved quantity and one may associate this with "mass".

We habitually couch things in terms of classical dynamics from now on and so we introduce the classical path $X(s)$ in terms of the flow Φ. We have $X(s) = \Phi_s \Phi_t^{-1} x$, where we accept that $x_0(x,t) = \Phi_t^{-1} x$ is not necessarily unique. Again we remark that given some regularity, the global inverse function theorem, [TZH], gives a caustic time $T(\omega)$ such that, for $s < T(\omega)$, Φ_s is a random diffeomorphism.

Recall that we had

$$\dot{\Phi}_t = \nabla S_0(\Phi_t, t),$$
$$v^\mu(x,t) = \nabla S^\mu(x,t),$$

and so, as we shall see,

$$v^0(x,t) = \nabla \left. S^\mu(x,t) \right|_{\mu=0} = \dot{\Phi}_t \Phi_t^{-1} x$$

is a formal solution of Burgers equation in the case $\mu = 0$, which is well defined up to the caustic time because $x_0(x,t)$ is unique.

Non-uniqueness of $x_0(x,t)$ will be associated with discontinuities in $v^0(x,t)$ and $u^0(x,t)$.

For a non-degenerate critical point, when the multiplicity of $x_0(x,t)$ is finite, so that

$$\Phi_t^{-1}\{x\} = \left\{x_0^1(x,t), x_0^2(x,t), \ldots, x_0^n(x,t)\right\},$$

we can deduce that

$$u^\mu(x,t) \sim \sum_{i=1}^{n} \theta_i \exp\left\{-S_0^i(x,t)/\mu^2\right\},$$

where $S_0^i(x,t) = S_0(x_0^i(x,t)) + A(x_0^i(x,t), x, t)$ for $i = 1, 2, \ldots, n$ and θ_i is an asymptotic series in μ^2. Needless to say the dominant term in the above comes from the minimising $x_0(x,t)$ and so we have

$$S(x,t) = \min_{i=1,2,\ldots,n} S_0^i(x,t)$$

in line with the results of Freidlin et al.

Of course $u^0(x,t)$ can switch discontinuously from being exponentially large to exponentially small as we cross parts of the caustic since the minimising S_0^i can disappear due to coalescence of pre images $x_0^{i_1}(x,t)$ and $x_0^{i_2}(x,t)$.

Stochastic Hamiltonians, C_t and H_t

Define $A(x_0, p_0, t)$, the stochastic action, by

$$\frac{1}{2}\int_0^t \dot{X}^2(s)\,ds - \int_0^t [c(X(s))ds + \epsilon k(X(s),s)dW_s],$$

a.s., with $X(s) = X(s, x_0, p_0)$ satisfying

$$d\dot{X}(s) = -\nabla c(X(s))\,ds - \epsilon \nabla k(X(s),s)\,dW_s,$$

$s \in [0,t]$, $X(0) = x_0$, $\dot{X}(0) = p_0$, $x_0, p_0 \in \mathbb{R}^d$. We shall assume that X_s is unique and as usual is $\mathcal{F}_s$ measurable. We also allow for p_0 to be an as yet unspecified function of x_0. Then, for ∇c, ∇k Lipschitz, with Hessians $\nabla^2 c$, $\nabla^2 k$ and all second derivatives with respect to space variables of c and k bounded, according to Kunita [KUN], $\partial X_s/\partial x_0^\alpha$ satisfies

$$\frac{d}{ds}\left(\frac{\partial X_s}{\partial x_0^\alpha}\right) = \frac{\partial \dot{X}_0}{\partial x_0^\alpha} - \int_0^s \left[\nabla^2 c(X(r))\frac{\partial X(r)}{\partial x_0^\alpha}\,dr + \epsilon \nabla^2 k(X(r),r)\frac{\partial X(r)}{\partial x_0^\alpha}\,dW_r\right]$$

Moreover,

$$\dot{X}_s = \dot{X}_0 - \int_0^s \left[\nabla c(X(r))\,dr + \epsilon \nabla k(X(r),r)\,dW_r\right].$$

Lemma. *Assume $S_0, c \in C^2$ and $k \in C^{2,0}$, ∇c, ∇k are Lipschitz, with Hessians $\nabla^2 c$, $\nabla^2 k$ and all second derivatives with respect to space variables of c and k bounded. If $\dot{X}_s$ satisfies the last equation and we have p_0, possibly x_0 dependent, then almost surely*

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \dot{X}(t) \cdot \frac{\partial X(t)}{\partial x_0^\alpha} - \dot{X}_\alpha(0).$$

Remark. *Observe that, if we fix $X(t)$, we obtain almost surely*

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = -\dot{X}_\alpha(0),$$

for $\alpha = 1, 2, \ldots, d$.

Let $X(s, x_0, x) = X(s, x_0, p_0)|_{p_0 = p(x_0, x, t)}$ where $p_0 = p(x_0, x, t)$ is the (! random) minimiser of $A(x_0, p_0, t)$ with $X(t, x_0, p_0) = x$.

Set $A(x_0, x, t) = A(x_0, p_0, t)|_{p_0 = p(x_0, x, t)}$ and define Hamilton's principal function corresponding to the initial momentum $\nabla S_0(x_0)$ to be

$$\mathcal{A}(x_0, x, t) = A(x_0, x, t) + S_0(x_0).$$

We define a prelevel surface of Hamilton's principal function by eliminating x between the equations

$$\mathcal{A}(x_0, x, t) = c \qquad \text{and} \qquad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0,$$

$\alpha = 1, 2, \ldots, d$, and a level surface H_t by eliminating x_0. So the prelevel surface is $\Phi_t^{-1} H_t$. Similarly we define the caustic C_t and the precaustic $\Phi_t^{-1} C_t$ by eliminating x_0 or x between

$$\text{Det}\left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}(x_0, x, t)\right) = 0 \text{ and } \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0,$$

$\alpha = 1, 2, \ldots, d$.

Lemma. *The classical flow map $x = \Phi_t(x_0)$ is a differentiable map from $\Phi_t^{-1} H_t$ to H_t with Frechet derivative*

$$D\Phi_t(x_0) = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}(x_0, x, t)\right)^{-1} \left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}(x_0, x, t)\right),$$

if $\mathcal{A}$ is C^3 in space derivatives.

Proposition. *We consider the random prelevel surface obtained by eliminating x between the equations*

$$\mathcal{A}(x_0, x, t) = c \qquad \textit{and} \qquad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0,$$

$\alpha = 1, 2, \ldots, d$. Then the normal to the prelevel surface at the point x_0 is to within a scalar multiplier given by

$$n(x_0) = -\left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}\right)\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1} \dot{X}(t, x_0, \nabla S_0(x_0)).$$

Corollary. *In three dimensions at any point $x_0 \in \Phi_t^{-1} C_t \cap \Phi_t^{-1} H_t$ where $n(x_0) \neq 0$ and*

$$\operatorname{Ker}\left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}(x_0, x, t)\right)\Bigg|_{x=\Phi_t(x_0)} = \langle e_0 \rangle,$$

e_0 being the zero eigenvector, T_{x_0} the tangent plane to the prelevel surface is spanned by e_0 and $(n(x_0) \wedge e_0)$.

Deterministic Case with $S_0(x_0, y_0) = x_0^2 y_0/2$.

Figure 1 : Precaustic and Prelevel Surface

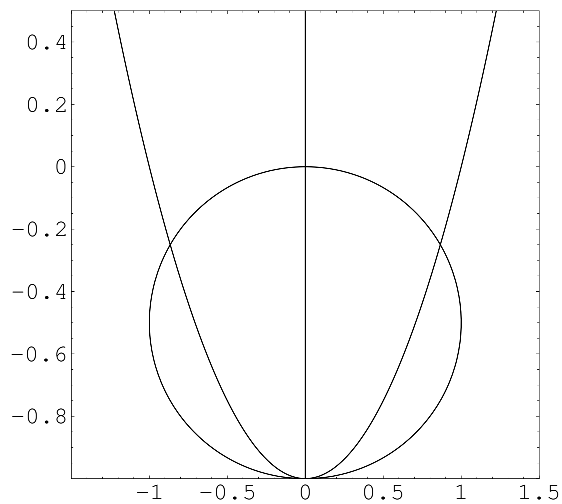
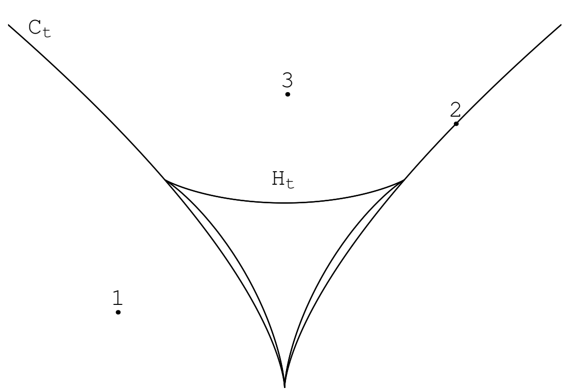


Figure 2 : Cusp and Tricorn



Here we illustrate the stochastic case where $k(x,t) \equiv x$ and initial data $S_0(x_0, y_0) = x_0^2 y_0/2$.

positive $\mathcal{A}$

Figure 3 :

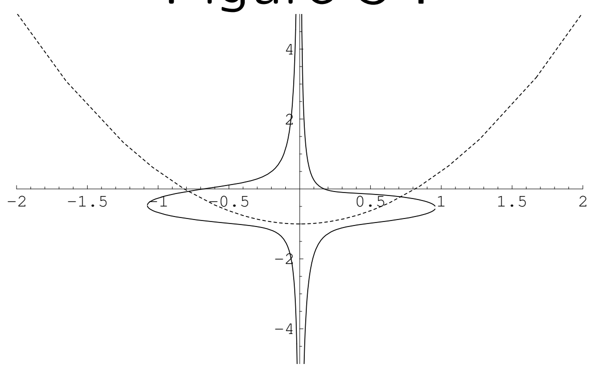
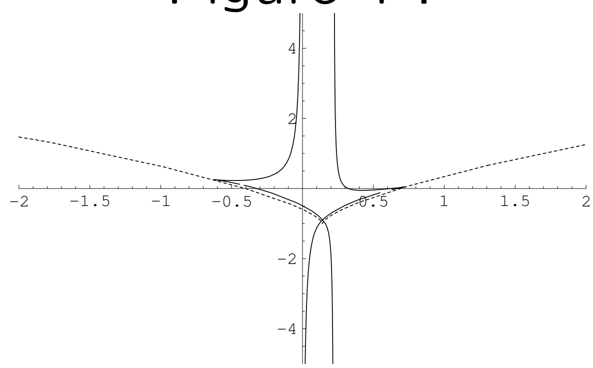


Figure 4 :



negative $\mathcal{A}$

Figure 5 :

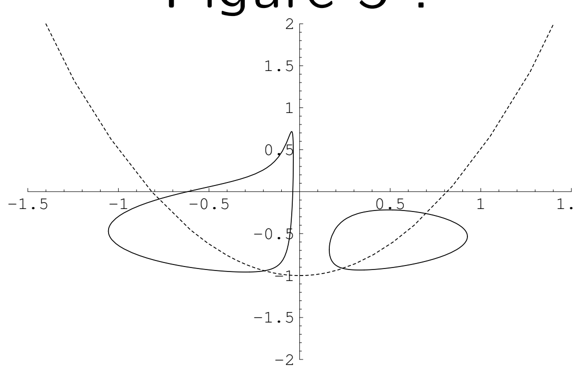
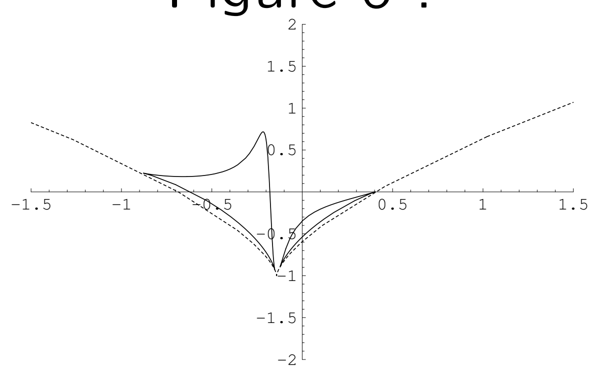


Figure 6 :



Note that the caustic has been displaced but maintains its shape.

Definition. *A curve* $x = x(\gamma)$, $\gamma \in N(\gamma_0, \delta)$, *is said to have a generalised cusp at* $\gamma = \gamma_0$, γ *being arc length if* $\left.\frac{dx(\gamma)}{d\gamma}\right|_{\gamma=\gamma_0} = 0$.

Definition. *The cusped part of the level surface* H_t *in three dimensions is given by*

$$\mathsf{Cusp}(H_t) = \Big\{ x \in H_t : x \in \Phi_t\left(\Phi_t^{-1} C_t \cap \Phi_t^{-1} H_t\right), \\ x = \Phi_t(x_0),\ n(x_0) \neq 0 \Big\}.$$

Theorem. *Any point* x *on the level surface* H_t, $x = \Phi_t(x_0)$ *with* $x_0 = \Phi_t^{-1}(x)$ *on the pre-level surface, can only be a generalised cusp of a curve on* H_t *if* x_0 *is a generalised cusp of the precurve on the prelevel surface or if* $x_0 \in \Phi_t^{-1} C_t$ *the precaustic.*

In three dimensions the planar cross section with normal e_p *of the level surface* H_t *through a point* x *where it meets* C_t *the caustic surface, will indeed have a genuine cusp at* x *if* $x \in \mathsf{Cusp}(H_t)$ *and there exists some non-zero vector* δx *(determined by three simple equations).*

Closeness to Classical

Having considered the general stochastic case in the previous section it is worthwhile demonstrating the closeness of the stochastic X_t, C_t and H_t to their deterministic counterparts. Let X^ϵ, with ϵ highlighted, satisfy

$$\begin{aligned} & d\dot{X}^\epsilon(x_0, s) \\ & = -\nabla c(X^\epsilon(x_0, s))ds - \epsilon \nabla k(X^\epsilon(x_0, s))dW_s, \end{aligned}$$

with $X^\epsilon(x_0, 0) = x_0$ and $\dot{X}^\epsilon(x_0, 0) = \nabla S_0(x_0)$, for $0 < s < t$. Let $X^0(x_0, s) = \Phi_s x_0$ be the deterministic version and let $\mathcal{G}$ be given by $\mathcal{G}_{ij} = \left\{ X_i^0(u), X_j^0(s) \right\} \theta(s-u)$,

Lemma

With X^0 defined as above $\mathcal{G}$ satisfies the matrix Jacobi equation

$$\left(\frac{d^2}{ds^2} + \nabla^2 c(X^0(x_0, s)) \right) \mathcal{G}(x_0, s, u) = 0,$$

with boundary conditions

$$\mathcal{G}(x_0, s_+, s) = 0, \qquad \frac{d\mathcal{G}}{ds}(x_0, s, u)\Big|_{s=u_+} = I.$$

Theorem Subject to certain conditions for continuity and boundedness of c and k and their derivatives, define

$$\tilde{X}^\epsilon(x_0, s) = \Phi_s x_0 - \epsilon \int_0^s \mathcal{G}(x_0, s, u) \nabla k(\Phi_u x_0)\, dW_u,$$

for $s \in [0, t]$. Then there exists a constant $M > 0$ such that for any $\delta > 0$ and sufficiently small $\epsilon > 0$,

$$P\{\epsilon^{-\frac{3}{2}} \sup_{x_0 \in \mathbb{R}^d} |X^\epsilon(x_0, s) - \tilde{X}^\epsilon(x_0, s)| > \delta, \quad \text{some } s \in [0, t]\} < \frac{M\epsilon^2}{\delta^4},$$

and

$$P\{\epsilon^{-\frac{3}{2}} \sup_{x_0 \in \mathbb{R}^d} |\nabla X^\epsilon(x_0, s) - \nabla \tilde{X}^\epsilon(x_0, s)| > \delta, \quad \text{some } s \in [0, t]\} < \frac{M\epsilon}{\delta^2}.$$

In particular, $X^\epsilon(x_0, s) - \tilde{X}^\epsilon(x_0, s) = \circ(\epsilon^{\frac{3}{2}})$ and $\nabla X^\epsilon(x_0, s) - \nabla \tilde{X}^\epsilon(x_0, s) = \circ(\epsilon^{\frac{3}{2}})$ as $\epsilon \to 0$ in probability.

In the case of the caustic C_t we define

$$\mathcal{D}^{\epsilon}_{pre} = \{(t, x_0) : \text{Det}\, \nabla_{x_0} X^{\epsilon}_t(x_0) \neq 0, 0 \leq t \leq T\},$$
$$\mathcal{D}_{pre} = \{(t, x_0) : \text{Det}\, \nabla_{x_0} \Phi_t x_0 \neq 0, 0 \leq t \leq T\}.$$

and we then state the result.

Lemma
As $\epsilon \to 0$, $[0, T] \times \mathbb{R}^d - \mathcal{D}^{\epsilon}_{pre} \to [0, T] \times \mathbb{R}^d - \mathcal{D}_{pre}$ in probability for any given $T > 0$. That is to say the precaustic surface of the stochastic dynamics converges to the precaustic surface of the classical mechanics as $\epsilon \to 0$ in probability.

Theorem

Denote for any given $T > 0$

$$\mathcal{D}^{\epsilon} = \{(t, x) : \text{Det}(\nabla (X^{\epsilon})^{-1}_t x) \neq 0, 0 \leq t \leq T\},$$
$$\mathcal{D} = \{(t, x) : \text{Det}(\nabla \Phi^{-1}_t x) \neq 0, 0 \leq t \leq T\}.$$

Then $[0, T] \times \mathbb{R}^d - \mathcal{D}^{\epsilon} \to [0, T] \times \mathbb{R}^d - \mathcal{D}$, as $\epsilon \to 0$. That is to say the caustic surface of the stochastic dynamics with noise converges in probability to the caustic surface of the classical mechanics without noise as $\epsilon \to 0$.

In the case of the wavefront H_t we have:

Theorem

Let ϕ_s be the minimizer of

$$\frac{1}{2}\int_0^t |\dot{\phi}_s|^2 ds + S_0(\phi_t) - \int_0^t c(\phi_s) ds,$$

satisfying $\phi_t = x$ and ϕ_s^ϵ be the minimizer of $\frac{1}{2}\int_0^t |\dot{\phi}_s^\epsilon|^2 ds + S_0(\phi_t^\epsilon) - \int_0^t c(\phi_s^\epsilon) ds - \epsilon \int_0^t k(\phi_s^\epsilon) dW_s$, satisfying $\phi_t^\epsilon = x$ for almost all $\omega \in \Omega$. Then we have for almost all $\omega \in \Omega$

$$S(x,t) - \epsilon \int_0^t k(\phi_s^\epsilon) dW_s$$
$$\leq S^\epsilon(x,t) \leq S(x,t) - \epsilon \int_0^t k(\phi_s) dW_s.$$

In particular, as $\epsilon \to 0$, $S^\epsilon(x,t) \to S(x,t)$ a.s.. Furthermore, for the classical mechanics , assume there exists a unique x_0 for fixed t and x such that $\Phi_t x_0 = x$. The random wavefront for the heat equation has (x,t) equation

$$S^0(x,t) = \epsilon \int_0^t k(\Phi_s x_0)\, dW_s + \circ(\epsilon),$$

where $S^0(x,t)$ is Hamilton's principal function for the path $X^0(x,t)(s)$.

Burgers Fluid in two dimensions

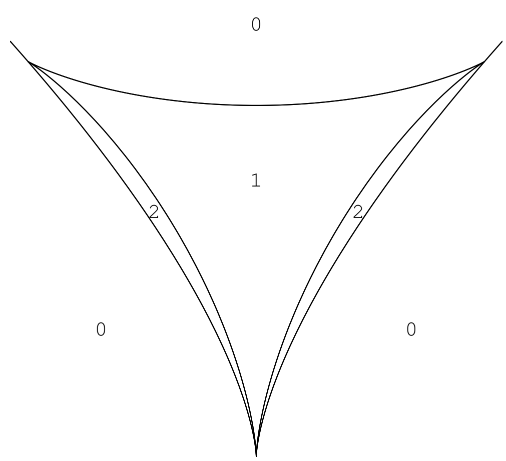
For smooth S_0, in two dimensions, we can divide one side of the caustic into *hot* and *cool* parts. We consider points of intersection α with the level surface $S(x,t) = S(\alpha,t)$ cusped at α: if the part of the level surface cusped at α corresponds to the minimising $x_0(\alpha,t)$ then the Burgers velocity field $v^0(x,t) = \nabla S(x,t) \to \nabla S(\alpha,t) = 0$ as $x \to \alpha$ from the cusped side of the caustic.

Moreover, this entails the minimising surface changing as we cross the caustic. Indeed the cusp occurs because the minimising surface on the cusped side of the caustic cannot be continued across the caustic. This is because two $x_0(x,t)$'s coalesce to a common minimiser at the cusp and then disappear. So $u^0(x,t)$ is necessarily exponentially discontinuous as we cross such parts of the caustic.

Points where the Burgers fluid has zero velocity on one side of the caustic, are called cool. *Chris Reynolds* has greatly expanded the classification techniques for higher dimensions in his recent Ph.D. thesis.

The whole of one side of the generic semicubical parabolic Cusp is cool. The exponential discontinuity in u^0 can be easily seen, at least for part of the cusp, by inspecting the following diagram giving the number of negative S_0^i's in the different regions. Similar results apply in three dimensions.

Figure 7 : Number of Negative S_0



Singularities

Recall that our random map $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$d_s\dot{\Phi}_s = -\nabla c(\Phi_s) - \epsilon\nabla k(\Phi_s)\, dW_s,$$

with $\Phi_0 = I$ and $\dot{\Phi}_0 = \nabla S_0$.

Proposition The random Burgers velocity field in the limit of zero viscosity is

$$v^0(x,t) = v^0(\Phi_t^{-1}x, 0) - \int_0^t \nabla c(\Phi_s\Phi_t^{-1}x)\, ds - \int_0^t \epsilon\nabla k(\Phi_s\Phi_t^{-1}x)\, dW_s,$$

with $v^0(x,0) = \nabla S_0(x)$. This v^0 is finite almost surely $0 \le t \le T(\omega)$. The corresponding solution of the continuity equation, the random "density" $\rho(x,t) = \left|\mathrm{Det}(\nabla\Phi_t^{-1}x)\right|$, is finite for $0 \le t \le T(\omega)$ but 'blows up' on the random caustics. Evidently $\nabla \wedge v^0 \equiv 0$, almost surely.
Remark. *More detailed information on the behaviour of v^0 and ρ on the caustic is given in Elworthy, Truman and Zhao [ETZ].*

Numerical simulation for $S_0(x_0, y_0) = x_0^2 y_0 / 2$.

Figure 8 : $|v^\mu|^2$ for small μ

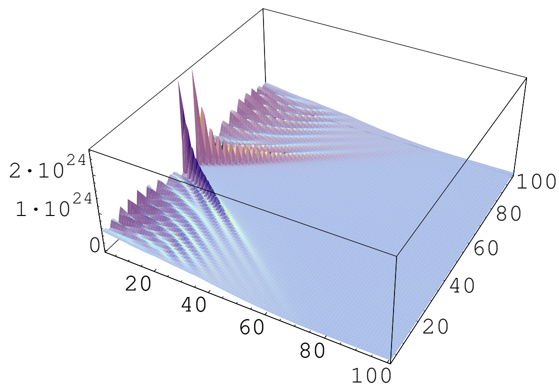
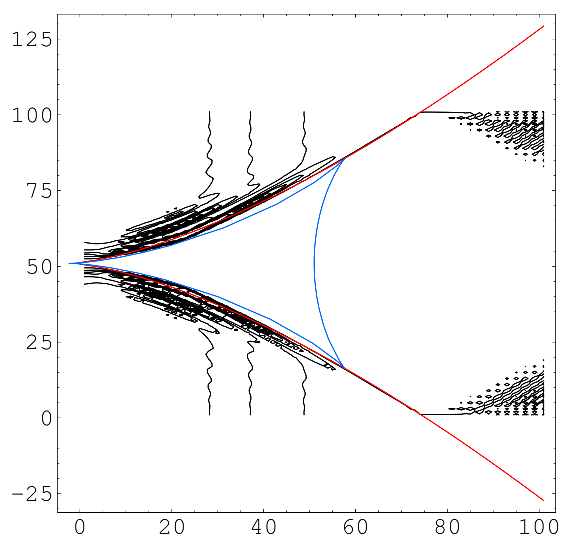


Figure 9 : Contour Plot of $|v^\mu|^2$ with Cusp and Tricorn



Stochastic Turbulence

We emphasise that new features emerge if one uses the information on caustic-wavefront interaction presented above as a basis for investigating stochastic turbulence. We can associate turbulent behaviour with a change in the number of cusped curves on the level surface. For a large family of S_0, in the two dimensional deterministic case, the positions and times of changes in caustic-wavefront interaction can be computed exactly. The times t when this occurs are just the times when the prelevel surfaces *touch* the precaustic. Furthermore, the times t when this number of curves changes in the deterministic case are simply the zeros of a deterministic function ζ, usually isolated zeros.

In the stochastic case we can analyse the caustic-wavefront interaction in detail by employing a knowledge of the propagator for the stochastic heat equation. In this case ζ is a stochastic process whose zeros usually form a perfect set, i.e. an infinite set containing no isolated points. At times when the prelevel surfaces *touch* the precaustic the number of cusped curves changes with infinite frequency because of the infinitely rapid oscillation of the stochastic process ζ. This is in line with what one would expect for turbulent behaviour. When the stochastic process ζ is *recurrent* this stochastic turbulent behaviour is "intermittent" so that the scale of turbulent fluctuations varies in a random periodic way. The analysis may be applied to the family of S_0 mentioned previously.

Intermittence in two dimensions

The example presented below is one of many detailed in the doctoral thesis of *Chris Reynolds*. More general results will feature in a future joint paper.

Let $c \equiv 0$, $k_t(x,y) \equiv x$ and $S_0(x_0,y_0) = f(x_0) + g(x_0)y_0$, where f, g, f' and g' are zero at $x_0 = a$, $g''(a) \neq 0$. The turbulent times t at which $n_c(t)$, the number of cusps on the zero pre-level surface of the Hamilton-Jacobi function changes are the zeros of the stochastic turbulence process ζ_0

$$\zeta_0(t) = -a\varepsilon W_t + \varepsilon^2 W_t \int_0^t W_s\, ds - \frac{\varepsilon^2}{2}\int_0^t W_s^2\, ds\ .$$

$\{t : \zeta_0(t) = 0\}$ is a perfect set and $\zeta(t)$ is recurrent to 0.

$\zeta_c(t) = \zeta_0(t) - c$ has exactly the same properties, where zeros of $\zeta_c(t)$ are times at which the number of cusps on the c pre-level surface of the Hamilton-Jacobi function changes.

Lemma. *Let W be a $BM(\mathbb{R})$ process starting at 0 and c a real constant. Define*

$$Y_t := -a\varepsilon W_t + \varepsilon^2 W_t \int_0^t W_s\, ds - \frac{\varepsilon^2}{2}\int_0^t W_s^2\, ds - c\ .$$

Then with probability one there exists a sequence of times (a_n) with $a_n \nearrow \infty$ such that $Y_{a_n} = 0$ for every n.

We begin by finding a sequence of times tending to infinity at which $Y_t \geq 0$. Define $f(r) := r$ for $0 \leq r \leq 1$ so that clearly f is absolutely continuous, $f(0) = 0$ and $\int_0^1 f'(u)^2\, du \leq 1$. Thus $f(r)$ is a Strassen function, $f \in K$.

Hence by Strassen's Law of the Iterated Logarithm we know that after throwing away a null set of paths, we can path-wise find a sequence t_n such that if

$$h(t) := (2t \ln\ln t)^{\frac{1}{2}}\ ,$$

then

$$h(t_n)^{-1} W_{rt_n} \to f(r)\ ,$$

uniformly over r in $[0,1]$.

We show that for each ω with $t_n = t_n(\omega)$ we have $h(t_n)^{-2} t_n^{-1} Y_{t_n} \to \frac{\varepsilon^2}{3}$. Let us consider each of the terms that comprise the stochastic process $Y_t(\omega)$.

$$a\varepsilon h(t_n)^{-2} t_n^{-1} W_{t_n} \to 0$$

$$\begin{aligned}
&h(t_n)^{-2} t_n^{-1} W_{t_n} \int_0^{t_n} W_u \, du \\
&= h(t_n)^{-1} W_{t_n} \int_0^1 h(t_n)^{-1} W_{r t_n} \, dr \\
&\to f(1) \int_0^1 f(r) \, dr = \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{2} h(t_n)^{-2} t_n^{-1} \int_0^{t_n} W_u^2 \, du \\
&= -\frac{1}{2} \int_0^1 \left(h(t_n)^{-1} W_{r t_n} \right)^2 \, dr \\
&\to -\frac{1}{2} \int_0^1 f(r)^2 \, dr = -\frac{1}{6}.
\end{aligned}$$

Combining the above we see that for each ω with $t_n = t_n(\omega)$ we have

$$h(t_n)^{-2} t_n^{-1} Y_{t_n} \to \varepsilon^2 \left(\frac{1}{2} - \frac{1}{6} \right) = \frac{\varepsilon^2}{3} \ .$$

To conclude we must find a sequence of times tending to infinity at which $Y_t \le 0$. If $c > 0$ then we simply choose times when $W_t = 0$. For $c \le 0$ we must choose a Strassen function such that

$$h(t_n)^{-2} t_n^{-1} W_{t_n} \int_0^{t_n} W_s\, ds \to f(1) \int_0^1 f(r)\, dr < 0 \ .$$

Taking

$$f(r) = \begin{cases} r & 0 \le r \le \frac{1}{3} \ , \\ \frac{2}{3} - r & \frac{1}{3} \le r \le 1 \ , \end{cases}$$

it may be easily shown that $f \in K$ and $f(1) \int_0^1 f(u)\, du = -\frac{1}{54} < 0$.

We have shown the recurrence of Y_t and thus the intermittence of stochastic turbulence for this large family of examples. We must again stress that there is no analogue of this phenomenon in the deterministic case.

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A Plethora of Cusps

We now explain why we see generalised cusps on planar sections of level surfaces in three dimensions. If δx is of second order of small quantities, solving

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0, \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0 + \delta x_0, x + \delta x, t) = 0,$$

for $\alpha = 1, 2, 3$ to second order reduces to

$$\delta x = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}\right)^{-1} \left\{ \left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}\right) \delta x_0 + \frac{1}{2} \left(\delta x_0 \cdot \nabla_{x_0}\right)^2 \frac{\partial \mathcal{A}}{\partial x_0} \right\}.$$

Now write

$$\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1} \dot{X}(t, x_0, \nabla S_0(x_0)) = -\left(\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2\right)$$

and set

$$\delta x_0 = \varepsilon(\lambda e_0 + \mu e_0^\perp) + \varepsilon^2(\xi e_0 + \eta e_0^\perp + \zeta n).$$

We now obtain Equations (C_1), (C_2) and (C_3). Note that (C_3) does not depend on η allowing us to solve for $\zeta : \lambda^2$ and $\eta : \lambda^2$.

$$\delta x \cdot e_p = 0, \qquad (C_1)$$

$$\delta x = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}\right)^{-1} \Big\{\varepsilon^2 \theta\Big(\eta\lambda_1\lambda_2(\alpha_2 e_1 - \alpha_1 e_2) + \zeta(\alpha_1\lambda_1^2 e_1 + \alpha_2\lambda_2^2 e_2)\Big) + \frac{\varepsilon^2}{2}\lambda^2\partial_0^2\frac{\partial \mathcal{A}}{\partial x_0}\Big\}, \qquad (C_2)$$

$$\delta x \cdot \frac{\partial \mathcal{A}}{\partial x} = (\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2) \cdot \left(\theta\Big(\eta\lambda_1\lambda_2(\alpha_2 e_1 - \alpha_1 e_2) + \zeta(\alpha_1\lambda_1^2 e_1 + \alpha_2\lambda_2^2 e_2)\Big) + \frac{\lambda^2}{2}\partial_0^2\frac{\partial \mathcal{A}}{\partial x_0}\right) = 0. \qquad (C_3)$$

$\lambda_0 = 0$, α_0, α_1, α_2 and θ are known and $\perp$ eigenvectors e_i satisfy

$$\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2} e_i = \lambda_i e_i, \qquad i = 0, 1, 2,$$