A Level Set Approach for Inverse Problems and Optimal Design Problems

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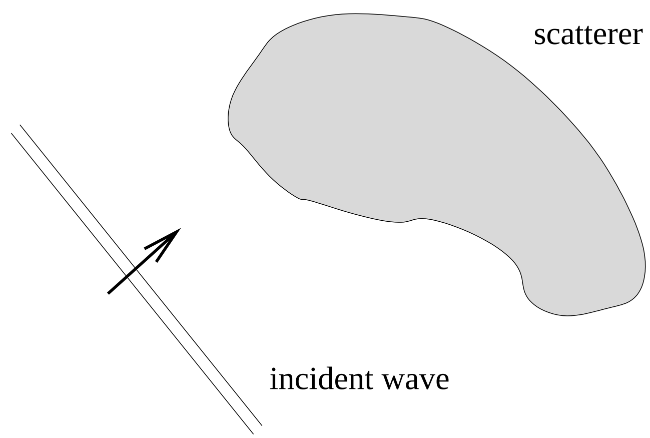
Bruno Luong, Schlumberger, Houston
Stanley Osher, UCLA

Context

Many inverse problem involve the determination of an unknown geometry from observed data.

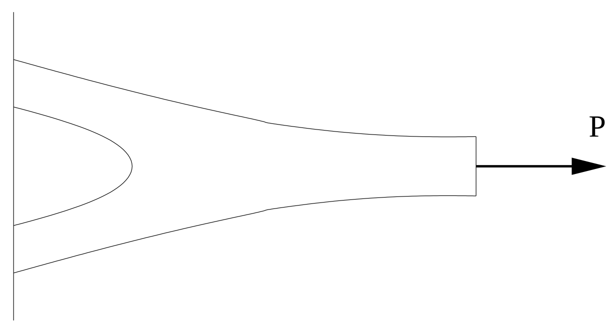
Examples: Inverse scattering, nondestructive testing

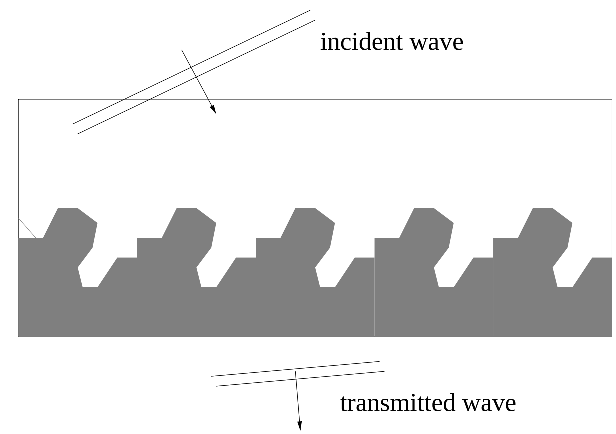


The relation between unknown and data is given by a partial differential equation. We can pose a least-squares problem to minimize data fit.

Many optimal design problem involves finding a geometry that maximizes an objective.

Examples: structures, diffractive optics.





The physical phenomenon is modeled by a partial differential equation; the objective is found from the solution of the PDE. We pose an optimization problem associated with the objective.

What is level set?

Let S be a region in 2D. If S is star-shaped, we can represent it in polar coordinates

$$r = f(\theta).$$

If S is more general, we can put marker points on the boundary ∂D

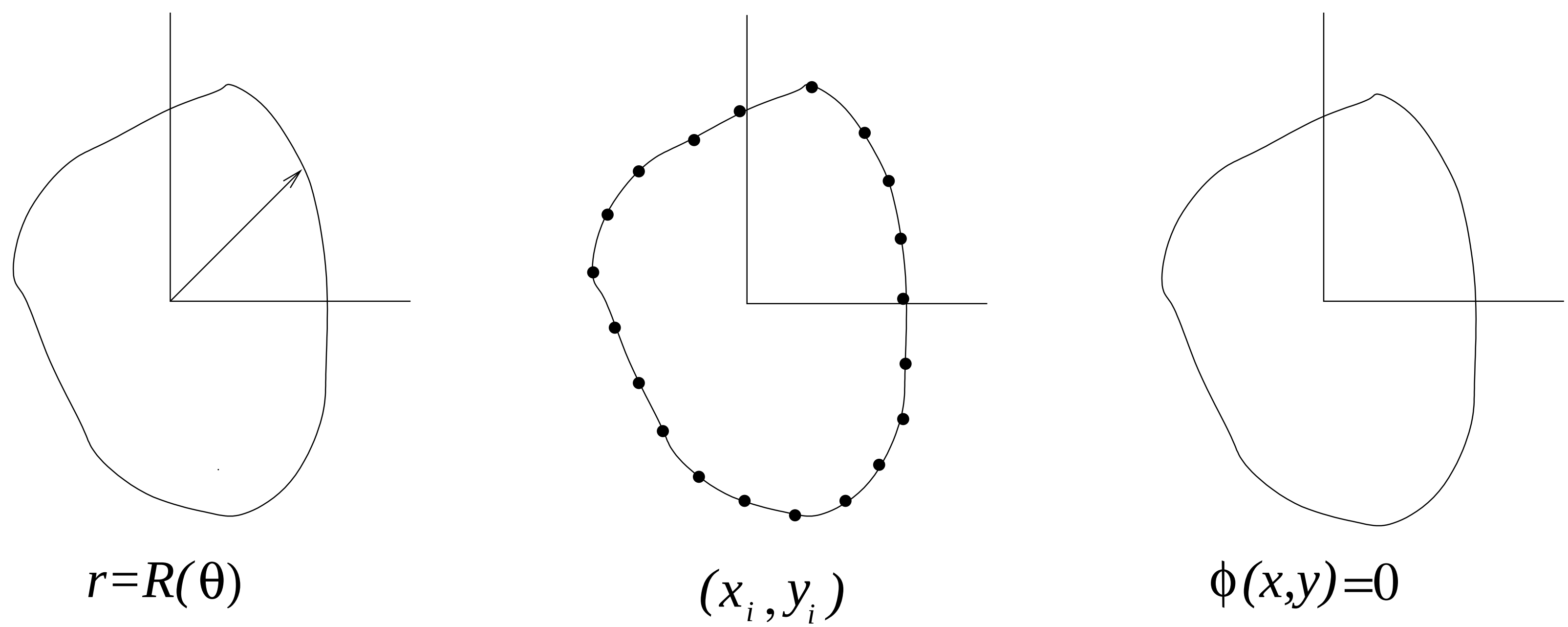
$$(x_i, y_i) \in \partial S,$$

and assume the points are connected by some spline.

In level set representation, we consider a function $\phi(x, y)$ so that

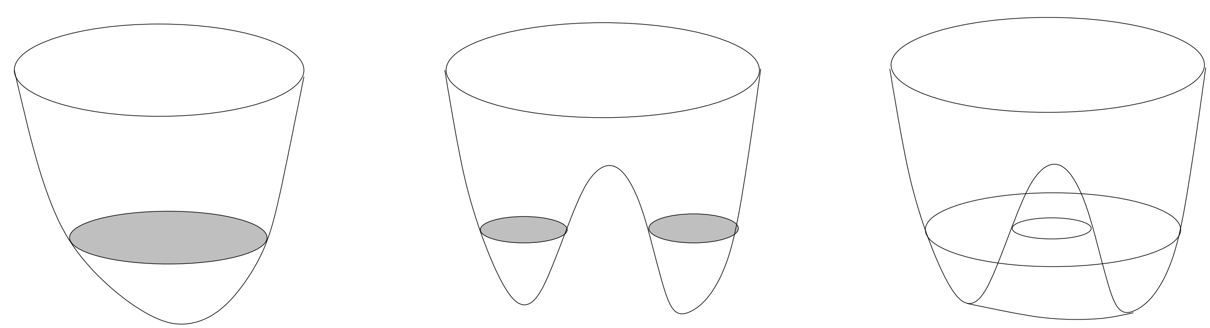
$$S = \{(x, y) \ : \ \phi(x, y) < 0\}.$$

Parameterizations of a curve ∂S in polar, ‘marker points’, and level set.



Why level set?

When we solve an inverse or design problem, we may start with an initial shape and iterate towards solution. However, depending on our starting shape, we may run into a situation where the shape changes topology. Star shape description may be too restrictive, and marker may require intervention to merge or break domains.



Level set method easily accommodates topology of the unknown.

Formulation of inverse problems

We parameterize the unknown domain S by the level set function $\phi(x)$

$$S = \{x : \phi(x) < 0\}.$$

In the inverse problem, we denote by $A(\phi)$ the mapping from unknown to data. We are given measured data g, therefore the inverse problem is to find ϕ in the equation

$$A(\phi) = g.$$

We choose a least-squares formulation

$$\min_{\phi} \|A(\phi) - g\|^2 .$$

Formulation of optimal design problems

In a design problem, the unknown is a geometry S, represented in the level set formulation as

$$S = \{x : \phi(x) < 0\}.$$

Associated with this geometry, we have an objective, and a constraint (or many constraints)

$$F(\phi) \quad \text{and} \quad G(\phi) = 0.$$

We wish to solve

$$\min_{\phi} F(\phi) \quad \text{subject to} \;\; G(\phi) = 0.$$

Gradient calculation

In any gradient based descent algorithm for optimization, we will need gradients of the functions involved. This is especially tricky with problems involving geometry because we need to differentiate with respect to geometry parameters.

Consider a function ρ whose value is defined through the domain S

$$\rho(x) = \begin{cases} \rho_1 & \text{for} \quad x \notin S \\ \rho_2 & \text{for} \quad x \in S \end{cases} .$$

In level set formulation

$$\rho(x) = \begin{cases} \rho_1 & \text{for} \quad \{x : \phi(x) < 0\} \\ \rho_2 & \text{for} \quad \{x : \phi(x) > 0\} \end{cases} .$$

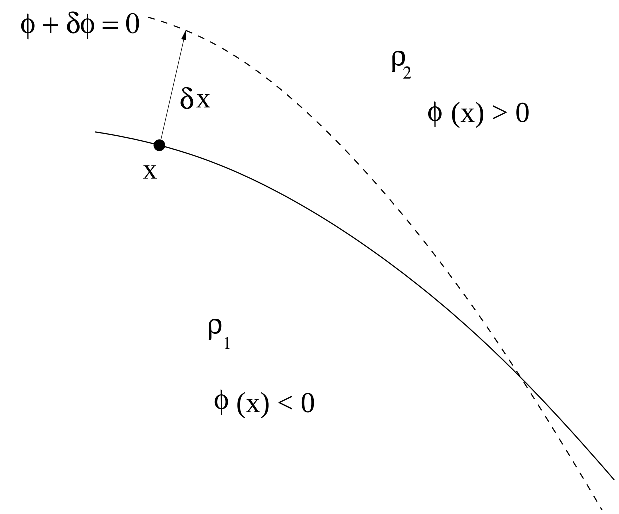
Let the function be

$$F(\rho(\phi)).$$

We use chain rule

$$D_\phi F = D_\rho F \;\; D_\phi \, \rho.$$

The first term is "easy", we know how to take variation with respect to a function for a fixed geometry. The second term requires a little work.



The variation $\delta\rho$ integrated against a test function $f(x)$

$$
\begin{aligned}
< \delta\rho, f > \quad &:= \quad \int_{\Omega} \delta\rho(x)\, f(x)\, dx \\
&= \quad \int_{\text{symdiff}(S,S')} \delta\rho(x)\, f(x)\, dx.
\end{aligned}
$$

Outward normal $n(x) = \nabla\phi / |\nabla\phi|$.

Also,

$$\delta\rho(x) = \begin{cases} +(\rho_2 - \rho_1) & \delta x \cdot n(x) < 0 \\ -(\rho_2 - \rho_1) & \delta x \cdot n(x) > 0 \end{cases} .$$

Therefore,

$$< \delta\rho, f >= -\int_{\partial S} (\rho_2 - \rho_1)\, \delta x \cdot n(x)\, f(x)\, ds(x).$$

Identify $\delta\rho$

$$\delta\rho = -\,(\rho_2 - \rho_1)\, \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \cdot \delta x \Big|_{x \in \partial S} .$$

From $\phi(x) = 0$, we can take a variation

$$\delta\phi + \nabla\phi \cdot \delta x = 0.$$

The previous formula now becomes

$$\delta\rho = D_\phi\ \rho \cdot \delta\phi = \ (\rho_2 - \rho_1)\, \frac{\delta\phi}{|\nabla\phi|}\bigg|_{x\in\partial D}\ .$$

This relationship decribes the the variation in the function $\rho(x)$ when you make a variation on $\phi(x)$.

So, for $F(\rho(\phi))$, we can now calculate the directional derivative

$$\begin{aligned} D_\phi F \cdot \delta\phi &= D_\rho\, F \cdot D_\phi\, \rho \cdot \delta\phi \\ &= D_\rho\, F \cdot \delta\rho. \end{aligned}$$

Putting it together, we have

$$D_\phi F \cdot \delta\phi = (\rho_2 - \rho_1) D_\rho F \cdot \left. \frac{\delta\phi}{|\nabla\phi|} \right|_{x \in \partial S}.$$

Because $D_\rho F$ is a linear operator, it must have a representation

$$D_\rho F \cdot \left. \frac{\delta\phi}{|\nabla\phi|} \right|_{x \in \partial S} = \int_{\partial S} K(x)\, \frac{\delta\phi}{|\nabla\phi|}\, ds(x).$$

Hence, the steepest descent direction associated with F is

$$\delta\phi(x) = -(\rho_2 - \rho_1)\ K(x)\ |\nabla\phi(x)| \quad \text{for} \;\; x \in \partial S.$$

This states that steepest descent direction is specified by *any* $\delta\phi(x)$ whose value on the boundary satisfy above. Since $K(x)$ is defined for all of x, we can extend the definition to all of x so that

$$\delta\phi(x) = -(\rho_2 - \rho_1)\ K(x)\ |\nabla\phi(x)| \quad \text{for all} \;\; x.$$

In the case where there is a constraint

$$G(\phi) = 0.$$

We modify the gradient with Lagrange multiplier ν (to be determined)

$$\delta\phi = -(\rho_2 - \rho_1)\ K(x)\ |\nabla\phi| + \nu L(x)\ |\nabla\phi|$$

where

$$D_\phi G(\phi) \cdot \delta\phi = \int_{\partial S} L(x) \frac{\delta\phi}{|\nabla\phi|} \, ds(x).$$

The multiplier can be chosen so that

$$G(\phi + \delta\phi) \approx 0.$$

We can perform similar calculation in the case of inverse problems since we can view

$$F(\phi) = \|A(\phi) - g\|^2 .$$

Other approaches:

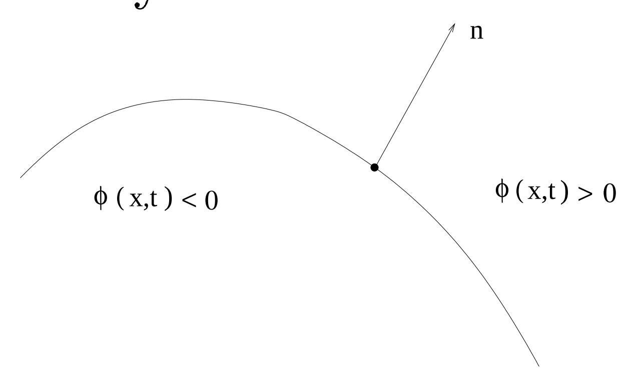
- Variational level set method (Chan, Merriman, Osher and Zhao)
- Differentiation of PDEs with respect to geometry (Zolésio); a more general procedure with well-developed theory

Curve evolution

We view level set function to be evolving, so $\phi(x,t)$. As $\phi(x,t)$ evolves, the zero level set $\{x : \phi(x,t) = 0\}$ moves. Its movement can be described by

$$\frac{\partial \phi}{\partial t} + v(x,t)|\nabla \phi| = 0,$$

where $v(x,t)$ is the velocity normal to the curve at x.



We need to find $v(x,t)$ that optimizes the objective and also respects the constraints. Extending $v(x,t)$ to the domain allows evolution of all level curves.

Recall that we have

$$\delta\phi(x) = -(\rho_2 - \rho_1)\, K(x)\, |\nabla\phi(x)| \quad \text{for all } \; x.$$

We view this as a curve evolution rule with velocity

$$v(x,t) = -(\rho_2 - \rho_1)\, K(x).$$

Special care needs to be taken in computing $|\nabla\phi(x)|$ (Osher and Sethian).

Note recent work by Burger where evolution is given by

$$\delta\phi(x,t) = -V(x,t) \cdot \nabla\phi(x,t),$$

the form of V does not restrict motion to the normal to the curve.

Algorithm

$$\text{optimize} \;\; F(\phi)$$
$$\text{subject to} \;\; G(\phi) = 0$$

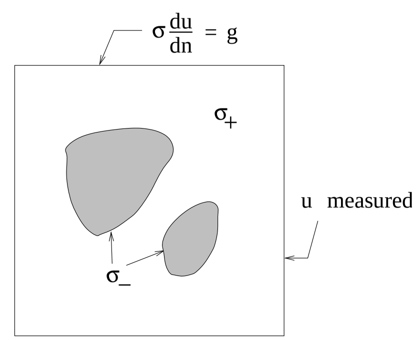
initial guess for $\phi(x)$

do while not optimal

- compute $D_\phi F(\phi)$ and $D_\phi G(\phi)$
- solve for Lagrange multiplier ν by linear approximation (when needed, solve for ν via Newton's method)
- get velocity $v(x)$
- update $\phi(x)$

Electrical impedance tomography

with Bruno Luong (circa 1997)



Let $u(x)$ be the electrical potential

$$\nabla \cdot \rho(x) \nabla u = 0 \quad \text{in} \;\; \Omega.$$

Assume that we measure $u|_{\partial\Omega}$ when current is applied

$$\nabla u \cdot n = f \quad \text{on} \;\; \partial\Omega.$$

We also know that

$$\rho(x) = \begin{cases} \rho_1 & \text{for} \quad x \notin S \\ \rho_2 & \text{for} \quad x \in S \end{cases} .$$

We do this experiment for two f's, (f_1, f_2) and measure the corresponding u on the boundary, denoted by (g_1, g_2).

The inverse problem is to find S from the measurements.

Here the mapping $A(\phi)$ takes S to (g_1, g_2). We solve the least squares problem

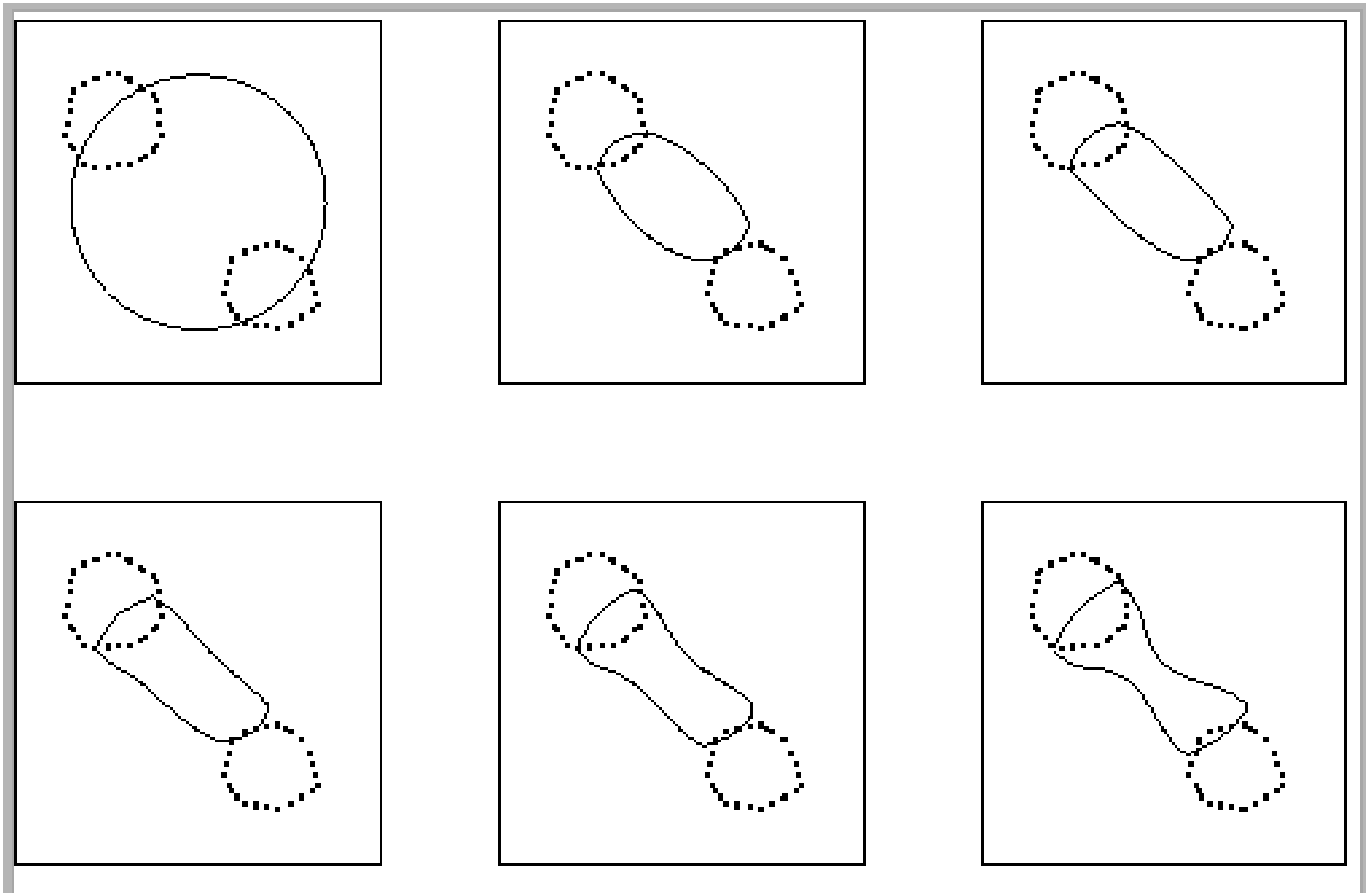
$$\min \|A(\phi) - [g_1, g_2]\|^2 .$$

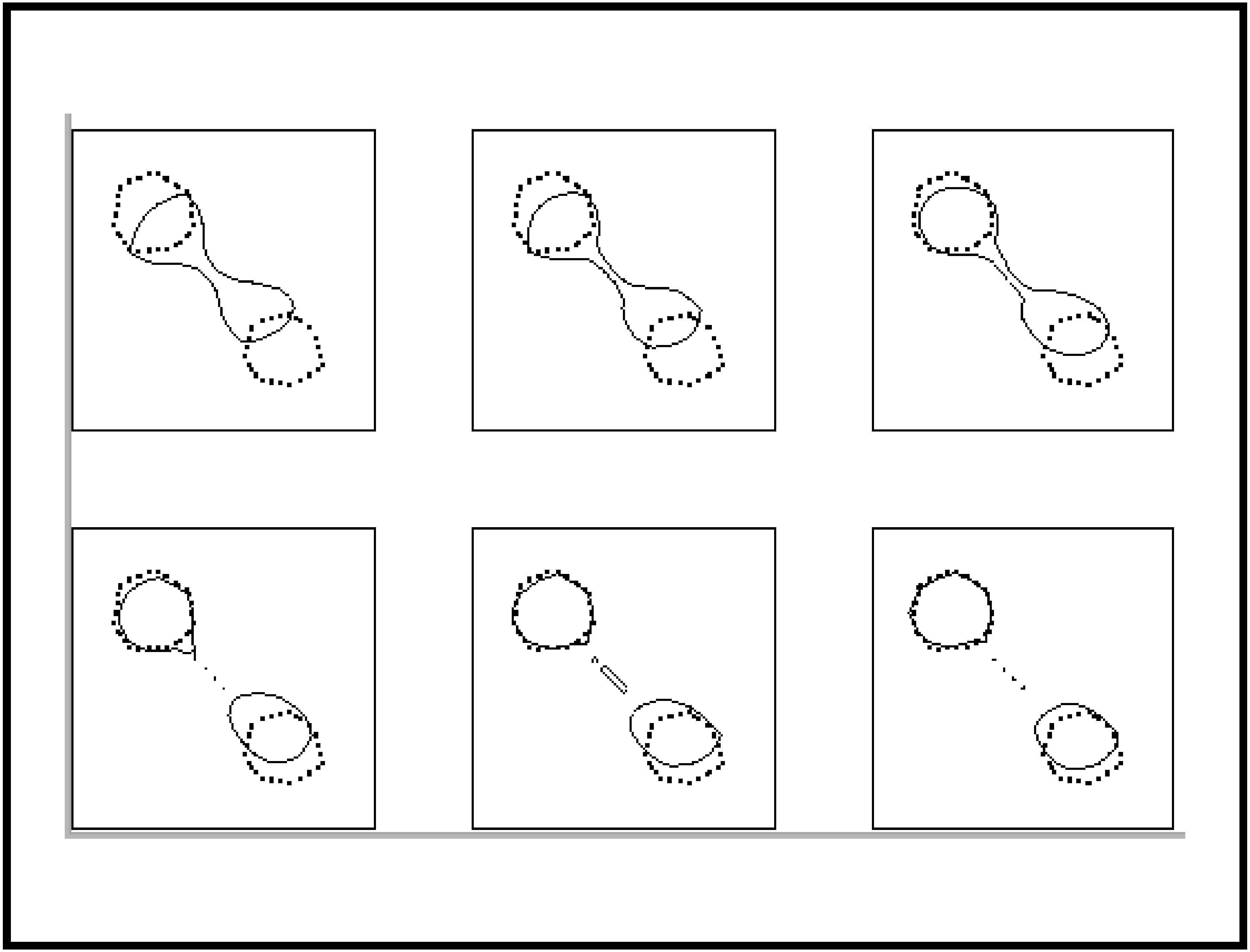
Computational details:

- Forward problem solved by Immersed Interface Method
- Gradient calculation by adjoint state method

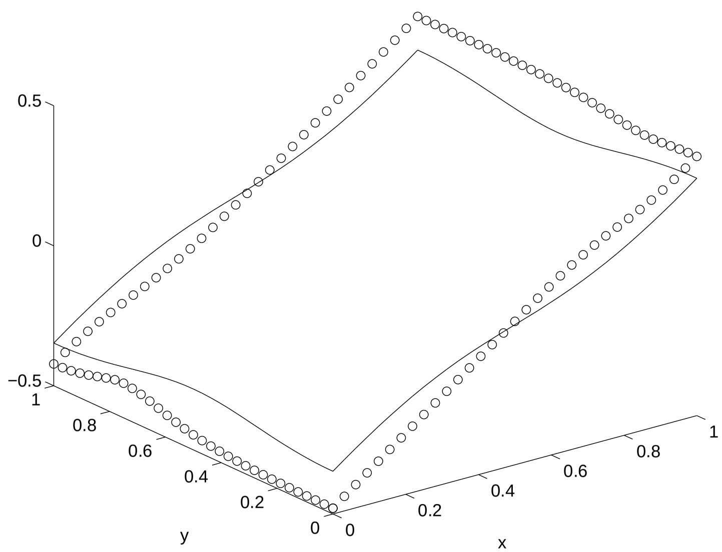
Convergence is *very* slow.

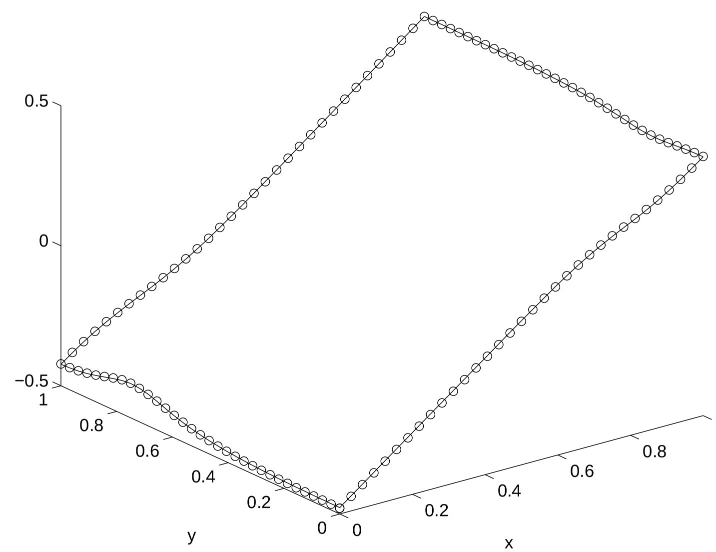
See also result by Ito, Kunisch, and Li.





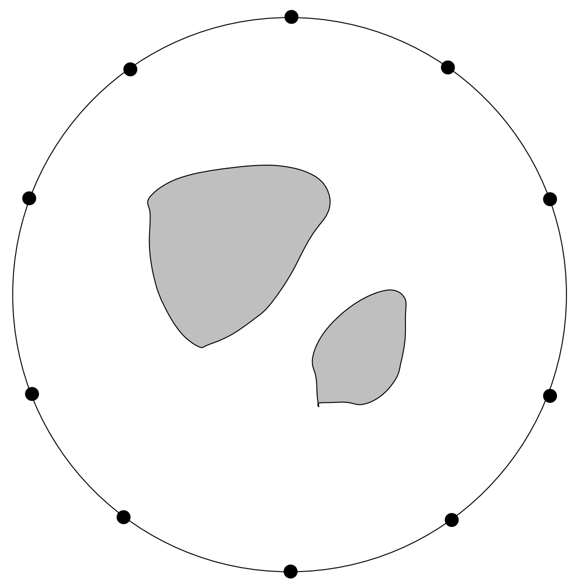
Data fit: initial vs. final





Reconstruction of sound speed anomaly

with Amelie Litman and Dominique Lesselier



Let $u(x)$ represent the amplitude in Helmholtz's equation

$$\triangle u + k^2 \rho(x)^2 u = 0.$$

We solve a point source problem and record $u(x_j)$ at all the 10 receiver points.

Again, we assume

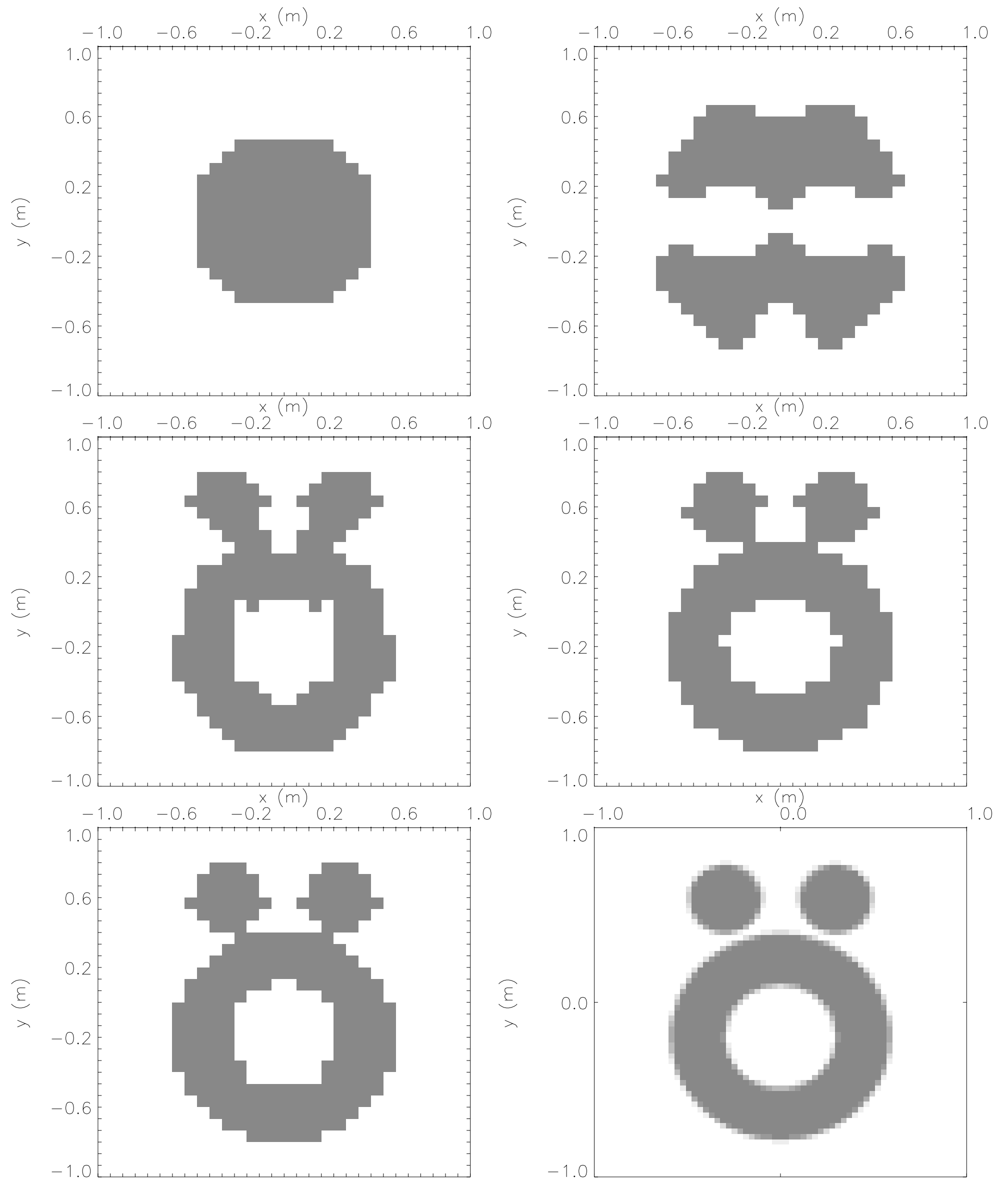
$$\rho(x) = \begin{cases} \rho_1 & \text{for} \quad x \notin S \\ \rho_2 & \text{for} \quad x \in S \end{cases} .$$

The inverse problem is to find S from the measured data.

We solve the forward problem using the Lippmann-Schwinger equation

$$u(x) = u_{\text{inc}}(x) - k^2(\rho_2^2 - \rho_1^2) \int_S G(x,y)u(y)\, dy.$$

Also recent results by Dorn, Miller and Rappaport.


x (m)
y (m)
−1.0
−0.6
−0.2
0.2
0.6
1.0
0.0

An eigenvalue optimization problem

with Stanley Osher

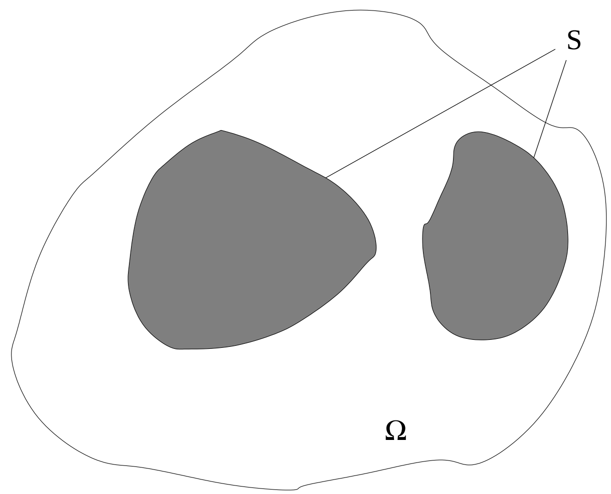
Let Ω be a domain in $\mathbb{R}^2$. Let $u(x)$ satisfy

$$-\triangle u = \lambda \rho(x) u \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

We assume that $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} \rho_1 & \text{for} \quad x \notin S \\ \rho_2 & \text{for} \quad x \in S \end{cases}.$$

Suppose that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \cdots$ for all S.



Problem 1.

$$\max \lambda_1 \quad \text{or} \quad \max \lambda_2 - \lambda_1$$

$$\text{subject to} \quad \|S\| = K.$$

or minimization.

Problem 2.

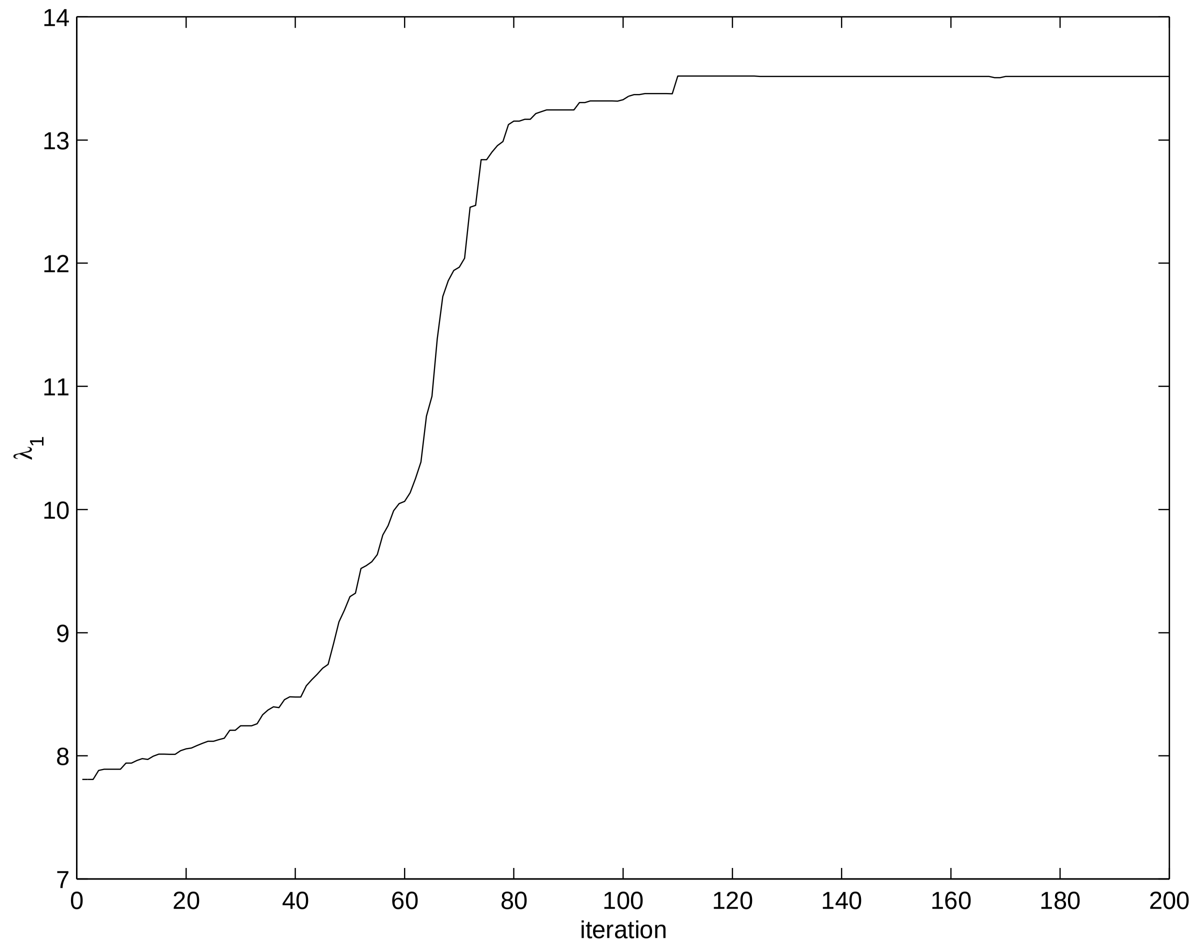
$$\min \|S\| \quad \text{subject to} \quad \lambda_2 - \lambda_1 = L$$

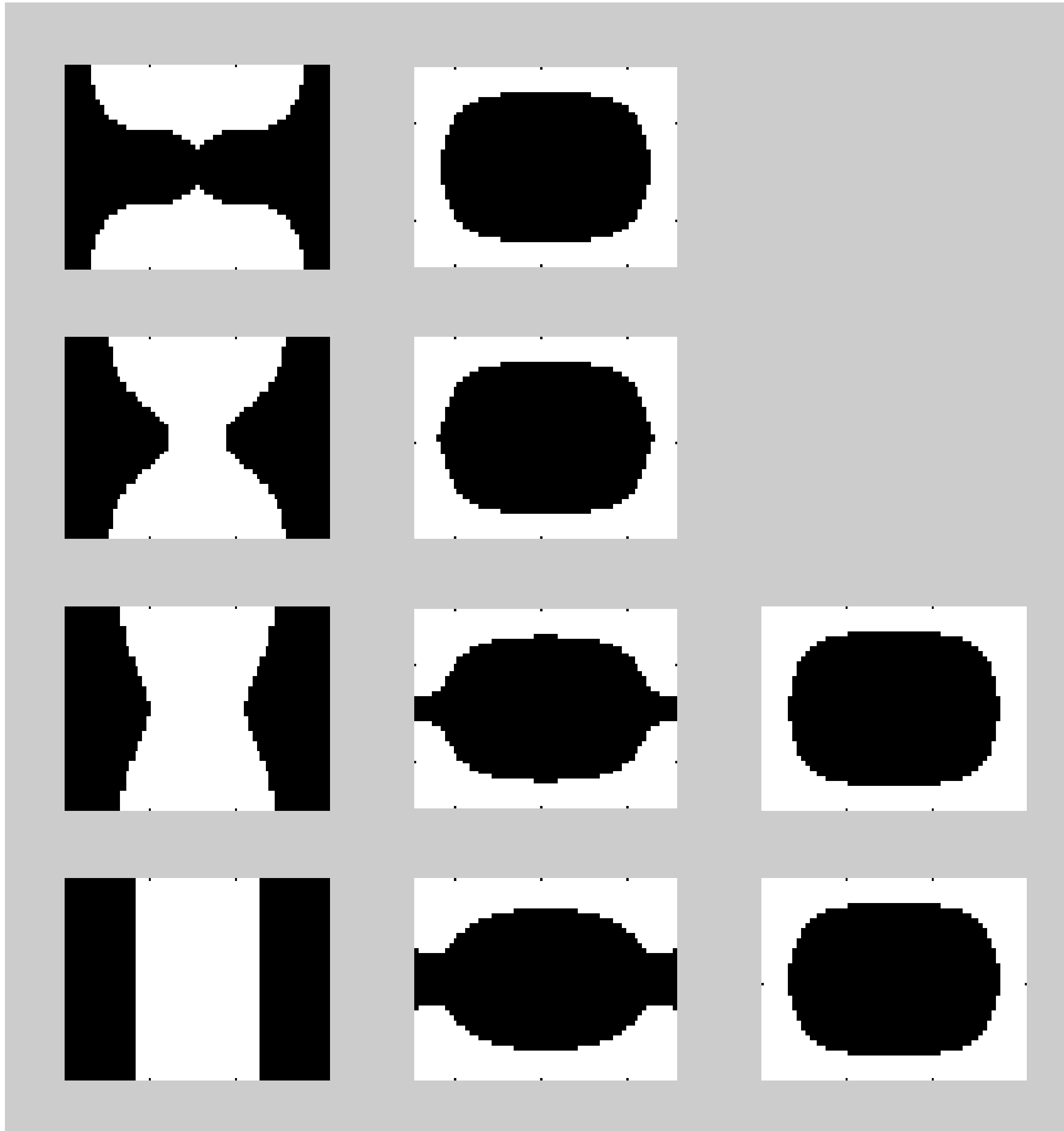
Linearized Lagrange multiplier update in Problem 1 has a nice interpretation. It states that to preserve area, the total velocity over the curve must be zero.

Curve evolution equation can be intepreted as the classical Feasible Arc Method with reduced gradient.

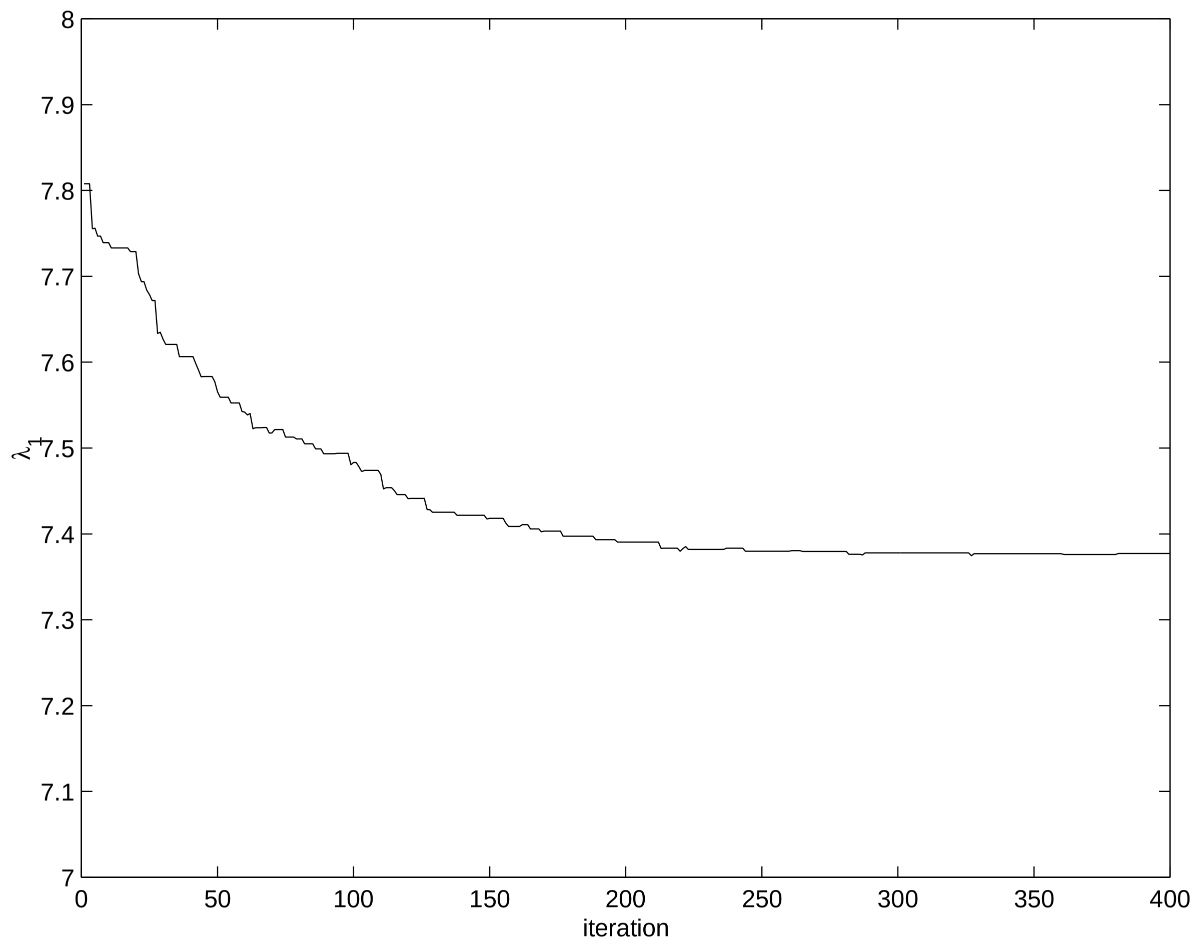
Sethian and Wiegmann considered classical structural optimization using level set; they did not use gradients to obtain velocity.

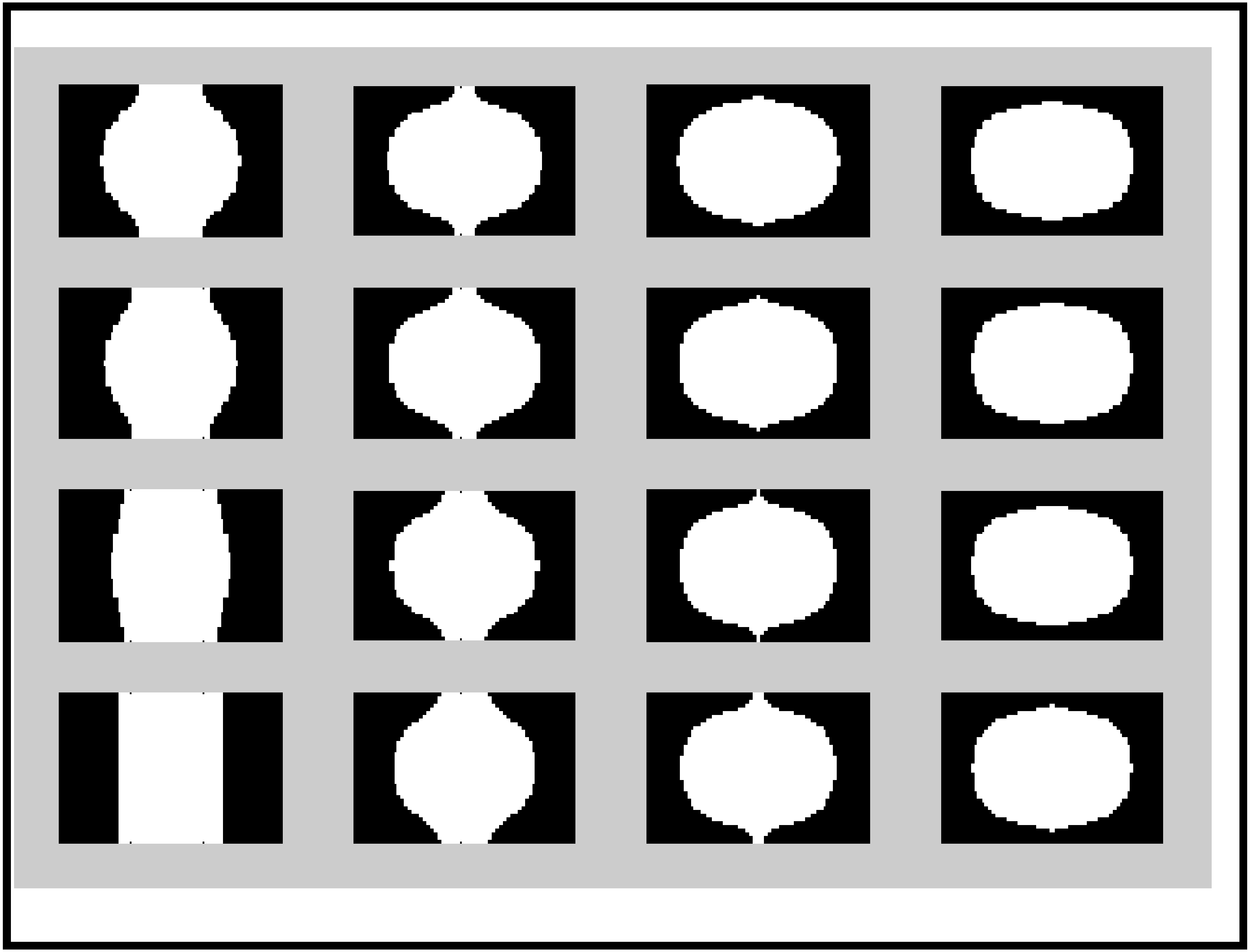
Example 1: Maximize λ_1



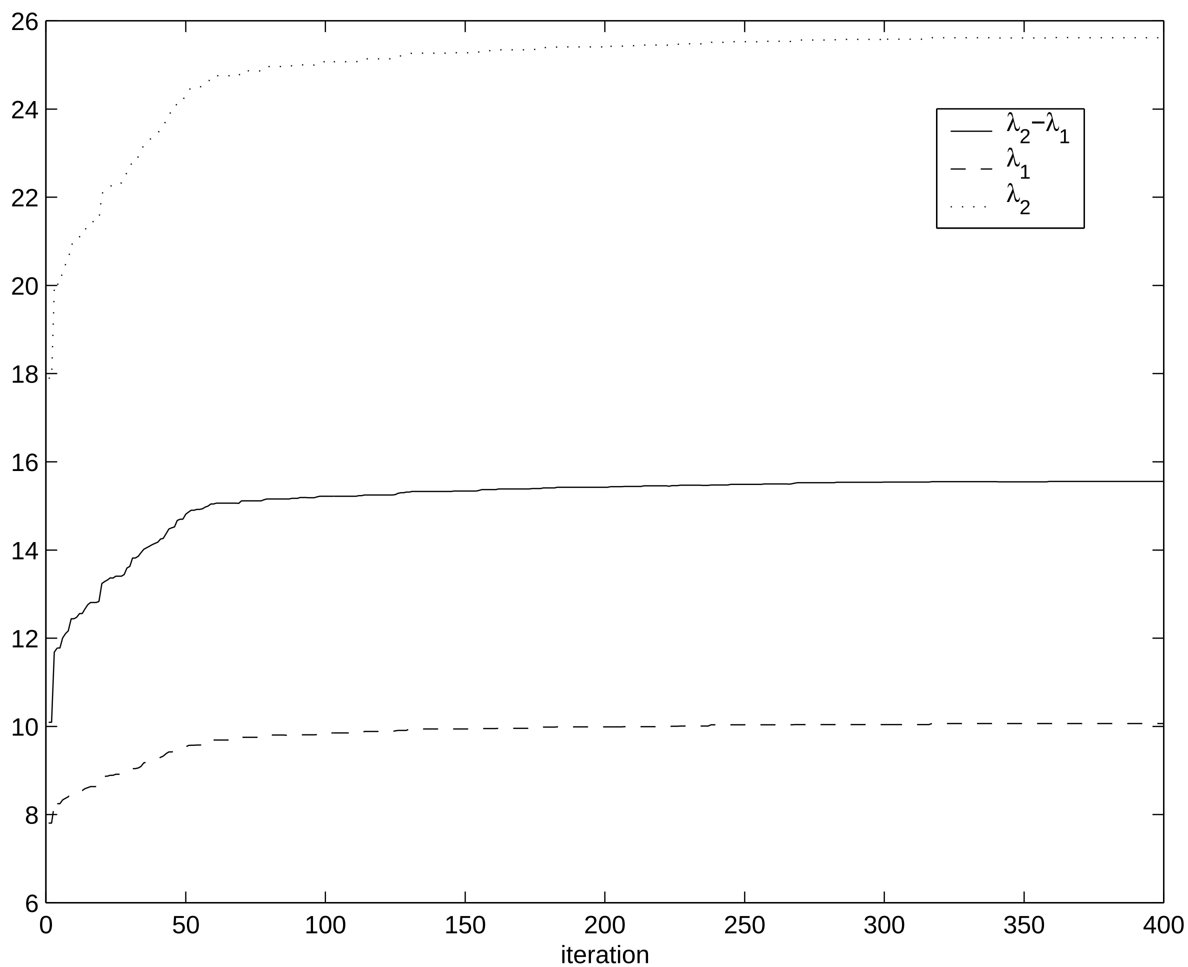


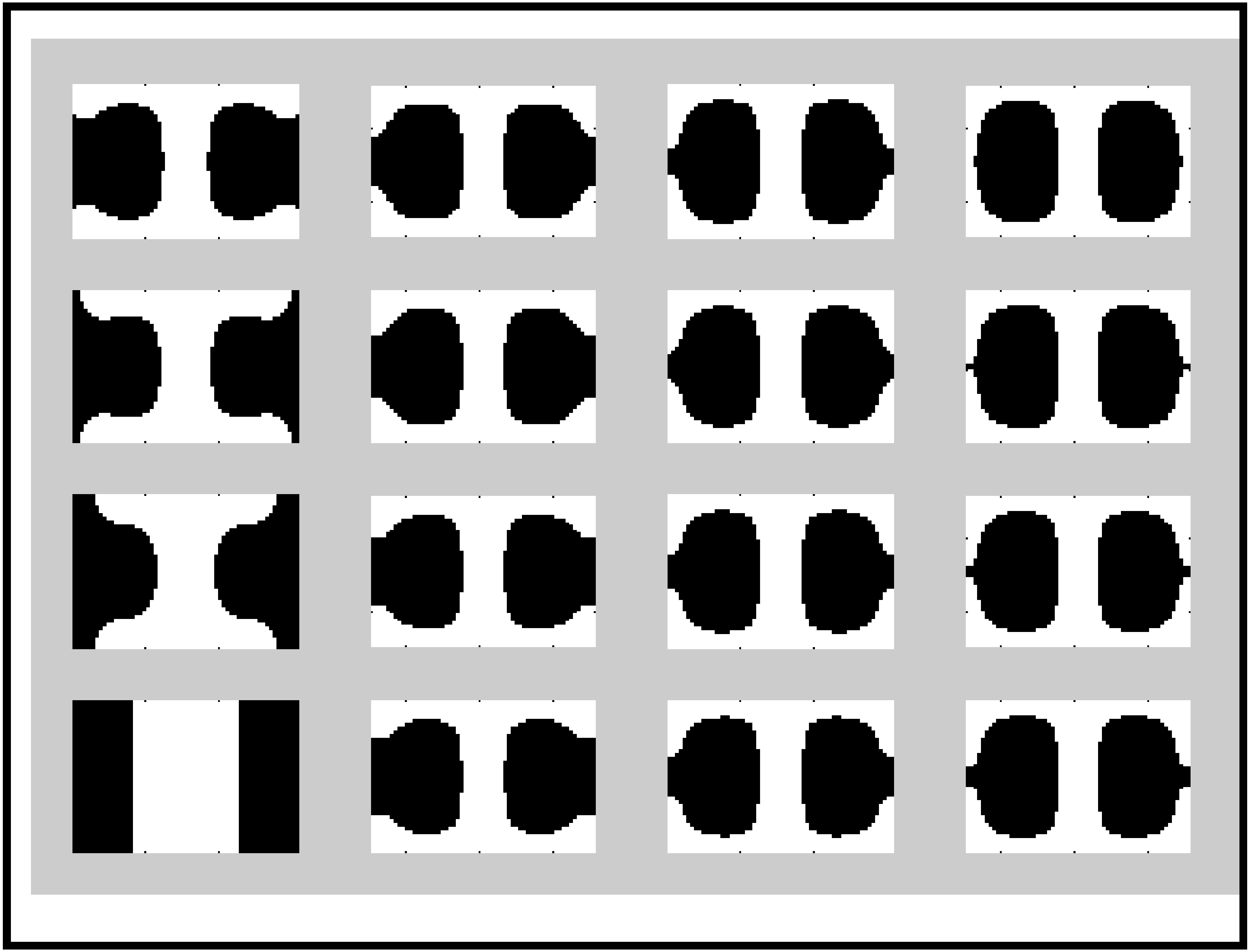
Example 2: Minimize λ_1



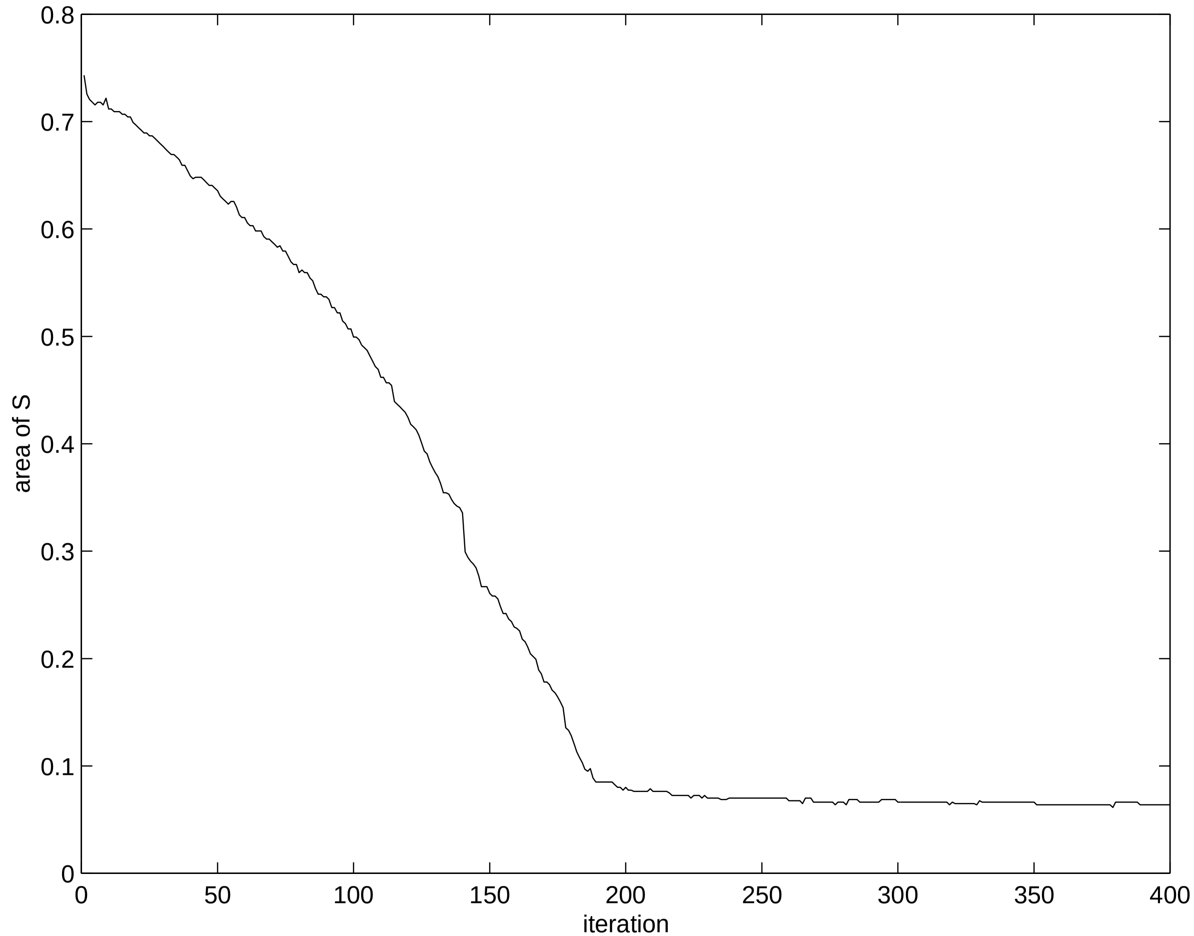


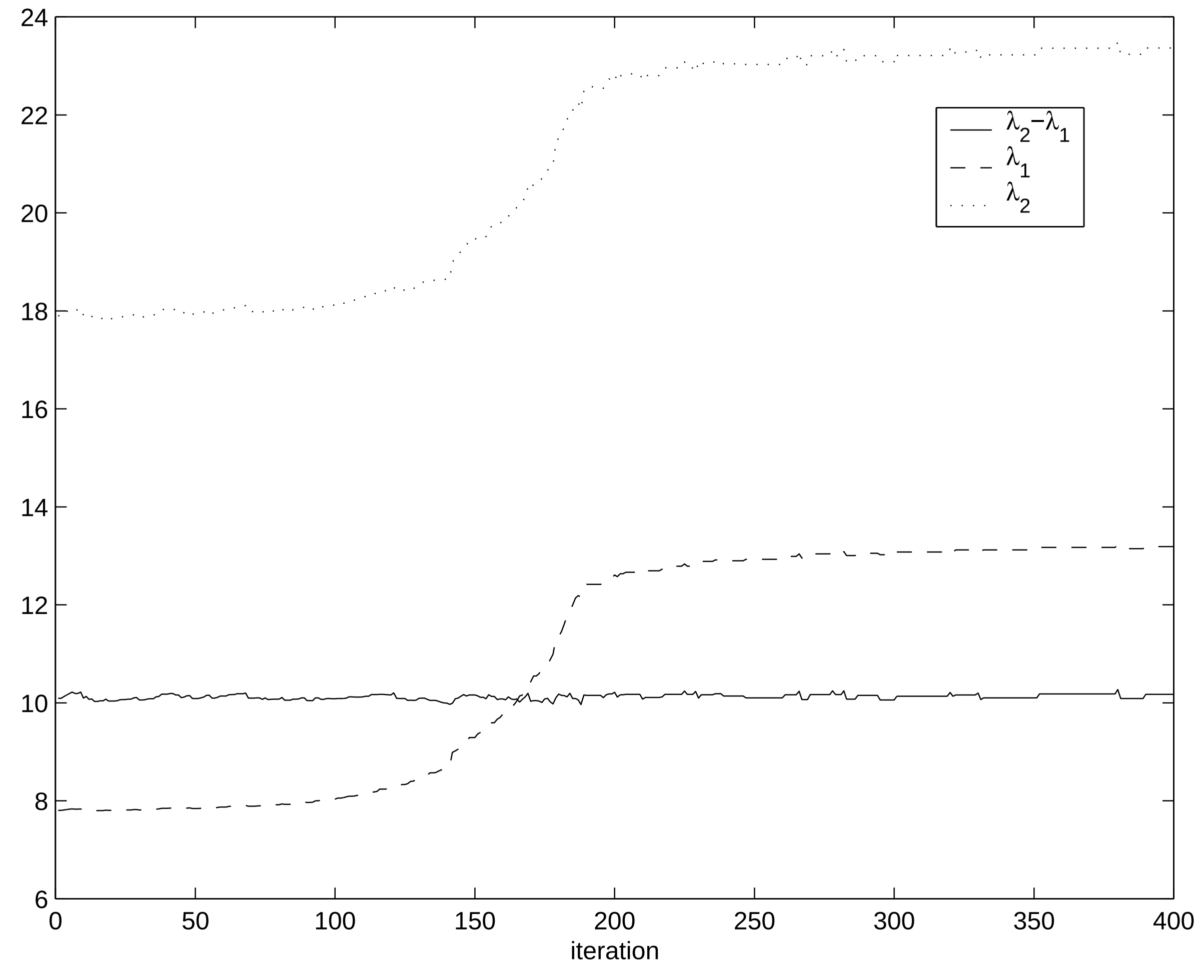
Example 3: Maximize $\lambda_2 - \lambda_1$





Example 4: Fix $\lambda_2 - \lambda_1$ and minimize area

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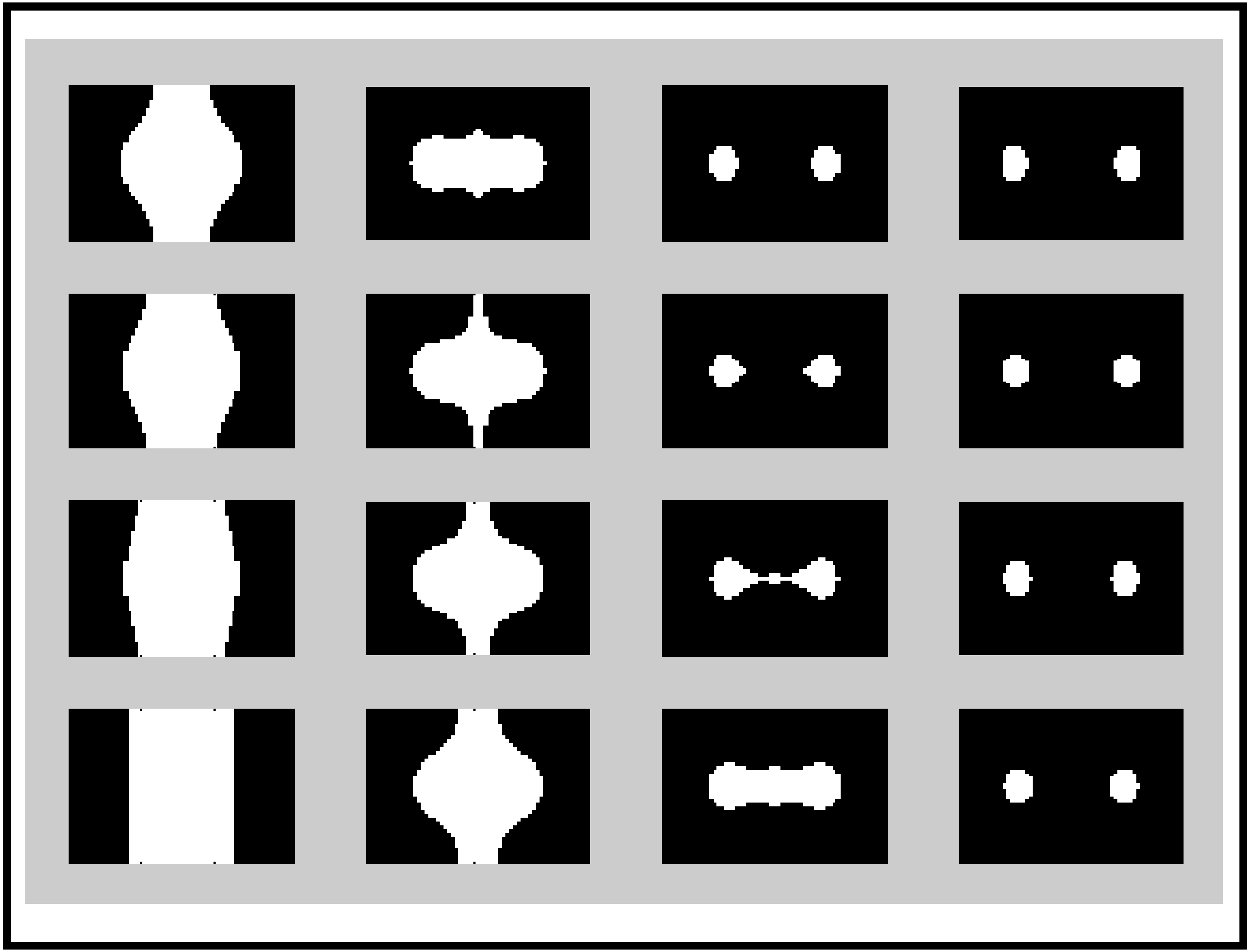
λ2−λ1
λ1
λ2



Example 5: Maximize spectral gap around a value

Pick a number c, find S such that the gap around c, namely

$$c - \lambda_m \quad \text{and} \quad \lambda_{m+1} - c$$

is maximized.

Ad hoc algorithm:

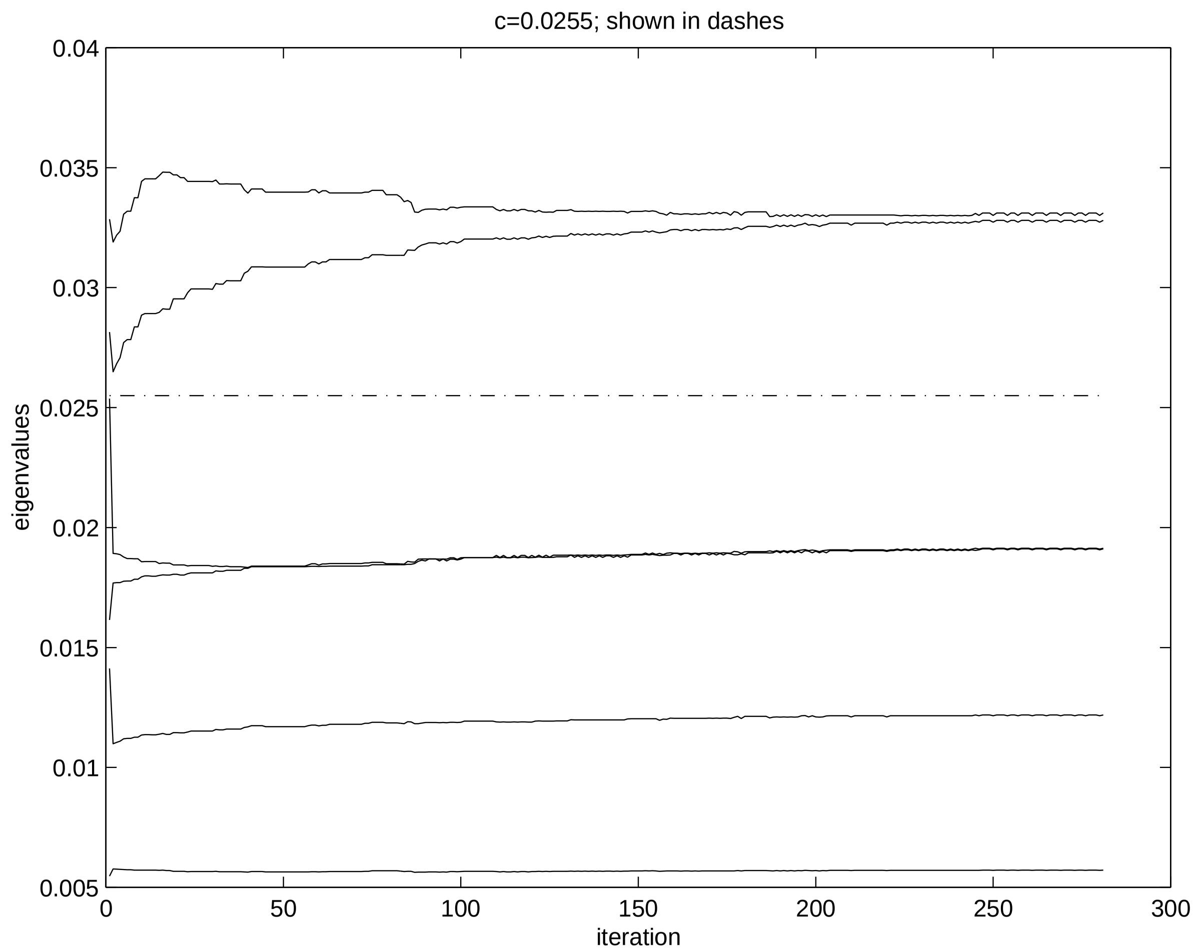
`initial guess for` $\phi(x)$

`do while not optimal`

- `solve eigenvalue problem to determine` m
- `take a reduced gradient step for objective`

$$\log(c - \lambda_m) + \log(\lambda_{m+1} - c)$$

- `take a Newton step towards constraint`


c=0.0255; shown in dashes
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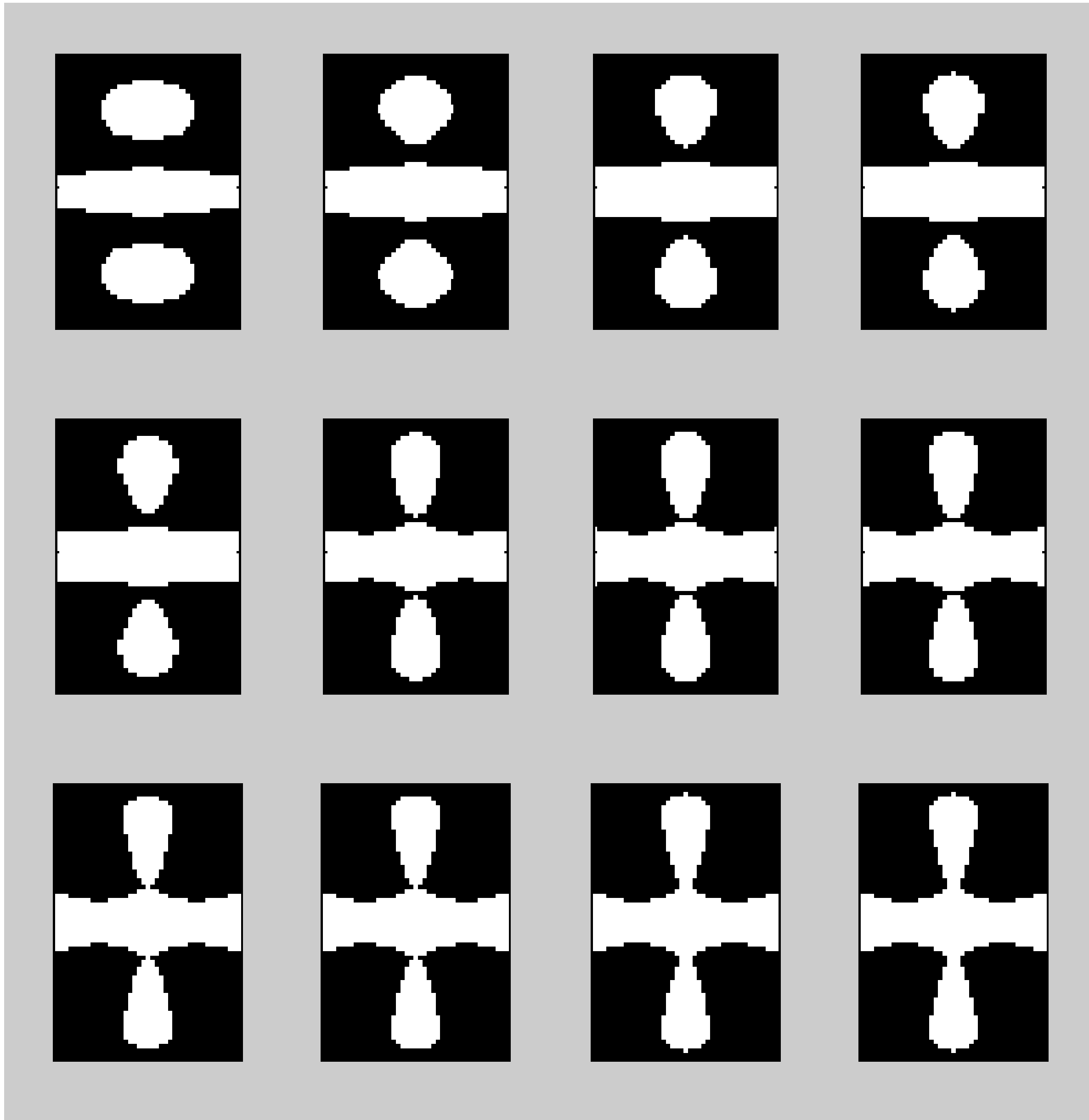
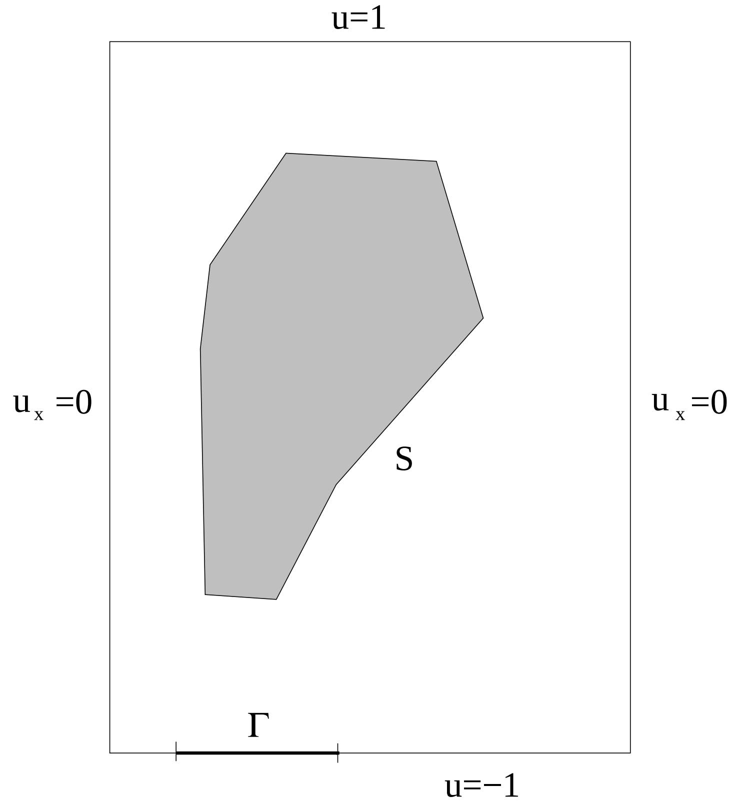
iteration



Optimal design in heat conduction

Joint work with L-T. Cheng (UCSD) and S. Osher (UCLA)



Let $u(x)$ be temperature,

$$\nabla \cdot \rho(x) \nabla u = 0 \quad \text{in} \Omega.$$

We specify mixed boundary conditions – Dirichlet on the top and bottom, flux-free on the sides.

Again, we give

$$\rho(x) = \begin{cases} \rho_1 & \text{for} \quad x \notin S \\ \rho_2 & \text{for} \quad x \in S \end{cases} .$$

Moreover, we constrain the size of S

$$\|S\| = K.$$

Optimal design problem is

$$\max_{\|S\|=K} \int_\Gamma \rho \nabla u \cdot n \, ds.$$

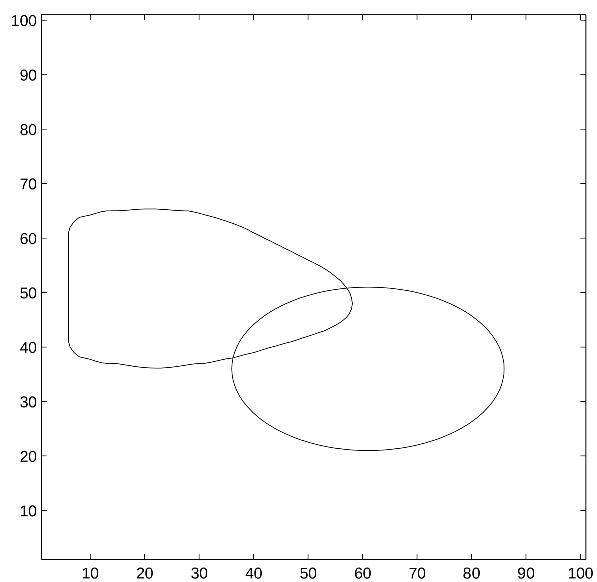
That is, find S such that the flux over Γ is maximized.

Note: A small change in the partial differential equation allows us to consider optimal design of bars with the maximum torsional rigidity. We can also modify the problem to consider structural optimization problem.

The relaxed solution to the problem is a composite. Our approach looks for a non-homogenized solution by penalizing the perimeter of S.

Computational details:

- Forward problem solved by ‘Ghost Fluid’ Method
- Gradient calculation by adjoint state method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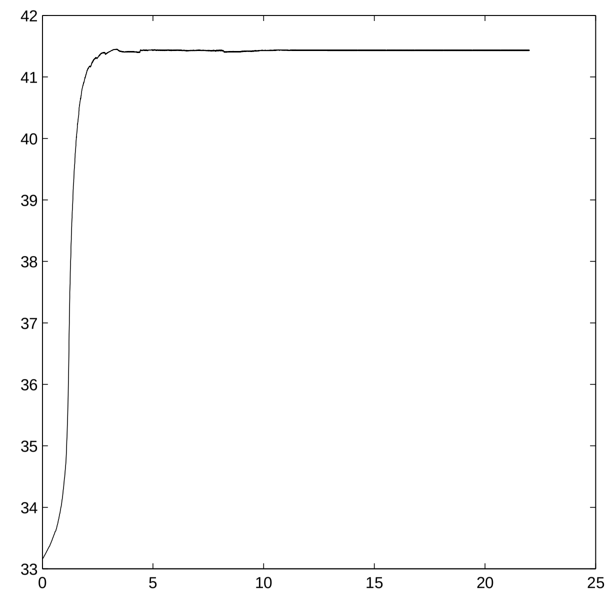
10 20 30 40 50 60 70 80 90 100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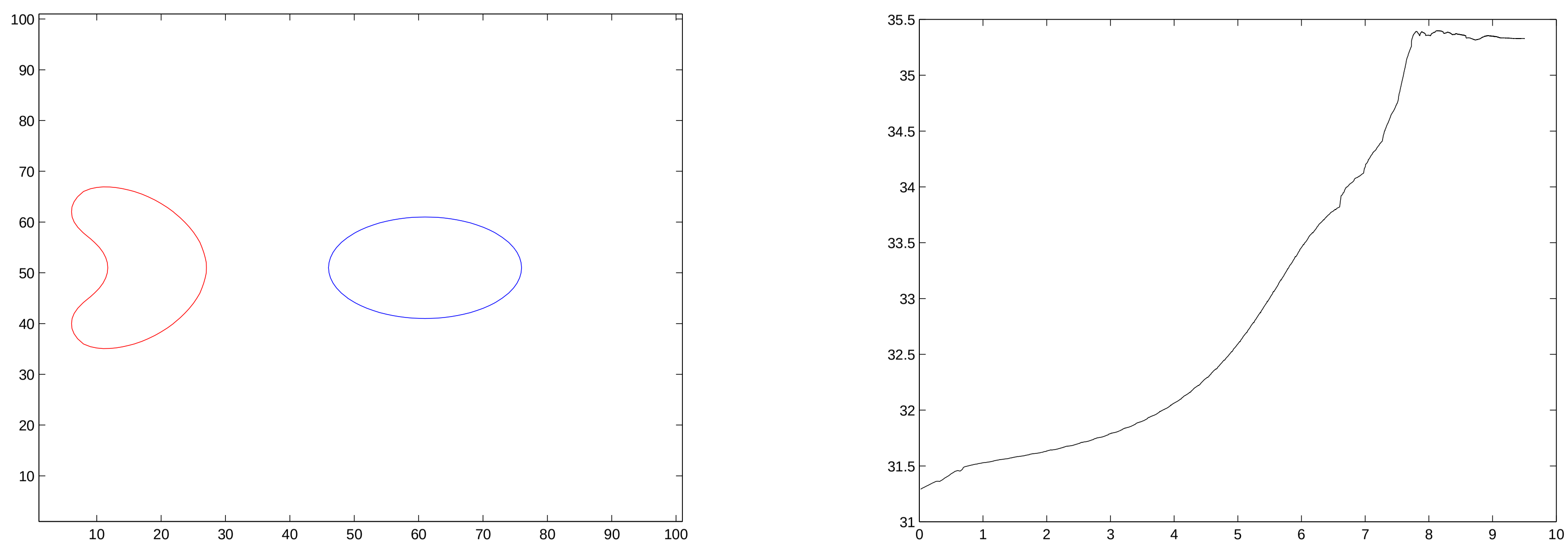
0 5 10 15 20 25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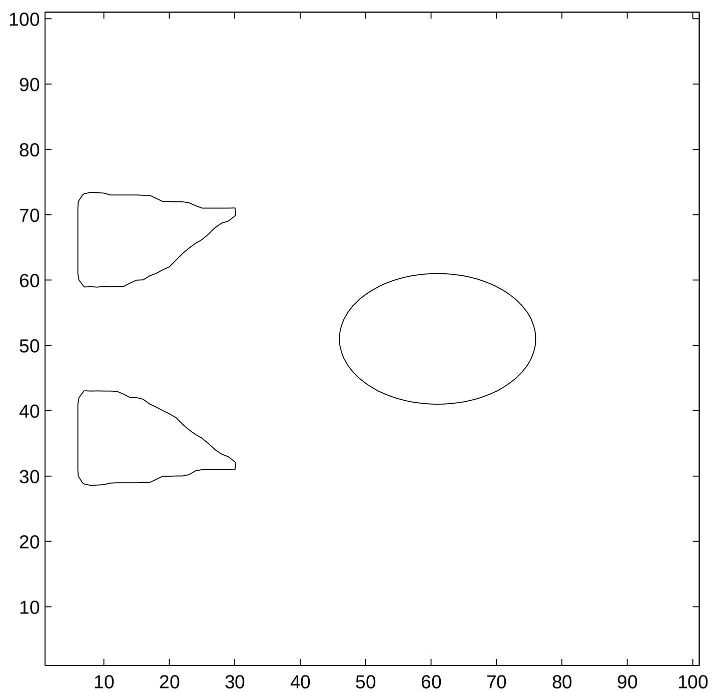
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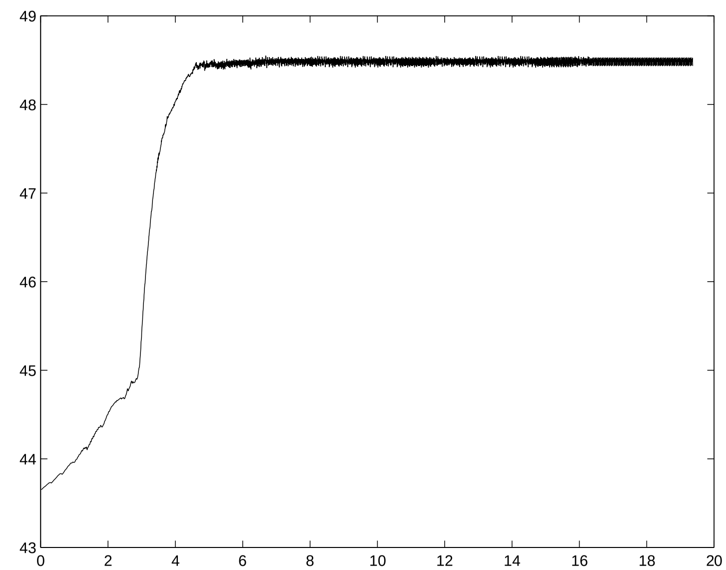
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Discussion

- A level set approach for solving inverse and design problem involving geometry.
- Geometrical and other constraints are handled by projection approach.
- Many improvements and refinement needed.
- Lots of theoretical questions need attention.
- Lots of application in design and inverse problems.
- Other geometrical constraints, such as lower bound on size of domain, distance between sets, are being implemented.