Inverse boundary spectral problems and gauge transformations

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Let $\Omega \subset \mathbf{R}^n$,

$u(x,t)$ satisfy a wave equation in $\Omega \times \mathbf{R}$

Inverse problem:

Can we determine the coefficients of the wave equation, i.e., physical model in Ω by observing

$$u(x,t) \text{ near } \partial\Omega \times \mathbf{R}$$

for all possible solutions $u(x,t)$?

Problems in formulation:

- We can change definition of x-coordinate: Let

  $$v(x,t) = u(\phi(x),t)$$

  where

  $$\phi : \Omega \to \Omega, \quad \phi|_{\partial\Omega} = id$$

- We can change scale of u-coordinate: Let

  $$w(x,t) = \kappa(x)u(x,t)$$

  where $\kappa(x) > 0$.

All functions u, v and w model the same physical process.

1. Invariant formulation

- Let us consider Ω as Riemannian manifold

  $d_g(x,y) =$ travel time between x and y.

- Let u satisfy the wave equation

$$u_{tt} + a(x,D)u = 0.$$

  Then the gauge transformation of u,

$$w(x,t) = \kappa(x)u(x,t)$$

  satisfy

$$w_{tt} + a_\kappa(x,D)w = 0,$$

  where

$$a_\kappa(x,D)w = \kappa a(x,D)\kappa^{-1}\,w$$

The gauge equivalence class of $a(x,D)$ is

$$[a(x,D)] = \{a_\kappa(x,D) \;:\quad \kappa > 0\}$$

2. Wave equation

Let us consider the wave equation

$$\begin{aligned} u_{tt}(x,t) + Au(x,t) &= 0, \quad \text{in} \quad M \times \mathbf{R}_+, \\ u|_{t=0} &= 0, \quad u_t|_{t=0} = 0, \\ u|_{\partial M \times \mathbf{R}^+} &= f \end{aligned}$$

where M is a n-dimensional manifold and

$$Au = -\sum_{j,k=1}^{n} a^{jk} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b^j \frac{\partial u}{\partial x^j} + cu,$$

where a^{jk}, b^j, c are real, smooth, $[a^{jk}(x)] > 0$. Assume that there is dV such that A is selfadjoint in $L^2(M, dV)$ with

$$\mathcal{D}(A) = H^2(M) \cap H^1_0(M).$$

Now

$$g^{jk} = a^{jk} \text{ defines a metric tensor on } M.$$

This makes (M, g) a Riemannian manifold.

3. Invariant inverse problem

The Dirichlet-to-Robin map is

$$\Lambda : u|_{\partial M \times \mathbf{R}_+} \mapsto (\partial_\nu u + \sigma u)|_{\partial M \times \mathbf{R}_+}.$$

Dynamical inverse problem:

Let ∂M and the map Λ be given. Can we determine

$$(M, g) \text{ and } [A(x, D)]?$$

4. Energy flux through boundary

The energy of the wave at time t is

$$E(u,t) =$$

$$= \int_M \big(|\partial_t u(t)|^2 + |\operatorname{Grad} u(t)|_g^2 + q|u(t)|^2\big) dV +$$

$$+ \int_{\partial \mathcal{M}} \sigma |u(t)|^2 dS.$$

For $f = u|_{\partial M \times \mathbf{R}_+} \in C_0^\infty(\partial M \times \mathbf{R}_+)$ let

$$\Pi(f) = \lim_{t \to \infty} E(u,t).$$

Inverse problem for energy flux:

Let ∂M and map Π be given. Can we determine

(M, g) and $[A(x, D)]$?

5. Boundary spectral problem

We can write

$$A = -\Delta_g + P + q$$

where Δ_g is a Laplace-Beltrami operator, P is a vector field, q a function. Operator A has in $L^2(M, dV)$ orthonormal eigenfunctions φ_j,

$$(-\Delta_g + P + q - \lambda_j)\varphi_j \; = \; 0,$$
$$\varphi_j|_{\partial M} = 0.$$

Inverse boundary spectral problem:

Let boundary spectral data

$$\{\partial M, \quad \lambda_j, \quad \partial_\nu \varphi_j|_{\partial M}, \quad j = 1, 2, \dots\}$$

be given. Can we determine

$$(M, g) \text{ and } [A(x, D)]?$$

6. Relations of different data

There is $m(x) > 0$ such that

$$dV = m(x) dV_g.$$

The energy flux Π and map Λ have relation

$$\Pi(f) = \int_0^\infty \int_{\partial M} f_t \, \Lambda f \, m dS_g \, dt.$$

Lemma 1 *Assume that ∂M and boundary volume form $m(x) dS_g$ are given. Then Λ and Π determine boundary spectral data.*

Lemma 2 *Assume that ∂M is given. Then Λ and Π determine the gauge-equivalent boundary spectral data*

$$\{\partial M, \quad \lambda_j, \quad \kappa_0 \partial_\nu \varphi_j|_{\partial M}, \quad j = 1, 2, \dots\}$$

where $\kappa_0 > 0$ is unknown function.

7. Schrödinger operator

Lemma 3 *Consider gauge equivalence class* $[A(x, D)]$ *of operator* $A(x, D)$*. Then there is a unique Schrödinger operator*

$$-\Delta_g + q \in [A(x, D)].$$

Inverse problem:

Let the gauge-equivalent boundary spectral data of $-\Delta_g + q$ be given

$$\{\partial M, \quad \lambda_j, \quad \kappa_0 \partial_\nu \varphi_j|_{\partial M}, \quad j = 1, 2, \dots\}.$$

Can we find

$$(M, g) \text{ and } q?$$

Earlier results:

- First global result for $\Delta + q$ in $\mathbf{R}^n$, by using exponentially growing solutions, Nachman-Sylvester-Uhlmann '88, Novikov '88.
- $c(x)^2\Delta$ in $\mathbf{R}^n$ by boundary control method, Belishev '87 , Belishev-Kurylev '87.
- Δ_g on manifold, Belishev-Kurylev '92.
- Local controllability, Tataru '95.
- Gaussian beams, Babich, Ralston
- Wave equations with dissipation, operator pencils L.-Kurylev '97, '00

8. Main ideas

1. Constructions of inner products
Denote by

$$u^f = u^f(x,t)$$

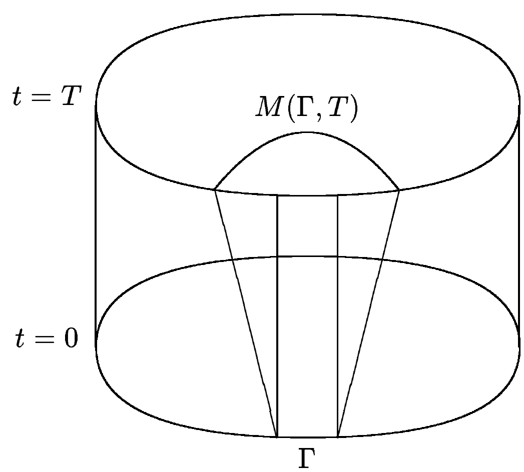
the solutions of

$$u_{tt} - \Delta_g u + qu = 0,$$
$$u|_{\partial M \times \mathbf{R}} = f$$

Lemma 4 *(Blagovestchenskii relation) Let $f, h \in C_0^\infty(\partial M \times \mathbf{R}_+)$. Then gauge-equivalent boundary spectral data determine*

$$\langle u^{\kappa_0 f}(t), u^{\kappa_0 h}(s)\rangle = \int_M u^{\kappa_0 f}(x,t) u^{\kappa_0 h}(x,t)\, dV_g.$$

Figure 1:



2. Controllability result. Let

$$M(\Gamma, t) = \{x \in M \ : \ \text{dist}(x, \Gamma) \le t\}.$$

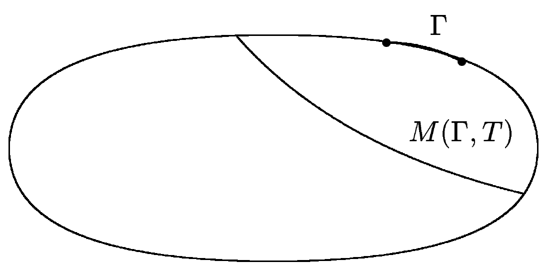
Tataru-Holmgren-John unique continuation theorem yields:

Lemma 5 *Let* $\Gamma \subset \partial M$ *be open. Then*

$$\{u^{\kappa_0 f}(T) : \ f \in C_0^\infty(\Gamma \times [0, T])\} \subset L^2(M(\Gamma, T))$$

is dense.

Figure 2:



Lemma 6 *Let gauge-equivalent boundary spectral data be given. We can find*

$$||\chi_{M(\Gamma,T)}u^{\kappa_0 f}(s)||_{L^2(M)}$$

for any Γ, T, f, *and* s.

Proof. Minimize

$$||u^{\kappa_0 f}(s) - u^{\kappa_0 h}(T)|| : \quad h \in C_0^\infty(\Gamma \times [0,T]).$$

9. Gaussian beams

Let $\gamma(t)$ be a geodesic. The Gaussian beam propagating along $\gamma(t)$ is a family of solutions of wave equation

$$U_\epsilon^N(\mathbf{x},t) =$$

$$= \epsilon^{-\frac{n}{4}} \exp\left(-\frac{1}{i\epsilon}\theta(x,t)\right) \sum_{n=0}^{N} u_n(x,t)(i\epsilon)^n + o(\epsilon^{N-\frac{n}{4}}),$$

where

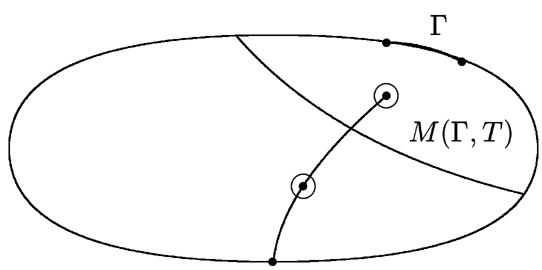
$$\operatorname{Im}\theta(x,t) \geq c\left(d(x,\gamma(t))\right)^2.$$

Lemma 7 *Let $z_0 \in \partial M$. Define in local coordinates of ∂M*

$$f_\epsilon(z,t) = \epsilon^{-\frac{n}{4}} \exp(-\frac{1}{i\epsilon}\big((z-z_0)^2 + t^2\big)).$$

Then $u^{\kappa_0 f_\epsilon}(x,t)$ is a Gaussian beam propagating along normal geodesic from z_0.

Figure 3:



10. How to follow the path of a Gaussian beam?

Lemma 8 *Let* $z_0 \in \partial M$*,* $t > 0$ *and* $\gamma_{z_0,\nu}$ *be the normal geodesic. Then the gauge equivalent boundary spectral data determine*

$$d(z, \gamma_{z_0,\nu}(t))$$

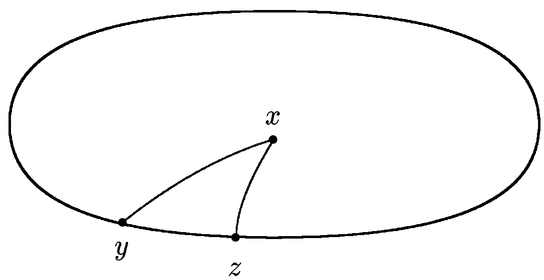
for $z \in \partial M$.

Proof. Let $u^{\kappa_0 f_\epsilon}(x,t)$ be the Gaussian beam. We can find

$$\lim_{\epsilon \to 0} ||\chi_{M(z,T)} u^{\kappa_0 f_\epsilon}(t)||_{L^2(M)}$$

This shows when $\gamma_{z_0,\nu}(t) \in M(z,T)$. □

Figure 4:



11. Boundary distance functions

For $x \in M$ define

$$r_x(y) \ = \ d(x,y), \ y \in \partial M.$$

Let

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Lemma 9 *We can determine*

$$R(M) = \{r_x \in C(\partial M) : \quad x \in M\}.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.

Lemma 10 *(Kurylev) The set $R(M)$ has a Riemann manifold structure which is isometric to M.*

Proof R is one-to-one and onto. As M is compact R is homeomorphism.
Next we construct metric tensor. Consider evaluation functions

$$E_z : R(M) \to \mathbf{R}, \quad E_{\mathbf{z}}(r) = r(\mathbf{z}), \;\; z \in \partial M.$$

Then

$$E_z \circ R : x \mapsto \mathrm{dist}(x, z).$$

Thus differentials $d(E_z \circ R)$ are unit vectors and form an open set of unit sphere bundle S^*M. This determines metric tensor g. □

12. Determination of potential

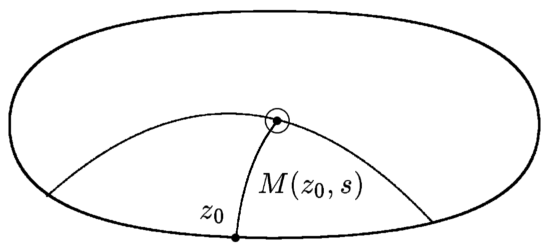
Lemma 11 *Gauge equivalent boundary spectral data determines* κ_0 *and* q.

Proof. Let $u^{\kappa_0 f_\epsilon}(x,t)$ be the Gaussian beam propagating along $\gamma_{z_0,\nu}$. Then

$$\lim_{\epsilon \to 0} ||u^{\kappa_0 f_\epsilon}(x,t)|| = c\kappa_0(z_0)$$

which determine κ_0.

Figure 5:



Let $y = \gamma_{z_0,\nu}(s)$ and $u^h(x,t)$ any solution. Then we can determine

$$u^h(y,t) = \lim_{\epsilon \to 0} c\epsilon^{-\frac{m+2}{4}} \int_M \chi(x) u^{f_\epsilon}(x,s) u^h(x,t) dV_g$$

where

$$\chi(x) = \chi_{M(z_0,s)}(x).$$

Thus we can find all solutions $u^h(y,t)$ in $(y,t) \in M \times \mathbf{R}_+$, which determine q. $\square$

13. Non-selfadjoint case

Consider a hyperbolic equation

$$\begin{aligned} u_{tt}(x,t) + \eta u_t(x,y) + Au(x,t) &= 0 \text{ in } M \times \mathbf{R}_+, \\ u_t(x,0) = u(x,0) &= 0, \\ u|_{\partial M \times \mathbf{R}_+} &= f \end{aligned}$$

In local coordinates

$$Au = -\sum_{j,l=1}^{n} g^{jl}(x)\partial_j\partial_l u + \sum_{j=1}^{n} b^j(x)\partial_j u + c(x)u$$

where g^{jl} defines a Riemannian metric and b^j, c are complex, smooth functions.

Definition 12 Bardos-Lebeau-Rauch condition:
All rays of geometrical optic "leave" the domain in time T

We have

Theorem 13 *(Kurylev-L.) Let* (M_1, A_1) *and* (M_2, A_2) *satisfy Bardos-Lebeau-Rauch -condition. Then the maps* Λ_1 *and* Λ_2 *are gauge-equivalent if and only if* $M_1 = M_2$, $\eta_1 = \eta_2$ *and there is* $\kappa \neq 0$, *such that*

$$A_2 = \kappa\, A_1\, \kappa^{-1}.$$

14. Main steps of construction

- Compute from boundary data the products

$$[u,v]_T =$$

$$= \int_M \big(\overline{u}(T)v_t(T) - \overline{u}_t(T)v(T) + \eta\overline{u}(T)v(T)\big)dV$$

of solutions u of wave equation and solutions v of the adjoint wave equation corresponding to A^* and $-\overline{\eta}$.

- Test when

$$\text{supp}\ (u(t), u_t(t)) \subset S,$$
$$S = \bigcup M(\Gamma_j, T_j + \epsilon) \setminus M(\Gamma_j, T_j)$$

- Choose $S^j \to \{x_0\}$. Find all waves u^j such that

$$\text{supp}\ (u^j(T_0), u^j_t(T_0)) \subset S_j$$

and

$$\lim_{j\to\infty} [u^j, v]_T$$

exists for H^s-smooth v.

- Then we find gauge transformations of waves,

$$\lim_{j\to\infty} [u^j, v]_T = \kappa(x_0) v(x_0, T).$$

- Find class $[a(x, D)]$ and η from gauge transformations of waves

$$\kappa(x) v(x, T).$$