Applications of the Malliavin Calculus to the Monte Carlo Analysis of SPDE's

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IMA July 25, 2003

What Kinds of Applications

Mathematicians have used Malliavin calculus mostly for:

- Smoothness of Solutions of SDE's
  - For SPDE's (Nualart, R.C., Sanz, ... , Mattingly, Pardoux)
- Martingale Representations (Clark-Ocone-Karatzas)
  - For SPDE's (R.C. & Tehranchi)
- Design Monte Carlo computations of sensitivities Greeks(Lasry, Lions, Touzi, ...)
  - For SPDE's (R.C. & L. Wang, Work in Progress)

Models for Time Reversal Mirrors

Fouque-Papanicolaou-Ryzhik and Solna

Wave Equation

$$\frac{1}{c(x,y,z)^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

$c(x,y,z)$ propagation speed

Fourier Transform in Time

$$\tilde{u}(\omega,x,y,z) = \int_{\mathbb{R}} e^{-i\omega t} u(t,x,y,z)\, dt$$

The wave equation becomes

$$\Delta \tilde{u} + k^2 n(x,y,z)^2 \tilde{u} = 0$$

with

$$k = \frac{\omega}{c_0} \qquad \text{and} \qquad n(x,y,z) = \frac{c_0}{c(x,y,z)}$$

Solve for each fixed k, and inverse-Fourier the result !!!!!

Parabolic Approximation

Search for a Solution of the Form

$$\tilde{u}(k,x,y,z) = e^{ikz}\psi(k,x,y,z)$$

Then $\psi(k,x,y,z)$ needs to satisfy

$$2ik\psi_z + \psi_{zz} + \Delta_{\mathbf{x}}\psi + k^2[n(x,y,z)^2 - 1]\psi = 0$$

with $\mathbf{x} = (x,y)$.

$a << L$ implies validity of narrow beam approximation

We assume ψ slowly varying in z: i.e. $k|\psi_z| >> |\psi_{zz}|$

$$\begin{cases} 2ik\psi_z + \Delta_{\mathbf{x}}\psi + k^2(n(x,y,z)^2 - 1)\psi = 0 \\ \psi(\mathbf{x}, z=0) = \psi_0(\mathbf{x}) \end{cases}$$

2-D Schrödinger Equation in $\mathbf{x}$ (z playing the role of time !!!) with a time dependent potential !!!

Central Limit Theorem

- Wavelength λ short compared to the propagation distance L

$$\epsilon \equiv \frac{\lambda}{L} << 1$$

- Fluctuations of index of refraction weak and isotropic

$$\mathbb{E}\{(n^2 - 1)\} = O(\epsilon)$$

- Correlation Length ℓ of Fluctuations comparable with wavelength λ

$$\ell \ \sim \ \lambda$$

$$n(z, \mathbf{x}) = 1 + \epsilon\mu(z, \mathbf{x})$$

Stochastic Schrödinger Equation

Formally

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2k}\Delta_{\mathbf{x}}\psi + \dot{W}(t,\mathbf{x})\psi$$

Rigorously

$$i d\psi(t) = -\frac{1}{2k}\Delta_{\mathbf{x}}\psi(t)\,dt + \psi(t)\circ dW(t,\,\cdot\,)$$

with $\{W(t,\mathbf{x})\}_{t,\mathbf{x}}$ mean-zero Gaussian field

$$\mathbb{E}\{W(s,\mathbf{x})W(t,\mathbf{y})\} = (s\wedge t)\; q(\mathbf{x}-\mathbf{y})$$

Existence of mild solution (Dawson-Papanicolaou) in $\mathbb{R}^d$ with

$$\psi_0 \in L^2(\mathbb{R}^d)$$

T.R. Mirrors need Green's Function

$$\psi^B(\mathbf{y}) = \int_A G(L, \mathbf{y}; \mathbf{x}) \overline{\phi_0(\psi(L, \mathbf{x}))} d\mathbf{x}.$$

where

- $A \subset \mathbb{R}^2$ mirror
- ϕ_0 cut-off function: $\phi_0(z) = z\mathbf{1}_{\{|z|>s_0\}}$
- $\psi(t, \mathbf{x})$ solution of stochastic Schrödinger equation
- $G(t, \mathbf{x}, \mathbf{y})$ Green's function, i.e.

  for each fixed $\mathbf{y}$, solution of stochastic Schrödinger equation with initial condition

$$G(t = 0, \mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$$

Time Reversal Mirror Greeks

Solve

$$d\psi(t,\mathbf{x}) = [\frac{i}{2k}\Delta\psi(t,\mathbf{x}) - \frac{1}{8}q(\mathbf{x},\mathbf{x})\psi(t,\mathbf{x})]dz - \frac{i}{2}\psi(t,\mathbf{x})dW(t,\mathbf{x}),$$

For a given parameter γ, compute:

$$\frac{\partial}{\partial\gamma}\mathbb{E}\{\psi^B(\mathbf{y})\}$$

where

$$\psi^B(\mathbf{y}) = \int_A G(L,\mathbf{y};\mathbf{x})\overline{\phi_0(\psi(L,\mathbf{x}))}d\mathbf{x}$$

Examples:

- $\gamma = k$
- $\gamma = \sigma^2$
- γ parameter in ψ_0

SPDE Setup

- Bounded domain $D = [-l, +l] \times [-l, l]$ for $\mathbf{x} = (x, y)$
- Homogeneity of spatial part of $W(t, \mathbf{x})$ modulo l,

$$q(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} - \mathbf{y})$$

With notation $\sigma^2 = q(0)$

$$d\psi(t, \mathbf{x}) = [\frac{i}{2k}\Delta\psi(t, \mathbf{x}) - \frac{\sigma^2}{8}\psi(t, \mathbf{x})]dt - \frac{i}{2}\psi(t, \mathbf{x})dW(t, \mathbf{x}),$$

- Choose periodic boundary conditions for Δ

SPDE Setup

- Spectral density $\alpha = \{\alpha_{m,n}\}_{m,n}$

$$q(\mathbf{x}) = \sum_{m,n=-\infty}^{\infty} \alpha_{m,n} B_{m,n}(\mathbf{x})$$

where $\alpha_{m,n} \geq 0$, $\sum_{m,n} \alpha_{m,n} = 1$, and:

$$B_{m,n}(x, y) = \frac{1}{2l} e^{i(m\pi x + n\pi y)/l}$$

- q and Δ diagonalized by the same set of eigenfunctions !!

Infinite Dimensional Wiener Process

- Notation $< f, g >= \int_D f(\mathbf{x})\overline{g(\mathbf{x})}\, d\mathbf{x}$
- $f \in H = H_Q$ means $\|f\|_H^2 = \sum_{m,n} \alpha_{m,n}^{-1} < f, B_{m,n} >^2 < \infty$
- $f \in L^2(D)$ means $\|f\|^2 = \sum_{m,n} < f, B_{m,n} >^2 < \infty$
- $f \in H = H_Q$ means $\|f\|_{H^*}^2 = \sum_{m,n} \alpha_{m,n} < f, B_{m,n} >^2 < \infty$

With these notation:

- $\{W(t)\}_t$ cylindrical Wiener process on H
- H Reproducing Kernel Hilbert Space of $\{W(t)\}_t$
- Stochastic integrals $\int_0^t \sigma_s dW(s)$ make sense as elements of Hilbert space F whenever σ_s Hilbert-Schmidt from H into F s.t.

$$\mathbb{E}\{\int_0^t \|\sigma_s\|_{HS}^2\, ds\} < \infty$$

Evolution Form (Search for Mild Solutions)

Solve:

$$\begin{cases} \frac{\partial \psi}{\partial t} = \frac{i}{2k}\Delta_{\mathbf{x}}\psi(t,\mathbf{x}) - \frac{\sigma^2}{8}\psi(t,\mathbf{x}) \\ \psi(t=0,\mathbf{x}) = \psi_0(\mathbf{x}) \end{cases}$$

Denote the solution by $\psi(t,\mathbf{x}) = [U(t)\psi_0](\mathbf{x})$

- $\{e^{\sigma^2 t/8}U(t)\}_t$ is a unitary group on $L^2(D)$
- Conservation law, NO SMOOTHING

Mild solution

$$\psi(t) = U(t)\psi_0 - \frac{i}{2}\int_0^t U(t-s)\psi(s)dW(s)$$

WARNING: $\psi(s)$ interpreted as a multiplication operator !

Solution

Iteration:

$$\begin{cases} \psi^{(0)}(t) = U(t)\psi_0 \\ \psi^{(N)}(t) = -\frac{i}{2}\int_0^t U(t-s)\psi^{(N-1)}(s)dW(s) \end{cases}$$

- Pick ψ_0 in a Hilbert space F
- Prove the stochastic integrals make sense (integrand is H-S from H into F
- Estimate *a norm* for $\psi^{(N)}$ by induction and prove convergence (Gronwald) in F

$$\psi(t) = \sum_{N=0}^{\infty} \psi^{(N)}(t)$$

Existence of a Mild solution

- If $\psi_0 \in L^2(D)$, convergence in $L^2(D)$ (Dawson-Papnicolaou)
- If $\psi_0 \in H_Q^*$, convergence in H_Q^*
- H_Q^* contains the Dirac functions
  so we have the Green function as element of H_Q^*

Malliavin Calculus

- Malliavian derivative of random variable $\xi \in L^2(\Omega; F)$ is an element of

$$L^2([0,T] \times \Omega; \mathcal{L}_{\rm HS}(H,F))$$

- $\{D_t\xi\}_{t\in[0,T]}$ is valued in $\mathcal{L}_{\rm HS}(H,F)$ and in fact

$$\mathbb{E}\left\{\int_0^T \|D_t\xi\|^2_{\mathcal{L}_{\rm HS}(H,F)} dt\right\} < +\infty$$

- Notation $\mathbb{H}^1(F)$ for the Hilbert space of derivable variables

$$\|\xi\|^2_{\mathbb{H}^1(F)} = \mathbb{E}\{\|\xi\|^2_F\} + \mathbb{E}\left\{\left(\int_0^T \|D_t\xi\|^2_{\mathcal{L}_{\rm HS}(H,F)} dt\right)\right\}$$

- Integration by parts

$$\mathbb{E}\left\{\int_0^T \langle D_t\xi, \beta_t\rangle_H dt\right\} = \mathbb{E}\left\{\xi \int_0^T \langle \beta_t^*, dW_t\rangle_H\right\}$$

Malliavin Derivation of the Solution

Set:

$$\psi_N(t) = \sum_{n=0}^{N} \psi^{(n)}(t)$$

Then

$$\psi_{N+1}(t) = U(t)\psi_0 - \frac{i}{2}\int_0^t U(t-s)M(\psi_N(s))dW(s)$$

with the notation $M(f)$ for the operator of multiplication by the function f

Formally

$$D_s\psi_{N+1}(t) = \frac{i}{2}U(t)M(\psi_N(s)) - \frac{i}{2}\int_0^t U(t-s)\tilde{M}(\psi_N(s))dW(s), \quad 0 \le s \le t$$

Existence of Malliavin derivatives

- If $\psi_0 \in L^2(D)$, $\psi(t) \in \mathbb{H}^1(L^2(D))$
- If $\psi_0 \in H_Q^*$, $\psi(t) \in \mathbb{H}^1(H_Q^*)$

Black Magic

- $\mathbf{1}_D \in L^2(D)$ and $\mathbf{1}_D \in H = H_Q$ (D bounded)
- Set $F = L^2(D)$ when $\psi_0 \in L^2(D)$ and $F = H_Q^*$ when $\psi_0 \in H_Q^*$
- $D_s\psi(t) \in \mathcal{L}_{\mathrm{HS}}(H, F)$

$$D_s\psi(t)\mathbf{1}_D = -\frac{i}{2}\psi(t)$$

Nothing magic

SPDE is linear so something like this HAS TO HOLD !

First Example of Sensitivity Computation

Choose ϕ_m smooth approximation of ϕ_0 ($\lim_{m\to\infty} \phi_m = \phi_0$)

$$\frac{d}{d\gamma}\mathbb{E}\{\psi_m^B(\mathbf{y})\} = \int_{\mathbf{x}\in A} \mathbb{E}\{G(T,\mathbf{y};\mathbf{x})\overline{\phi_m'(\psi(T,\mathbf{x}))\frac{d\psi(T,\mathbf{x})}{d\gamma}}d\mathbf{x}\}$$

We use the Jacobian flow

$$\frac{d\psi(T)}{d\gamma} = Y_{0,T}\frac{d\psi_0}{d\gamma},$$

where $Y_{s,t}$ is the strong random operator solution of

$$Y_{s,t} = -\frac{i}{2}U(t-s) - \frac{i}{2}\int_s^t U(t-v)D_s\psi(v)W(dv).$$

in the sense of Skorohod. More notation:

$$\beta(T,\mathbf{x}) = \begin{cases} 2i\psi(T,\mathbf{x})^{-1}\frac{d\psi(T,\mathbf{x})}{d\gamma}, & \text{if } \psi(T,\mathbf{x}) \neq 0 \\ 0, & \text{if } \psi(T,\mathbf{x}) = 0 \end{cases}$$

First Example of Sensitivity Computation

Fixing $\mathbf{x} \in A$ and $\mathbf{y} \in G$, we have

$$
\begin{aligned}
&\mathbb{E}\{G(T,\mathbf{y};\mathbf{x})\overline{\frac{d\phi_m(\psi(T,\mathbf{x}))}{d\gamma}}\} \\
&= \mathbb{E}\{G(T,\mathbf{y};\mathbf{x})\overline{\phi'_m(\psi(T,\mathbf{x}))\frac{d\psi(T,\mathbf{x})}{d\gamma}}\} \\
&= \mathbb{E}\{\frac{1}{T}\int_0^T \overline{\phi'_m(\psi(T,\mathbf{x}))[D_t\psi(T)1_G](\mathbf{x})}G(T,\mathbf{y};\mathbf{x})\overline{\beta(T,\mathbf{x})}dt\} \\
&= \mathbb{E}\{\frac{1}{T}\int_0^T \overline{D_t\phi_m(\psi(T))G(T,\mathbf{y};\mathbf{x})}\beta(T,\mathbf{x})1_G(\mathbf{x})dt\} \\
&= \mathbb{E}\left\{\frac{\overline{\phi(\psi(T,\mathbf{x}))}}{T}\int_0^T G(T,\mathbf{y};\mathbf{x})\overline{\beta(T,\mathbf{x})}1_G^* W(dt)\right\}.
\end{aligned}
$$

First Example of Sensitivity Computation

Therefore, taking the limit $m \to \infty$

$$
\begin{aligned}
\frac{d}{d\gamma}\mathbb{E}\{\psi^B(\mathbf{y})\} &= \mathbb{E}\left\{\int_{\mathbf{x}\in A} \overline{\phi(\psi(T,\mathbf{x}))}(G(T,\mathbf{y};\mathbf{x})d\mathbf{x}\right\} \\
&= \lim_{m\to\infty} \frac{d}{d\gamma}\mathbb{E}[\psi_m^B(\mathbf{y})] \\
&= \mathbb{E}\left\{\int_{\mathbf{x}\in A} \overline{\phi(\psi(T,\mathbf{x}))}\frac{1}{T}\int_0^T (G(T,\mathbf{y};\mathbf{x})\overline{\beta(T,\mathbf{x})}1_G)^* W(dt)d\mathbf{x}\right\} \\
&= \mathbb{E}\left\{\int_{\mathbf{x}\in A} \overline{\phi(\psi(T,\mathbf{x}))}\text{KERNEL}((T,\mathbf{y};\mathbf{x})d\mathbf{x}\right\}.
\end{aligned}
$$

Here comes Galerkin

C'est Fini, MERCI