Detecting an inclusion in an elastic body by boundary measurements *

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Abstract

We consider the problem of determining an inclusion D made of different elastic material in an elastic isotropic body Ω from boundary measurements of traction and displacement. We prove that the volume of D can be estimated, from above and below, by an easily expressed quantity related to work, only depending on the boundary traction and displacement.

1 Introduction

In this paper we address the following problem of *nondestructive testing*:

To determine, in an elastic body Ω, the possible presence of an inclusion D made of a different elastic material (either harder or softer) from measurements of traction and displacement taken at the exterior boundary of Ω.

In mathematical terms, if u denotes the displacement field in Ω, one wishes to recover $D \subset\subset \Omega$ in the *system of linearized elasticity*

$$\mathrm{div}\,((\chi_{\Omega\setminus D}\mathbb{C} + \chi_D\widetilde{\mathbb{C}})\nabla u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

by the knowledge of one pair of Cauchy data on $\partial\Omega$

$$(\mathbb{C}\nabla u)\nu = \varphi \quad \text{on } \partial\Omega, \tag{1.2}$$

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$$u = g \quad \text{on } \partial\Omega. \tag{1.3}$$

Here $\mathbb{C}$ and $\widetilde{\mathbb{C}}$ denote the elasticity tensor fields in $\Omega \backslash D$ and in D, respectively, ν is the unit exterior normal to $\partial\Omega$ and χ_E denotes the characteristic function of E.

This appears as an extremely difficult inverse problem. A similarly stated problem in the field of electrical impedance tomography (for which the direct problem involves a single scalar elliptic partial differential equation, rather than a system) has received great attention in recent years (see, for instance, Friedman [Fr87], Friedman and Gustafsson [FrG87], Friedman and Isakov [FrI89], Alessandrini, Rosset and Seo [ARS00] and also Alessandrini and Isakov [AI96], Alessandrini [Al99] for an extended reference list), but still many fundamental questions remain unanswered. See also for previous results on this problem Ikehata [I98].

Here, following the line of a research initiated in Alessandrini and Rosset [AR98], Kang, Seo and Sheen [KSS97], Alessandrini, Rosset and Seo [ARS00] in the electrostatic setting, we pose ourselves a relatively modest, but realistic goal:

Can we estimate the size (the volume) of the unknown inclusion D from one set of boundary measurements of traction and displacement?

In the present paper we restrict our attention to the Lamé system of linearized elasticity, corresponding to system (1.1) when the material is isotropic.

In order to illustrate our main results it is convenient to consider the solution u_0 to the Neumann problem (1.1)-(1.2) when D is replaced by the empty set.

Theorem 2.4 below states that if, for a given $h_1 > 0$, the following "fatness-condition" is satisfied

$$\mid \{x \in D \mid dist(x, \partial D) > h_1\} \mid \geq \frac{1}{2}|D| \tag{1.4}$$

then

$$C_1 \left| \frac{W - W_0}{W_0} \right| \leqq |D| \leqq C_2 \left| \frac{W - W_0}{W_0} \right|, \tag{1.5}$$

where C_1, C_2 are estimated in terms of the data. Here, the quantities $W = \int_{\partial\Omega} g \cdot \varphi$ and $W_0 = \int_{\partial\Omega} g_0 \cdot \varphi$ represent the work exerted by the surface forces φ when the boundary displacement fields are g and $g_0 = u_0|_{\partial\Omega}$, respectively. See Remark 2.6 for a discussion of the "fatness-condition" (1.4).

In Theorem 2.5 we treat the case when no a priori assumption is made on D. We find that for a suitable $p > 1$, we have

$$C_1 \left| \frac{W - W_0}{W_0} \right| \leqq |D| \leqq C_2 \left| \frac{W - W_0}{W_0} \right|^{\frac{1}{p}}. \tag{1.6}$$

(See Section 2 below for the precise statements.)

We believe that these estimates can be useful in practice as a decision tool in quality control tests. Namely, one can fix experimentally a threshold parameter $T > 0$ such that one can say that D is absent or negligible if $\left|\frac{W-W_0}{W_0}\right| \le T$, whereas D is significant if $\left|\frac{W-W_0}{W_0}\right| \ge T$.

The main underlying idea in these estimates is that the integral

$$\int_D |\widehat{\nabla} u_0|^2 \quad \textit{is} \quad \textit{comparable} \quad \textit{to} \quad |W - W_0|, \tag{1.7}$$

where $\widehat{\nabla} u_0 = \frac{1}{2}(\nabla u_0 + (\nabla u_0)^T)$ is the strain tensor field, see Ikehata [I98].

The next point is to control the above integral in terms of the measure (volume) of D. On one side, this task involves upper bounds on $|\widehat{\nabla} u_0|^2$, which is standard in the regularity theory of elliptic systems like (1.1). On the other side, it involves local lower bounds on $|\widehat{\nabla} u_0|^2$. Rather then to regularity theory, this task is more strictly related to the issue of unique continuation, namely, the study of the character of zeroes (order of vanishing, and size of the zero sets) of non trivial solutions to the system (1.1). This issue is very well studied and understood for the case of linear elliptic equations (see, for instance, Aronszajn, Krzywicki, and Szarski [AKS62], Garofalo and Lin [GL86], [GL87], Koch and Tataru [KT01]). Instead, until recently, very little was known on the unique continuation problem for the Lamé system of linearized elasticity, Dehman and Robbiano [DR93], Ang, Ikehata, Trong and Yamamoto [AITY98], Eller, Isakov, Nakamura and Tataru [EINT98], Alessandrini and Morassi [AM01]. In this paper we apply some of the estimates of unique continuation found in [AM01] (three spheres inequalities ([AM01], (5.2)) and strong unique continuation ([AM01], (5.5))) and we further elaborate on this topic. The main result in this direction here are new *doubling inequalities* for the reference solution u_0 (see Theorem 3.9) and for its symmetrized gradient $\widehat{\nabla} u_0$ (see Corollary 3.10). Such an inequality allows to prove for $|\widehat{\nabla} u_0|^2$ the property of being a Muckenhoupt weight (Coifman and Fefferman [CF74], Garcia Cuerva and Rubio de Francia [GCRDF85]). This is a property of homogeneity in the average at all scales which was first proved for solutions of scalar elliptic equations by Garofalo and Lin [GL86].

The plan of the paper is as follows. In Section 2 we introduce some notation and state our main results (Theorem 2.4 and Theorem 2.5).

Section 3 is devoted to the derivation of quantitative estimates of unique continuation for solutions to the Lamé system, following ideas introduced in Alessandrini and Morassi [AM01].

In Section 4 we first derive an interior average lower bound on $|\widehat{\nabla} u_0|^2$ on small balls contained inside Ω (see Proposition 4.1). Moreover, we rephrase the doubling inequalities obtained in the previous Section in terms of the boundary data (see Proposition 4.3), and we show that $|\widehat{\nabla} u_0|^2$ is a Muckenhoupt weight (see Proposition 4.4).

Finally, Section 5 contains the proofs of the main theorems.

2 Main results

Let us introduce some notation which will be useful in the sequel. We restrict our attention to the dimensions $n = 2, 3$ which are those physically relevant for elasticity.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, for any $h > 0$ we shall denote

$$\Omega_h = \{x \in \Omega \mid dist(x, \partial\Omega) > h\}. \tag{2.1}$$

When representing locally a boundary as a graph, it will be convenient to use the following notation. For every $x \in \mathbb{R}^n$ we shall set $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

Definition 2.1. Given a bounded domain $\Omega \subset \mathbb{R}^n$, we shall say that $\partial\Omega$ is of *class* $C^{1,1}$ *with constants* $r_0, M_0 > 0$, if, for any $x_0 \in \partial\Omega$, there exists a rigid transformation of coordinates under which we have $x_0 = 0$ and

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) \quad s.t. \quad x_n > \varphi(x')\},$$

where φ is a $C^{1,1}$ function on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,1}(B_{r_0}(0))} \leq M_0 r_0.$$

Remark 2.2. We have chosen to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above is meant as follows

$$\|\phi\|_{C^{1,1}(B_{r_0}(0))} = \|\phi\|_{L^\infty(B_{r_0}(0))} + r_0\|\nabla\phi\|_{L^\infty(B_{r_0}(0))} + \\ + {r_0}^2\|\nabla^2\phi\|_{L^\infty(B_{r_0}(0))}. \tag{2.2}$$

Similarly, given a function $f : \Omega \mapsto \mathbb{R}$, where $\partial\Omega$ satisfies Definition 2.1, we shall denote

$$\|f\|_{C^{1,1}(\Omega)} = \|f\|_{L^\infty(\Omega)} + r_0\|\nabla f\|_{L^\infty(\Omega)} + {r_0}^2\|\nabla^2 f\|_{L^\infty(\Omega)}. \tag{2.3}$$

Notice also that, when $\Omega = B_R(0)$, then Ω satisfies Definition 2.1 with $r_0 = R$.

We consider weak solutions $u \in H^1(\Omega, \mathbb{R}^n)$ to the *displacement equation of equilibrium* when body forces are absent

$$\mathrm{div}\,(\mathbb{C}(x)(\nabla u(x))) = 0 \quad \text{in } \Omega, \tag{2.4}$$

see Gurtin [Gur72].

We shall assume throughout that the elasticity tensor field $\mathbb{C} = \mathbb{C}(x)$ of the materials under consideration have components C_{ijkl} which satisfy the following conditions

$$C_{ijkl} \in L^\infty(\Omega, \mathbb{R}), \quad \forall i,j,k,l = 1, ..., n, \tag{2.5}$$

$$C_{ijkl} = C_{klij} = C_{klji}, \quad \forall i,j,k,l = 1, ..., n, \quad a.e. \quad in \quad \Omega. \tag{2.6}$$

We recall that the symmetry conditions (2.6) are equivalent to:

$$\mathbb{C}A = \mathbb{C}\widehat{A}, \tag{2.7}$$

$$\mathbb{C}A \quad is \quad symmetric, \tag{2.8}$$

$$\mathbb{C}A \cdot B = \mathbb{C}B \cdot A, \tag{2.9}$$

for every $n \times n$ matrices A,B.

Here, and in the sequel the following notation has been used

$$(\mathbb{C}A)_{ij} = \sum_{k,l=1}^{n} C_{ijkl}A_{kl}, \tag{2.10}$$

$$A \cdot B = \sum_{i,j=1}^{n} A_{ij}B_{ij}, \tag{2.11}$$

$$|A| = (A \cdot A)^{\frac{1}{2}}, \tag{2.12}$$

$$\widehat{A} = \frac{1}{2}(A + A^T), \tag{2.13}$$

for every $n \times n$ matrices A,B.

We shall also use the following conventions for inequalities. Given $\mathbb{C}$, $\widetilde{\mathbb{C}}$ satisfying (2.5), (2.6) we shall say that

$$\widetilde{\mathbb{C}} \leq \mathbb{C} \tag{2.14}$$

if and only if

$$\widetilde{\mathbb{C}} A \cdot A \leq \mathbb{C} A \cdot A \tag{2.15}$$

for every *symmetric* $n \times n$ matrix A.

We shall say that:

$\widetilde{\mathbb{C}}$ is *strongly convex* in Ω if there exists a positive constant ξ_0 such that

$$\mathbb{C}(x) A \cdot A \geq \xi_0 |A|^2 \quad for \quad a.e. \quad x \in \Omega, \tag{2.16}$$

for any symmetric $n \times n$ matrix A.

$\widetilde{\mathbb{C}}$ is said *strongly elliptic* in Ω if there exists a positive constant κ_0 such that

$$\mathbb{C}(x) A \cdot A \geq \kappa_0 |A|^2 \quad for \quad a.e. \quad x \in \Omega, \tag{2.17}$$

for any matrix A of the form $A_{ij} = a_i b_j$, where a and b are $n-$vectors.

It is well known that *if* $\mathbb{C}$ *is strongly convex then it is also strongly elliptic.*

When the elastic material is *isotropic*, then the elasticity tensor $\mathbb{C}$ takes the following form

$$C_{ijkl}(x) = \lambda(x) \delta_{ij} \delta_{kl} + \mu(x)(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}), \tag{2.18}$$

where $\lambda = \lambda(x)$ and $\mu = \mu(x) \in L^\infty(\Omega, \mathbb{R})$ are the *Lamé moduli.* Hence, in this case, denoting by I_n the $n \times n$ identity matrix, we have

$$\mathbb{C}(x) A = \lambda(x)(A \cdot I_n) I_n + 2\mu(x) \widehat{A}, \tag{2.19}$$

and the displacement equation of equilibrium (2.4) becomes the Lamé system

$$\mathrm{div}\,(2\mu \widehat{\nabla} u) + \nabla(\lambda \mathrm{div}\, u) = 0 \quad \text{in } \Omega. \tag{2.20}$$

In the isotropic case, the *strong convexity* condition takes the form

$$\mu(x) \geq \alpha_0, \quad 2\mu(x) + n\lambda(x) \geq \gamma_0 \quad for \quad a.e. \quad x \in \Omega \tag{2.21}$$

and the *strong ellipticity* condition is expressed by

$$\mu(x) \geq \alpha_0, \quad 2\mu(x) + \lambda(x) \geq \beta_0 \quad for \quad a.e. \quad x \in \Omega, \tag{2.22}$$

where α_0, β_0, γ_0 are positive constants.

Let Ω be a bounded domain whose boundary is of class $C^{1,1}$ with given constants $r_0, M_0 > 0$. Let D be a measurable, possibly disconnected, subset of Ω such that, given $d_0 > 0$,

$$dist(D, \partial\Omega) \geq d_0. \tag{2.23}$$

Given elasticity tensors $\mathbb{C}$, $\widetilde{\mathbb{C}}$ satisfying (2.5), (2.6) we shall consider traction problems in Ω when the elasticity tensor is either $\chi_{\Omega\setminus D}\mathbb{C} + \chi_D\widetilde{\mathbb{C}}$ or $\mathbb{C}$.

We shall prescribe a boundary traction field $\varphi \in L^2(\partial\Omega, \mathbb{R}^n)$ satisfying the compatibility conditions

$$\int_{\partial\Omega} \varphi \cdot r = 0 \tag{2.24}$$

for every *infinitesimal rigid displacement* r, that is $r(x) = c + Wx$, where c any constant $n-$vector and W is any constant skew $n \times n$ matrix. Namely we shall consider weak solutions $u, u_0 \in H^1(\Omega, \mathbb{R}^n)$ of the following problems:

$$\mathrm{div}\,((\chi_{\Omega\setminus D}\mathbb{C} + \chi_D\widetilde{\mathbb{C}})\nabla u) = 0 \quad \text{in } \Omega, \tag{2.25}$$

$$(\mathbb{C}\nabla u)\nu = \varphi \quad \text{on } \partial\Omega; \tag{2.26}$$

$$\mathrm{div}\,(\mathbb{C}\nabla u_0) = 0 \quad \text{in } \Omega, \tag{2.27}$$

$$(\mathbb{C}\nabla u_0)\nu = \varphi \quad \text{on } \partial\Omega. \tag{2.28}$$

Regarding existence, we recall that, provided the compatibility condition (2.24) is satisfied, a solution of the traction problem exists as long as the involved elasticity tensor either satisfies the strong convexity condition or it is continuous and satisfies the strong ellipticity condition, see for instance Valent ([V88], §III).

With respect to uniqueness we recall that it is well-known that solutions u, u_0 to the above problems are uniquely determined up to an infinitesimal rigid displacement. In order to uniquely identify such solutions, we shall assume from now on that both u and u_0 satisfy the following normalization conditions

$$\int_\Omega u = 0, \quad \int_\Omega (\nabla u - (\nabla u)^T) = 0. \tag{2.29}$$

We set g, $g_0 \in H^{1/2}(\partial\Omega, \mathbb{R}^n)$ be the traces of u, u_0, respectively, on $\partial\Omega$.

Now we are in position to state our main result on the size estimates for the inclusion.

We shall use the following assumptions on the elasticity tensors $\mathbb{C}$, $\widetilde{\mathbb{C}}$:

i) $\mathbb{C}$ satisfies the isotropy condition (2.18) and the strong convexity (2.21);

ii) (bounds on the jump and uniform strong convexity for $\widetilde{\mathbb{C}}$)
either there exist $\eta > 0$ and $\delta > 1$ such that

$$\eta\mathbb{C} \leq \widetilde{\mathbb{C}} - \mathbb{C} \leq (\delta - 1)\mathbb{C} \quad a.e. \quad in \quad \Omega, \tag{2.30}$$

or there exist $\eta > 0$ and $0 < \delta < 1$ such that

$$-(1-\delta)\mathbb{C} \leq \widetilde{\mathbb{C}} - \mathbb{C} \leq -\eta\mathbb{C} \quad a.e. \quad in \quad \Omega; \tag{2.31}$$

iii) ($C^{1,1}$ regularity for $\mathbb{C}$)
there exists $M > 0$ such that

$$\|\mu\|_{C^{1,1}(\Omega)} + \|\lambda\|_{C^{1,1}(\Omega)} \leq M. \tag{2.32}$$

Remark 2.3. It is worth noticing that very mild assumptions are made on the unknown inclusion; namely, the inclusion D may consist of an anisotropic material which is either harder (case (2.30)) or softer (case (2.31)) than the surrounding material in Ω, and no additional regularity assumption is required on the elasticity tensor inside D.

Theorem 2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$, such that $\partial\Omega$ is of class $C^{1,1}$ with constants r_0, M_0. Let D be a measurable subset of Ω satisfying* (2.23) *and*

$$|D_{h_1}| \geq \frac{1}{2}|D|, \tag{2.33}$$

for a given positive constant h_1. Let $\mathbb{C}$, $\widetilde{\mathbb{C}}$ satisfy (i),(ii),(iii). If (2.30) *holds, then we have*

$$\frac{1}{\delta-1}C_1^+ \frac{\int_{\partial\Omega}(g_0 - g)\cdot\varphi}{\int_{\partial\Omega} g_0\cdot\varphi} \leq |D| \leq \frac{\delta}{\eta}C_2^+ \frac{\int_{\partial\Omega}(g_0 - g)\cdot\varphi}{\int_{\partial\Omega} g_0\cdot\varphi}, \tag{2.34}$$

if, conversely, (2.31) *holds, then we have*

$$\frac{\delta}{1-\delta}C_1^- \frac{\int_{\partial\Omega}(g - g_0)\cdot\varphi}{\int_{\partial\Omega} g_0\cdot\varphi} \leq |D| \leq \frac{1}{\eta}C_2^- \frac{\int_{\partial\Omega}(g - g_0)\cdot\varphi}{\int_{\partial\Omega} g_0\cdot\varphi}, \tag{2.35}$$

where C_1^+, C_1^- depend on d_0, $|\Omega|$, α_0, γ_0, r_0, M_0, M only, and C_2^+, C_2^- only depend on d_0, $|\Omega|$, α_0, γ_0, r_0, M_0, M, h_1 and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$.

Theorem 2.5. *Let Ω be as in Theorem 2.4 and let D be any measurable subset of Ω satisfying* (2.23). *Let $\mathbb{C}$, $\widetilde{\mathbb{C}}$ satisfy (i),(ii),(iii). If* (2.30) *holds, then we have*

$$\frac{1}{\delta-1}C_1^+\frac{\int_{\partial\Omega}(g_0-g)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi}\leq|D|\leq\left(\frac{\delta}{\eta}\right)^{\frac{1}{p}}C_2^+\left(\frac{\int_{\partial\Omega}(g_0-g)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi}\right)^{\frac{1}{p}}, \qquad (2.36)$$

if, conversely, (2.31) *holds, then we have*

$$\frac{\delta}{1-\delta}C_1^-\frac{\int_{\partial\Omega}(g-g_0)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi}\leq|D|\leq\left(\frac{1}{\eta}\right)^{\frac{1}{p}}C_2^-\left(\frac{\int_{\partial\Omega}(g-g_0)\cdot\varphi}{\int_{\partial\Omega}g_0\cdot\varphi}\right)^{\frac{1}{p}}, \qquad (2.37)$$

where C_1^+, C_1^- are the same as in Theorem 2.4 and $p>1$, C_2^+, C_2^- depend on d_0, $|\Omega|$, α_0, γ_0, r_0, M_0, M and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$.

Remark 2.6. Let us notice that the "fatness-condition" (2.33) is satisfied when mild a priori regularity assumptions are made on D. For instance, the constant h_1 can be easily estimated when D is a priori known to be Lipschitz, we refer to Alessandrini and Rosset ([AR98], Lemma 2.8) for related calculations. See also Alessandrini, Rosset and Seo [ARS00] for comments on the optimality of this kind of results in the case of a scalar elliptic equation.

3 Quantitative estimates of unique continuation

In this Section we shall prove quantitative estimates of unique continuation of the form of *three spheres inequalities* and of *doubling inequalities* for solutions $u\in H^1(\Omega,\mathbb{R}^n)$ to the Lamé system of linearized elasticity (2.20) in a bounded domain Ω satisfying Definition 2.1 with constants r_0, M_0. Throughout this Section the Lamé moduli $\mu=\mu(x)$, $\lambda=\lambda(x)$ are assumed to satisfy the strong ellipticity condition (2.22) and the regularity assumption (2.32). Following ideas introduced in [AM01], the first step consists in reducing the Lamé system (2.20) to a weakly coupled elliptic system with Laplacian principal part. We denote by $\mathbb{M}^{m\times n}$ the space of $m\times n$ real valued matrices.

Proposition 3.1. *Under the above assumptions, there exist $B\in L^\infty(\Omega, L(\mathbb{M}^{(n+1)\times n},\mathbb{R}^{n+1}))$ and $V\in L^\infty(\Omega,L(\mathbb{R}^{n+1},\mathbb{R}^{n+1}))$ such that, for every weak solution $u\in H^1(\Omega,\mathbb{R}^n)$ to* (2.20), *the $\mathbb{R}^{n+1}$-valued function U given by*

$$U=\begin{pmatrix}u\\ r_0\,div\,u\end{pmatrix} \qquad (3.1)$$

belongs to $W^{2,p}_{loc}(\Omega,\mathbb{R}^{n+1})$ *for every* $p<\infty$, *and satisfies*

$$-\Delta U + B(\nabla U) + V(U) = 0 \quad \textit{in } \Omega. \tag{3.2}$$

Moreover

$$r_0\|B\|_{L^\infty(\Omega,L(\mathbb{M}^{(n+1)\times n},\mathbb{R}^{n+1}))} + r_0^2\|V\|_{L^\infty(\Omega,L(\mathbb{R}^{n+1},\mathbb{R}^{n+1}))} \leqslant CM, \tag{3.3}$$

where $C>0$ *only depends on* α_0 *and* β_0.

Proof. The proof is essentially contained in [AM01], Theorem 2.1. Here the statement is slightly modified in order to encompass the scaling invariance of the norms introduced in the present paper. □

Three spheres inequalities and doubling inequalities for solutions u to systems of the form (3.2), under the assumption (3.3), were derived in [AM01], Theorem 3.1 and Theorem 4.1. Next, one can obtain analogous estimates for solutions u to the Lamé system (2.20), via the reduction described in Proposition 3.1

Proposition 3.2. ([AM01], Theorem 5.1) *Let* $\Omega = B_R = \{x\in\mathbb{R}^n \mid |x|<R\}$. *Under the above assumptions, there exist* $\overline{\theta}$, $0<\overline{\theta}\leqslant 1$, δ, $0<\delta<1$, *depending on* α_0, β_0, M *only, such that for every weak solution* $u\in H^1(B_R,\mathbb{R}^n)$ *to* (2.20) *and for every* r_1, r_2, r_3, $0<r_1<r_2<r_3\leqslant\overline{\theta}R$, *we have*

$$\int_{B_{r_2}}|u|^2 \leqslant C\left(\int_{B_{r_1}}|u|^2\right)^{\delta}\left(\int_{B_{r_3}}|u|^2\right)^{1-\delta}, \tag{3.4}$$

where $C>0$ *only depends on* α_0,β_0,M, $\frac{r_1}{r_3}$ *and* $\frac{r_2}{r_3}$.

Proof. The proof can be found in [AM01]. We notice that here, in view of our scaling on the norms (see Remark 2.2), the constant C does not explicitly depend on R. □

In view of the applications in Section 5, we need the analogous result for $\widehat{\nabla}u$.

Corollary 3.3. *Under the same hypotheses of Proposition 3.2, for every weak solution* $u\in H^1(B_R,\mathbb{R}^n)$ *to* (2.20) *and for every* r_1, r_2, r_3, $0<r_1<r_2<r_3\leqslant\overline{\theta}R$, *we have*

$$\int_{B_{r_2}}|\widehat{\nabla}u|^2 \leqslant C\left(\int_{B_{r_1}}|\widehat{\nabla}u|^2\right)^{\delta}\left(\int_{B_{r_3}}|\widehat{\nabla}u|^2\right)^{1-\delta}, \tag{3.5}$$

where $\overline{\theta}$, $0<\overline{\theta}\leqslant 1$, δ, $0<\delta<1$, *are the same as in Proposition 3.2 and* $C>0$ *only depends on* α_0,β_0,M, $\frac{r_1}{r_3}$ *and* $\frac{r_2}{r_3}$.

In order to prove Corollary 3.3 it is convenient to recall the following two inequalities:

Lemma 3.4. (Caccioppoli-type inequality) *If* $\mathbb{C}$ *satisfies* (2.18), (2.22), (2.32), *then for every solution* $u \in H^1(B_R, \mathbb{R}^n)$ *to* (2.4) *and for every* r, $0 < r < R$, *we have*

$$\int_{B_r} |\nabla u|^2 \leqslant \frac{C}{(R-r)^2} \int_{B_R} |u|^2 \tag{3.6}$$

where $C > 0$ *only depends on* α_0, β_0, M.

Proof. The proof follows by a standard cut-off argument from Gärding's inequality [V88]. ☐

Given $u \in H^1(B_R, \mathbb{R}^n)$ and r, $0 < r < R$, set

$$u_r = \frac{1}{|B_r|} \int_{B_r} u, \tag{3.7}$$

$$W_r = \frac{1}{2|B_r|} \int_{B_r} (\nabla u - (\nabla u)^T). \tag{3.8}$$

Lemma 3.5. (Korn inequality) *There exists an absolute constant* $C > 0$ *such that for every* $u \in H^1(B_R, \mathbb{R}^n)$ *and every* r, $0 < r < R$ *we have*

$$\int_{B_R} |\nabla u - W_r|^2 + \frac{1}{R^2} |u - u_r - W_r x|^2 \leqslant C \left(\frac{R}{r} \right)^{4n-2} \int_{B_R} |\widehat{\nabla} u|^2. \tag{3.9}$$

Remark 3.6. When $r = R$ this is the well-known second Korn inequality, which is known to hold in every sufficiently regular domain Ω (see [Fi72], [T99]). Here we introduce a minor variant, in which the averages of u and of the skew part of ∇u are taken on the smaller ball B_r. For the convenience of the reader, a sketch of the main arguments of a proof is outlined at the end of this Section.

Proof of Corollary 3.3. The function v defined in B_R by

$$v = u - u_{r_1} - W_{r_1} x \tag{3.10}$$

satisfies equation (2.20) and $\frac{1}{|B_{r_1}|} \int_{B_{r_1}} v = 0$, $\frac{1}{2|B_{r_1}|} \int_{B_{r_1}} \nabla v - (\nabla v)^T = 0$. By applying to v the Caccioppoli-type inequality (3.6) and the three spheres inequality (3.4), and using twice the Korn inequality (3.9), we have

$$\begin{aligned}\int_{B_{r_2}}|\widehat{\nabla}u|^2=\int_{B_{r_2}}|\widehat{\nabla}v|^2 &\leqslant \frac{C_1}{(r_3-r_2)^2}\int_{B_{\frac{r_2+r_3}{2}}}|v|^2\leqslant\\ &\leqslant \frac{C_2}{(r_3-r_2)^2}\left(\int_{B_{r_1}}|v|^2\right)^{\delta}\left(\int_{B_{r_3}}|v|^2\right)^{1-\delta}\leqslant\\ &\leqslant C_3\left(\int_{B_{r_1}}|\widehat{\nabla}u|^2\right)^{\delta}\left(\int_{B_{r_3}}|\widehat{\nabla}u|^2\right)^{1-\delta},\end{aligned}\tag{3.11}$$

where C_1, C_2, C_3 are constants only depending on α_0,β_0,M, $\frac{r_1}{r_3}$ and $\frac{r_2}{r_3}$. $\square$

In order to obtain the doubling inequality for solutions to the Lamé system (2.20), we need to state a slightly modified version of the doubling inequality for solutions U to (3.2) contained in [AM01], Theorem 4.1.

Proposition 3.7. *Let $\Omega=B_R$ and let $R\|B\|_\infty+R^2\|V\|_\infty\leqslant E$. There exists θ^*, $0<\theta^*\leqslant 1$, only depending on E, such that for every nonzero solution $U\in H^1(B_R,\mathbb{R}^{n+1})$ to (3.2) we have*

$$\int_{B_{2r}}|U|^2\leqslant C\int_{B_r}|U|^2,\ \textit{for every } r,\ 0<r\leqslant\frac{\theta^*R}{2},\tag{3.12}$$

where C only depends on E and, increasingly, on the quantity $N_0(\theta^*R)$, where

$$N_0(r)=N_0(U;r)=\frac{r\int_{B_r}|\nabla U|^2}{\int_{\partial B_r}|U|^2},\ 0<r\leqslant R.\tag{3.13}$$

Proof. By Theorem 4.1 in [AM01] and by a rescaling argument, it follows easily that (3.12) holds, with C only depending on E and, increasingly, on $N(\theta^*R)$, where

$$N(r)=N(U;r)=\frac{r\int_{B_r}|\nabla U|^2+B(\nabla U)\cdot U+V(U)\cdot U}{\int_{\partial B_r}|U|^2},\ 0<r\leqslant R.\tag{3.14}$$

Hence, we have to show that $N(r)$ can be bounded from above in terms of $N_0(r)$. It is convenient to recall the following notation introduced in [AM01]

$$G(r)=\int_{B_r}|U|^2,\tag{3.15}$$

$$H(r) = \int_{\partial B_r} |U|^2, \tag{3.16}$$

$$I(r) = \int_{B_r} |\nabla U|^2 + B(\nabla U)\cdot U + V(U)\cdot U, \tag{3.17}$$

$$D(r) = \int_{B_r} |\nabla U|^2, \tag{3.18}$$

for $0 < r \leqslant R$. We easily have

$$I(r) \leqslant D(r) + \frac{E}{R}(D(r)G(r))^{\frac{1}{2}} + \frac{E}{R^2}G(r) \leqslant C\left(D(r) + \frac{G(r)}{R^2}\right), \tag{3.19}$$

with C only depending on E. Moreover, from Lemma 3.3 in [AM01] we have

$$G(r) \leqslant rH(r), \text{ for } r \leqslant \theta^* R. \tag{3.20}$$

Hence, for $r \leqslant \theta^* R$ we have

$$N(r) = \frac{rI(r)}{H(r)} \leqslant C\left(N_0(r) + \frac{r^2}{R^2}\right) \leqslant C(N_0(r)+1), \tag{3.21}$$

where C depends on E only. □

Remark 3.8. Let us notice that, analogously to (3.21) we also have

$$N_0(r) \leqslant C(N(r)+1), \text{ for } r \leqslant \theta^* R, \tag{3.22}$$

where C depends on E only.

Theorem 3.9. *Let $\Omega = B_R$. There exists θ^*, $0 < \theta^* \leqslant 1$, only depending on α_0, β_0, M, such that for every nonzero weak solution $u \in H^1(B_R, \mathbb{R}^n)$ to* (2.20) *we have*

$$\int_{B_{2r}} |u|^2 \leqslant K \int_{B_r} |u|^2, \textit{ for every } r,\ 0 < r \leqslant \frac{\theta^* R}{2}, \tag{3.23}$$

where $K > 0$ only depends on α_0, β_0, M and, increasingly, on $\tilde{N}_0(\theta^* R)$, where

$$\tilde{N}_0(r) = \frac{r^2 \int_{B_r} |\nabla u|^2 + R^2 |\nabla(div\, u)|^2}{\int_{B_r} |u|^2 + R^2 |div\, u|^2},\ 0 < r \leqslant R. \tag{3.24}$$

Proof. By applying (3.12) of Proposition 3.7 to the solution U to (3.2) given by the position (3.1), and recalling (3.3), we have

$$\int_{B_{2r}} |u|^2 + R^2|\mathrm{div}\, u|^2 \leqslant C \int_{B_r} |u|^2 + R^2|\mathrm{div}\, u|^2, \text{ for every } r,\ 0 < r \leqslant \frac{\theta^* R}{2}, \tag{3.25}$$

where C only depends on α_0, β_0, M and, increasingly, on $N_0(\theta^* R)$, with $N_0(r)$ given by (3.13). By an iterated application of (3.25) and by the Caccioppoli-type inequality (3.6), we have

$$\int_{B_{2r}} |u|^2 \leqslant C\left(1 + \frac{R^2}{r^2}\right) \int_{B_r} |u|^2, \text{ for every } r,\ 0 < r \leqslant \frac{\theta^* R}{2}. \tag{3.26}$$

Let ρ be such that $0 < \rho \leqslant 1$ and let

$$u_\rho(x) = u(\rho x) \text{ in } B_{\frac{R}{\rho}}.$$

We have that u_ρ is solution in $B_{\frac{R}{\rho}}$ to the Lamé system (2.20) with Lamé moduli satisfying uniformly the bounds (2.22), (2.32). Therefore, by (3.26), we have that

$$\int_{B_{2r}} |u_\rho|^2 \leqslant C_\rho\left(1 + \frac{R^2}{\rho^2 r^2}\right) \int_{B_r} |u_\rho|^2, \text{ for every } r,\ 0 < r \leqslant \frac{\theta^* R}{2\rho}, \tag{3.27}$$

where C_ρ only depends on α_0, β_0, M and, increasingly, on $N_0(U_\rho; \frac{\theta^* R}{\rho})$, where U_ρ is given by (3.1) when u, r_0 are replaced with u_ρ, $\frac{R}{\rho}$ respectively. We have for any r, $0 < r \leqslant \frac{\theta^* R}{2\rho}$

$$\begin{aligned} N_0(U_\rho; r) &= \frac{r \int_{B_r} |\nabla u_\rho|^2 + \frac{R^2}{\rho^2}|\nabla \mathrm{div}\, u_\rho|^2}{\int_{\partial B_r} |u_\rho|^2 + \frac{R^2}{\rho^2}|\mathrm{div}\, u_\rho|^2} = \\ &= \frac{r \int_{B_r} \rho^2|\nabla u(\rho x)|^2 + \frac{R^2}{\rho^2} \cdot \rho^4|\nabla \mathrm{div}\, u(\rho x)|^2}{\int_{\partial B_r} |u(\rho x)|^2 + \frac{R^2}{\rho^2} \cdot \rho^2|\mathrm{div}\, u(\rho x)|^2} = \\ &= \frac{\rho r \int_{B_{\rho r}} |\nabla u|^2 + R^2|\nabla \mathrm{div}\, u|^2}{\int_{\partial B_{\rho r}} |u|^2 + R^2|\mathrm{div}\, u|^2} = N_0(U; \rho r). \end{aligned} \tag{3.28}$$

Hence, in particular,

$$N_0\left(U_\rho; \frac{\theta^* R}{\rho}\right) = N_0(U; \theta^* R). \tag{3.29}$$

Consequently, the quantity C_ρ appearing in (3.27) is uniformly bounded from above with respect to $\rho \in (0,1]$. Taking $r = \frac{\theta^* R}{2\rho}$ in (3.27) and setting $s = r\rho$, we obtain

$$\int_{B_{2s}} |u|^2 \leqslant K \int_{B_s} |u|^2, \text{ for every } s, \ 0 < s \leqslant \frac{\theta^* R}{2}, \tag{3.30}$$

where K only depends on α_0, β_0, M and, increasingly, on $N_0(U; \theta^* R)$. Recalling (3.13), (3.24), we have that $N_0(U; \theta^* R)$ equals $\tilde{N}_0(\theta^* R)$. □

From Theorem 3.9, by using arguments analogous to those employed in the proof of Corollary 3.3, the following doubling inequality for $\widehat{\nabla} u$ follows.

Corollary 3.10. *Let $\Omega = B_R$. There exists θ^*, $0 < \theta^* \leqslant 1$, only depending on α_0, β_0, M, such that for every nonzero weak solution $u \in H^1(B_R, \mathbb{R}^n)$ to* (2.20) *we have*

$$\int_{B_{2r}} |\widehat{\nabla} u|^2 \leqslant K \int_{B_r} |\widehat{\nabla} u|^2, \text{ for every } r, \ 0 < r \leqslant \frac{\theta^* R}{4}, \tag{3.31}$$

where $K > 0$ only depends on α_0, β_0, M and, increasingly, on $\tilde{N}_0(v; \theta^* R)$ given by (3.24)*, where $v = u - u_r - W_r x$, with u_r and W_r defined by* (3.7) *and* (3.8)*, respectively.*

Proof of Lemma 3.5. We adapt arguments from Tiero [T99]. Inequality (3.9) follows, through the introduction of the axial vector ω associated to the skew matrix $W = \frac{\nabla u - (\nabla u)^T}{2}$, from the two scalar inequalities

$$\int_{B_R} (\psi - \psi_r)^2 \leqslant C \left(\frac{R}{r}\right)^{2(n-1)} R^2 \int_{B_R} |\nabla \psi|^2, \text{ for every } \psi \in H^1(B_R), \tag{3.32}$$

$$\int_{B_R} (\psi - \psi_r)^2 \leqslant C \left(\frac{R}{r}\right)^{2n} \|\nabla \psi\|^2_{H^{-1}(B_R)}, \text{ for every } \psi \in L^2(B_R). \tag{3.33}$$

Here $C > 0$ is an absolute constant and the $H^{-1}(B_R)$-norm above is defined as follows

$$\|F\|_{H^{-1}(B_R)} = \sup \left\{ \int_{B_R} FG \mid G \in H^1(B_R, \mathbb{R}^n), \int_{B_R} |\nabla G|^2 = 1 \right\}.$$

It suffices to prove (3.32), (3.33) when $R = 1$ and $\psi \in C^1(\overline{B}_1)$, by usual scaling and density arguments. We recall that (3.32), (3.33) are well known when $r=1$, see for instance [MS58]. Let us estimate $\psi_1-\psi_r$, for $0<r<1$. We easily obtain

$$\psi_1-\psi_r=\frac{1}{n\omega_n}\int_r^1 ds\int_{B_1}\nabla\psi(sx)\cdot x dx,$$

and, changing variables and reversing the order of integration,

$$\psi_1-\psi_r=\frac{1}{n\omega_n}\int_{B_1}\nabla\psi\cdot z,$$

where

$$z(x)=\frac{1}{n}\{\max\{r,|x|\}^{-n}-1\}x.$$

We have

$$\int_{B_1}|z|^2\leqslant Cr^{1-n},$$

$$\int_{B_1}|\nabla z|^2\leqslant Cr^{-n}.$$

Hence

$$|\psi_1-\psi_r|^2\leqslant Cr^{2-2n}\int_{B_1}|\nabla\psi|^2,$$

$$|\psi_1-\psi_r|^2\leqslant Cr^{-2n}\|\nabla\psi\|^2_{H^{-1}(B_1)},$$

and (3.32), (3.33) follow. □

4 Estimates in terms of the boundary data

In this Section we shall consider the traction problem (2.27), (2.28) for a given $\varphi\in L^2(\partial\Omega,\mathbb{R}^n)$ satisfying (2.24). For simplicity of notation we shall denote by u the solution (instead of u_0). The normalization (2.29) is understood throughout.

Regarding the elasticity tensor $\mathbb{C}$ we assume the isotropy condition (2.18), the strong ellipticity (2.22) and the $C^{1,1}$ regularity (2.32).

Proposition 4.1. (Lipschitz propagation of smallness) *For every $\rho>0$ and for every $x\in\Omega_{\frac{4\rho}{\overline{\theta}}}$, we have*

$$\int_{B_\rho(x)}|\widehat{\nabla}u|^2\geqslant C_\rho\int_\Omega|\widehat{\nabla}u|^2, \tag{4.1}$$

where $\overline{\theta}$, $0<\overline{\theta}<1$, depends on α_0,β_0,M only and C_ρ depends on α_0, β_0, M, $|\Omega|$, r_0, M_0, $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$ and ρ only.

We adapt arguments from [ARS00], Theorem 2.2. We start with the following auxiliary Lemma.

Lemma 4.2.

$$\int_{\Omega\setminus\Omega_{\frac{5\rho}{\theta}}} |\widehat{\nabla} u|^2 \leqslant C\rho^{1/n}\|\varphi\|^2_{L^2(\partial\Omega)}, \tag{4.2}$$

where C depends on $\alpha_0, \beta_0, M, r_0, M_0, |\Omega|$ only.

Proof of Lemma 4.2. By Hölder's inequality

$$\|\widehat{\nabla} u\|^2_{L^2(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})} \leqslant |\Omega\setminus\Omega_{\frac{5\rho}{\theta}}|^{1/n}\|\widehat{\nabla} u\|^2_{L^{2n/(n-1)}(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})}, \tag{4.3}$$

and by the Sobolev inequality (see for instance [Ad75])

$$\|\widehat{\nabla} u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant C\|\widehat{\nabla} u\|^2_{H^{1/2}(\Omega)}, \tag{4.4}$$

we have

$$\|\widehat{\nabla} u\|^2_{L^2(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})} \leqslant C|\Omega\setminus\Omega_{\frac{5\rho}{\theta}}|^{1/n}\|u\|^2_{H^{3/2}(\Omega)}, \tag{4.5}$$

where C depends on r_0, M_0, $|\Omega|$ only.

Moreover, we have

$$\|u\|_{H^{3/2}(\Omega)} \leqslant C\|\varphi\|_{L^2(\partial\Omega)}, \tag{4.6}$$

where C depends on $\alpha_0, \beta_0, M, r_0, M_0, |\Omega|$ only. Inequality (4.6) follows, by interpolation (see [LM72]), from the global estimates for the Neumann problem

$$\|u\|_{H^1(\Omega)} \leqslant C_1\|\varphi\|_{H^{-1/2}(\partial\Omega)}, \tag{4.7}$$

$$\|u\|_{H^2(\Omega)} \leqslant C_2\|\varphi\|_{H^{1/2}(\partial\Omega)}, \tag{4.8}$$

where C_1 and C_2 depend on α_0, β_0, M, r_0, M_0, $|\Omega|$ only (see [ADN64]).

Moreover,

$$|\Omega\setminus\Omega_{\frac{5\rho}{\theta}}| \leqslant C\rho, \tag{4.9}$$

where C depends on r_0, M_0, $|\Omega|$ only (see $(A.3)$ in [AR98] for details). From (4.5), (4.6) and (4.9) the thesis follows. □

Proof of Proposition 4.1. Let us fix ρ_0, depending on r_0, M_0 only, such that $\Omega_{\frac{4\rho}{\overline{\theta}}}$ is connected for every $\rho \leqslant \rho_0$. Without loss of generality, we may assume, for this proof, $\rho \leqslant \rho_0$. Given any $y \in \Omega_{\frac{4\rho}{\overline{\theta}}}$, let γ be an arc in $\Omega_{\frac{4\rho}{\overline{\theta}}}$ joining x and y. Let us define $\{x_i\}$, $i = 1, ..., L$, as follows: $x_1 = x$, $x_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \mid |\gamma(t) - x_i| = 2\rho\}$ if $|x_i - y| > 2\rho$, otherwise let $i = L$ and stop the process. Then, by construction, the balls $B_\rho(x_i)$ are pairwise disjoint, $|x_{i+1} - x_i| = 2\rho$ for $i = 1, ..., L-1$, $|x_L - y| \leqslant 2\rho$.

Since $x_i \in \Omega_{\frac{4\rho}{\overline{\theta}}}$, we may apply (3.5) for $x = x_i$, $r_1 = \rho$, $r_2 = 3\rho$, $r_3 = 4\rho$, for $i = 1, ..., L-1$, obtaining

$$\frac{\|\widehat{\nabla} u\|_{L^2(B_\rho(x_{i+1}))}}{\|\widehat{\nabla} u\|_{L^2(\Omega)}} \leqslant C \left(\frac{\|\widehat{\nabla} u\|_{L^2(B_\rho(x_i))}}{\|\widehat{\nabla} u\|_{L^2(\Omega)}} \right)^{\delta} \tag{4.10}$$

where $C > 0$ and δ, $0 < \delta < 1$, only depend on α_0, β_0 and M. By induction we have

$$\frac{\|\widehat{\nabla} u\|_{L^2(B_\rho(y))}}{\|\widehat{\nabla} u\|_{L^2(\Omega)}} \leqslant C^{1/(1-\delta)} \left(\frac{\|\widehat{\nabla} u\|_{L^2(B_\rho(x))}}{\|\widehat{\nabla} u\|_{L^2(\Omega)}} \right)^{\delta^L} . \tag{4.11}$$

Let us notice that $L \leqslant \frac{|\Omega|}{\omega_n \rho^n}$.

Let us cover $\Omega_{\frac{5\rho}{\overline{\theta}}}$ with internally nonoverlapping closed cubes of side $l = 2\rho/\sqrt{n}\overline{\theta}$. Clearly, any such cube is contained in a ball of radius ρ and center in $\Omega_{\frac{4\rho}{\overline{\theta}}}$ and the number of such cubes is controlled by $N = \frac{|\Omega| n^{n/2} \overline{\theta}^n}{2^n \rho^n}$. Therefore, from (4.11) we have

$$\frac{\|\widehat{\nabla} u\|_{L^2(\Omega_{\frac{5\rho}{\overline{\theta}}})}}{\|\widehat{\nabla} u\|_{L^2(\Omega)}} \leqslant \frac{C}{\rho^{n/2}} \left(\frac{\|\widehat{\nabla} u\|_{L^2(B_\rho(x))}}{\|\widehat{\nabla} u\|_{L^2(\Omega)}} \right)^{\delta^L} , \tag{4.12}$$

where C only depends on $\alpha_0, \beta_0, M, |\Omega|$.

Now, let us estimate from below the left hand side of (4.12) by means of a positive constant. Let us set

$$\frac{\|\widehat{\nabla} u\|^2_{L^2(\Omega_{\frac{5\rho}{\overline{\theta}}})}}{\|\widehat{\nabla} u\|^2_{L^2(\Omega)}} = 1 - \frac{\int_{\Omega \setminus \Omega_{\frac{5\rho}{\overline{\theta}}}} |\widehat{\nabla} u|^2}{\int_\Omega |\widehat{\nabla} u|^2}. \tag{4.13}$$

By a trace inequality (see, for instance, [LM72]) and by the Korn inequality (3.9), we have

$$\|\varphi\|_{H^{-1/2}(\partial\Omega)} \leqslant C \|\widehat{\nabla} u\|_{L^2(\Omega)}, \tag{4.14}$$

where C depends on $\alpha_0, \beta_0, r_0, M_0, |\Omega|$ only. Hence, by (4.2) and (4.14), we have that there exists $\overline{\rho} > 0$, depending on $\alpha_0, \beta_0, M, r_0, M_0, |\Omega|$ and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$ only, such that

$$\frac{\|\widehat{\nabla} u\|^2_{L^2(\Omega_{\frac{5\rho}{\theta}})}}{\|\widehat{\nabla} u\|^2_{L^2(\Omega)}} \geqslant \frac{1}{2}, \tag{4.15}$$

for every ρ, $0 < \rho \leqslant \overline{\rho}$.

Finally, from (4.12) and (4.15) the thesis follows when $0 < \rho \leqslant \overline{\rho}$; for larger values of ρ, inequality (4.1) is trivial. □

Proposition 4.3. *There exists θ^*, $0 < \theta^* \leqslant 1$, only depending on α_0, β_0, M, such that for every $\overline{r} > 0$ and every $x_0 \in \Omega_{\overline{r}}$ we have*

$$\int_{B_{2r}(x_0)} |u|^2 \leqslant K \int_{B_r(x_0)} |u|^2, \text{ for every } r,\ 0 < r \leqslant \frac{\theta^* \overline{r}}{2}, \tag{4.16}$$

$$\int_{B_{2r}(x_0)} |\widehat{\nabla} u|^2 \leqslant K \int_{B_r(x_0)} |\widehat{\nabla} u|^2, \text{ for every } r,\ 0 < r \leqslant \frac{\theta^* \overline{r}}{4}, \tag{4.17}$$

where $K > 0$ depends on α_0, β_0, M, r_0, M_0, $|\Omega|$, $\overline{r}$ and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$ only, and θ^* is the quantity introduced in Corollary 3.10.

Proof. The proofs of (4.16) and (4.17) are similar. Let us prove (4.17), which takes a little bit more work. Given $x_0 \in \Omega_{\overline{r}}$ and r, $0 < r < \frac{\theta^* \overline{r}}{4}$, we may apply Corollary 3.10 with $R = \overline{r}$, obtaining (4.17) with K only depending on α_0, β_0, M and, increasingly, on

$$\tilde{N}_0(v; \theta^* \overline{r}) = \frac{(\theta^* \overline{r})^2 \int_{B_{\theta^* \overline{r}}(x_0)} |\nabla v|^2 + \overline{r}^2 |\nabla(\operatorname{div} v)|^2}{\int_{B_{\theta^* \overline{r}}(x_0)} |v|^2 + \overline{r}^2 |\operatorname{div} v|^2}, \tag{4.18}$$

where v is defined in $B_{\overline{r}}(x_0)$ by

$$v = u - c - W(x - x_0), \tag{4.19}$$

with

$$c = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u, \quad W = \frac{1}{2|B_r(x_0)|} \int_{B_r(x_0)} \nabla u - (\nabla u)^T. \tag{4.20}$$

We have that $\nabla v = \nabla u - W$, $\widehat{\nabla} v = \widehat{\nabla} u$ and $\operatorname{div} v = \operatorname{div} u$. Moreover, by interior regularity estimates (see [ADN64]), we have

$$|W| \leqslant \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |\nabla u| \leqslant \|\nabla u\|_{L^\infty(B_r(x_0))} \leqslant$$
$$\leqslant C\|u\|_{H^1(\Omega)} \leqslant C\|\varphi\|_{H^{-1/2}(\partial\Omega)}, \quad (4.21)$$

where C only depends on α_0, β_0, M, r_0 and $\overline{r}$. Hence

$$\int_{B_{\theta^*\overline{r}}(x_0)} |\nabla v|^2 \leqslant 2\int_{B_{\theta^*\overline{r}}(x_0)} |\nabla u|^2 + |W|^2 \leqslant C\|\varphi\|^2_{H^{-1/2}(\partial\Omega)}, \quad (4.22)$$

where C only depends on α_0, β_0, M, r_0 and $\overline{r}$. By the Caccioppoli-type inequality (3.6) we have

$$\int_{B_{\theta^*\overline{r}}(x_0)} |v|^2 \geqslant C\int_{B_{\frac{\theta^*\overline{r}}{2}}(x_0)} |\nabla v|^2 \geqslant C\int_{B_{\frac{\theta^*\overline{r}}{2}}(x_0)} |\widehat{\nabla} u|^2, \quad (4.23)$$

where C only depends on α_0, β_0, M, $\overline{r}$. If $\frac{\theta^*}{2} \leqslant \frac{\overline{\theta}}{4}$ we may apply Proposition 4.1 with $\rho = \frac{\theta^*\overline{r}}{2}$, whereas if $\frac{\theta^*}{2} \geqslant \frac{\overline{\theta}}{4}$ we may apply Proposition 4.1 with $\rho = \frac{\overline{\theta}\overline{r}}{4}$. In both cases by trace theorems and by the standard Korn inequality we obtain

$$\int_{B_{\frac{\theta^*\overline{r}}{2}}(x_0)} |\widehat{\nabla} u|^2 \geqslant C\int_\Omega |\widehat{\nabla} u|^2 \geqslant C\|\varphi\|^2_{H^{-1/2}(\partial\Omega)}, \quad (4.24)$$

where C only depends on α_0, β_0, M, r_0 M_0, $|\Omega|$, $\overline{r}$ and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$. By (4.22)–(4.24) and by interior regularity estimates (see [ADN64]) we have

$$\tilde{N}_0(v; \theta^*\overline{r}) \leqslant C, \quad (4.25)$$

where C only depends on α_0, β_0, M, r_0 M_0,$|\Omega|$, $\overline{r}$ and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$. Hence the thesis follows. □

Proposition 4.4. (A_p property) *For every* $\overline{r} > 0$ *there exist* $B > 0$ *and* $p > 1$ *such that for every* $x_0 \in \Omega_{\overline{r}}$ *we have*

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |\widehat{\nabla} u|^2\right)\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |\widehat{\nabla} u|^{-2/(p-1)}\right)^{p-1} \leqslant B,$$
$$\text{for every } r,\ 0 < r \leqslant \frac{\theta^*\overline{r}}{4}, \quad (4.26)$$

where θ^* *is the quantity introduced in Corollary 3.10 and where* B, p *only depend on* α_0, β_0, M, r_0, M_0,$|\Omega|$, $\overline{r}$ *and* $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$.

Proof. In view of the results in [CF74] it is enough to prove a reverse Hölder's inequality for $|\widehat{\nabla} u|^2$. Let $v = u - c - W(x - x_0)$, with $c = \frac{1}{|B_{2r}(x_0)|}\int_{B_{2r}(x_0)} u$, $W = \frac{1}{2|B_{2r}(x_0)|}\int_{B_{2r}(x_0)} \nabla u - (\nabla u)^T$. By interior regularity estimates, by the Korn inequality (3.9) and by Proposition 4.3 we have

$$\begin{aligned}\|\widehat{\nabla} u\|_{L^\infty(B_r(x_0))} = \|\widehat{\nabla} v\|_{L^\infty(B_r(x_0))} \leqslant \frac{C}{r^{(n+2)/2}}\|v\|_{H^1(B_{2r}(x_0))} \leqslant \\ \leqslant \frac{C}{r^{n/2}}\|\widehat{\nabla} u\|_{L^2(B_{2r}(x_0))} \leqslant \frac{C}{r^{n/2}}\|\widehat{\nabla} u\|_{L^2(B_r(x_0))},\end{aligned} \tag{4.27}$$

where C only depends on α_0, β_0, M, r_0, M_0,$|\Omega|$, $\overline{r}$ and $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$. □

5 Proofs of Theorem 2.4 and Theorem 2.5

We premise the proof of our main theorems with two auxiliary lemmas.

Lemma 5.1. *Let the elasticity tensor fields* $\mathbb{C}(x)$ *and* $\widetilde{\mathbb{C}}(x)$ *satisfy* (2.5), (2.6) *in* Ω. *Suppose that weak solutions* u, $u_0 \in H^1(\Omega, \mathbb{R}^n)$ *to the traction problems* (2.25)-(2.26), (2.27)-(2.28) *exist. The following identities hold:*

$$\begin{aligned}\int_\Omega (\chi_{\Omega\setminus D}\mathbb{C} + \chi_D\widetilde{\mathbb{C}})\nabla(u - u_0)\cdot\nabla(u - u_0) - \\ - \int_D (\widetilde{\mathbb{C}} - \mathbb{C})\nabla u_0 \cdot \nabla u_0 = \int_{\partial\Omega}(g - g_0)\cdot\varphi,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\int_\Omega \mathbb{C}\nabla(u - u_0)\cdot\nabla(u - u_0) + \\ + \int_D (\widetilde{\mathbb{C}} - \mathbb{C})\nabla u \cdot \nabla u = \int_{\partial\Omega}(g_0 - g)\cdot\varphi,\end{aligned} \tag{5.2}$$

$$\int_D (\widetilde{\mathbb{C}} - \mathbb{C})\nabla u \cdot \nabla u_0 = \int_{\partial\Omega}(g_0 - g)\cdot\varphi, \tag{5.3}$$

where g,$g_0 \in H^{1/2}(\partial\Omega, \mathbb{R}^n)$ *are the traces of* u,u_0, *respectively, on* $\partial\Omega$.

Proof of Lemma 5.1. Let us denote $\mathbb{H} = (\widetilde{\mathbb{C}} - \mathbb{C})$ in Ω. From the weak formulation of the problem (2.25)-(2.26) with $D = D_1$ and $D = D_2$ we get the identity

$$\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla u_1\cdot\nabla w = \\ = \int_\Omega (\mathbb{C}+\chi_{D_2}\mathbb{H})\nabla u_2\cdot\nabla w \qquad for \quad every \quad w\in H^1(\Omega,\mathbb{R}^n), \quad (5.4)$$

where u_i is the solution to (2.25)-(2.26) with $D=D_i$, $i=1,2$, respectively. Subtracting the quantity $\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla u_2\cdot\nabla w$ to both sides of (5.4) we have

$$\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla (u_1-u_2)\cdot\nabla w = \\ = \int_\Omega (\chi_{D_2}-\chi_{D_1})\mathbb{H}\nabla u_2\cdot\nabla w \qquad for \quad every \quad w\in H^1(\Omega,\mathbb{R}^n). \quad (5.5)$$

Choosing $w=u_1$ into (5.5) we get

$$\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla (u_1-u_2)\cdot\nabla u_1 = \int_\Omega (\chi_{D_2}-\chi_{D_1})\mathbb{H}\nabla u_2\cdot\nabla u_1. \qquad (5.6)$$

By using the weak formulation of the traction problems for u_1 and u_2, the left hand side of (5.6) can be rewritten as follows

$$\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla (u_1-u_2)\cdot\nabla u_1 = \int_{\partial\Omega} (g_1-g_2)\cdot\varphi, \qquad (5.7)$$

where $g_i=u_i\,|_{\partial\Omega}$, $i=1,2$, and therefore (5.6) becomes:

$$\int_{\partial\Omega} (g_1-g_2)\cdot\varphi = \int_\Omega (\chi_{D_2}-\chi_{D_1})\mathbb{H}\nabla u_2\cdot\nabla u_1. \qquad (5.8)$$

Choosing $w=u_1-u_2$ into (5.5) and using (5.8) we get

$$\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla (u_1-u_2)\cdot\nabla (u_1-u_2) = \\ = \int_\Omega (\chi_{D_1}-\chi_{D_2})\mathbb{H}\nabla u_2\cdot\nabla u_2 + \int_{\partial\Omega} (g_1-g_2)\cdot\varphi, \quad (5.9)$$

and finally we obtain the fundamental identity

$$\int_\Omega (\mathbb{C}+\chi_{D_1}\mathbb{H})\nabla (u_1-u_2)\cdot\nabla (u_1-u_2) + \int_{D_2\setminus D_1} \mathbb{H}\nabla u_2\cdot\nabla u_2 = \\ = \int_{\partial\Omega} (g_1-g_2)\cdot\varphi + \int_{D_1\setminus D_2} \mathbb{H}\nabla u_2\cdot\nabla u_2, \quad (5.10)$$

which is the analogue to the identity found in Kang, Seo and Sheen [KSS97] for the inverse conductivity problem.

By choosing $D_1 = D$ and $D_2 = \emptyset$ we obtain the first identity (5.1) of the Lemma. The second identity (5.2) follows from (5.10) for $D_1 = \emptyset$ and $D_2 = D$.

To get the third identity (5.3), we choose $w = u_0$ and $w = u$ in the weak formulation of the traction problem (2.25)-(2.26) for $D = D_1$ and $D = \emptyset$, respectively:

$$\int_\Omega (\mathbb{C} + \chi_D \mathbb{H})\nabla u \cdot \nabla u_0 = \int_{\partial\Omega} g_0 \cdot \varphi, \tag{5.11}$$

$$\int_\Omega \mathbb{C}\nabla u_0 \cdot \nabla u = \int_{\partial\Omega} g \cdot \varphi. \tag{5.12}$$

Subtracting (5.12) from (5.11) we obtain identity (5.3). $\square$

Lemma 5.2. *Let* $\mathbb{C}(x)$ *and* $\widetilde{\mathbb{C}}(x)$ *satisfy* (2.5), (2.6) *in* Ω. *Let* ξ_0, ξ_1, $0 < \xi_0 < \xi_1$, *be such that*

$$\xi_0 |A|^2 \leq \mathbb{C}(x)A \cdot A \leq \xi_1 |A|^2 \qquad for \quad a.e. \quad x \in \Omega, \tag{5.13}$$

for any symmetric $n \times n$ *matrix* A, *and let the jump* $(\widetilde{\mathbb{C}}(x) - \mathbb{C}(x))$ *satisfy either* (2.30) *or* (2.31). *Let* u, $u_0 \in H^1(\Omega, \mathbb{R}^n)$ *be the weak solutions to the traction problems* (2.25)-(2.26), (2.27)-(2.28), *respectively.*

If (2.30) *holds, then we have*

$$\frac{\eta\xi_0}{\delta} \int_D |\widehat{\nabla} u_0|^2 \leq \int_{\partial\Omega} (g_0 - g) \cdot \varphi \leq (\delta - 1)\xi_1 \int_D |\widehat{\nabla} u_0|^2; \tag{5.14}$$

if instead (2.31) *holds, then we have*

$$\eta\xi_0 \int_D |\widehat{\nabla} u_0|^2 \leq \int_{\partial\Omega} (g - g_0) \cdot \varphi \leq \frac{1-\delta}{\delta}\xi_1 \int_D |\widehat{\nabla} u_0|^2. \tag{5.15}$$

Proof of Lemma 5.2. Suppose that (2.30) holds. Then, from identity (5.1) we have

$$\int_{\partial\Omega} (g_0 - g) \cdot \varphi \leq \int_D \mathbb{H}\nabla u_0 \cdot \nabla u_0, \tag{5.16}$$

where $\mathbb{H} = (\widetilde{\mathbb{C}} - \mathbb{C})$ in Ω. The inequality below follows by the symmetry properties (2.7),(2.8),(2.9) and the positivity condition (2.30):

$$\int_D \mathbb{H}\nabla u_0 \cdot \nabla u_0 \leq (1+\epsilon) \int_D \mathbb{H}\nabla(u - u_0) \cdot \nabla(u - u_0) + \\ + (1 + \frac{1}{\epsilon}) \int_D \mathbb{H}\nabla u \cdot \nabla u \qquad for \quad every \quad \epsilon > 0. \tag{5.17}$$

Then, from (2.30) we have

$$\begin{aligned}&\int_D \mathbb{H}\nabla u_0\cdot\nabla u_0 \leq\\ &\leq (1+\epsilon)(\delta-1)\left[\int_D \mathbb{C}\nabla(u-u_0)\cdot\nabla(u-u_0)+\frac{1}{\epsilon(\delta-1)}\int_D \mathbb{H}\nabla u\cdot\nabla u\right].\end{aligned} \tag{5.18}$$

Choosing $\epsilon=\frac{1}{\delta-1}$ in (5.18) and by employing identity (5.2) we get

$$\int_D \mathbb{H}\nabla u_0\cdot\nabla u_0 \leq \delta\int_{\partial\Omega}(g_0-g)\cdot\varphi. \tag{5.19}$$

The double inequality (5.14) follows from (5.16), (5.19) and (5.13), (2.30).

In case (2.31) holds, from (5.1) we have

$$\int_{\partial\Omega}(g-g_0)\cdot\varphi \geq \int_D -\mathbb{H}\nabla u_0\cdot\nabla u_0. \tag{5.20}$$

From (5.3) we obtain $\int_{\partial\Omega}(g-g_0)\cdot\varphi=\int_D -\mathbb{H}\nabla u\cdot\nabla u_0$, and reasoning as in (5.17),

$$\begin{aligned}\int_{\partial\Omega}(g-g_0)\cdot\varphi \leq \frac{\epsilon}{2}\int_D -\mathbb{H}\nabla u\cdot\nabla u +&\\ +\frac{1}{2\epsilon}\int_D -\mathbb{H}\nabla u_0\cdot\nabla u_0 \qquad &for \quad every \quad \epsilon>0.\end{aligned} \tag{5.21}$$

By using (5.2), (2.31) and (5.1) we have

$$\begin{aligned}\int_D -\mathbb{H}\nabla u\cdot\nabla u &= \int_{\partial\Omega}(g-g_0)\cdot\varphi+\int_\Omega \mathbb{C}\nabla(u-u_0)\cdot\nabla(u-u_0)\leq\\ &\leq \int_{\partial\Omega}(g-g_0)\cdot\varphi+\frac{1}{\delta}\int_\Omega(\mathbb{C}+\mathbb{H})\nabla(u-u_0)\cdot\nabla(u-u_0)=\\ &= \int_{\partial\Omega}(g-g_0)\cdot\varphi+\frac{1}{\delta}\left[\int_{\partial\Omega}(g-g_0)\cdot\varphi+\int_\Omega \mathbb{H}\nabla u_0\cdot\nabla u_0\right]=\\ &= \frac{\delta+1}{\delta}\int_{\partial\Omega}(g-g_0)\cdot\varphi+\frac{1}{\delta}\int_{\mathbb{D}}\mathbb{H}\nabla u_0\cdot\nabla u_0.\end{aligned} \tag{5.22}$$

Replacing inequality (5.22) in (5.21) we obtain

$$\int_{\partial\Omega}(g-g_0)\cdot\varphi \leq \alpha(\epsilon)\int_D -\mathbb{H}\nabla u_0\cdot\nabla u_0, \tag{5.23}$$

where $\alpha(\epsilon)=\frac{\delta-\epsilon^2}{\epsilon(2\delta-\epsilon\delta-\epsilon)}$. The minimum of $\alpha(\epsilon)$ occurs when $\epsilon=\delta$ and in this case we have

$$\int_{\partial\Omega}(g-g_0)\cdot\varphi\leq\frac{1}{\delta}\int_D-\mathbb{H}\nabla u_0\cdot\nabla u_0. \tag{5.24}$$

The double inequality (5.15) follows from (5.20), (5.24) and (5.13), (2.31). □

Proof of Theorem 2.4. By (2.21), (5.13) follows, with $\xi_0=\min\{2\alpha_0,\gamma_0\}$, $\xi_1=\max\{2\alpha_1,\gamma_1\}$, so that Lemma 5.2 holds.

By standard regularity estimates for elliptic systems (see Agmon, Douglis and Nirenberg [ADN64]), by the Korn inequality, by (5.13) and the weak formulation of the Neumann problem (2.27)-(2.28), we have

$$\|\widehat{\nabla}u_0\|_{L^\infty(D)}\leq C\|u_0\|_{H^1(\Omega)}\leq C\|\widehat{\nabla}u_0\|_{L^2(\Omega)}\leq C\left(\int_{\partial\Omega}g_0\cdot\varphi\right)^{\frac{1}{2}}, \tag{5.25}$$

where the constant C only depends on α_0, γ_0, M, d_0 and $|\Omega|$.

The lower bound for $|D|$ in (2.34), (2.35) follows from the right hand side of (5.14),(5.15) and from (5.25).

Let us prove the upper bound for $|D|$ in (2.34), (2.35).

Let $\epsilon=\min\{\frac{\bar{\theta}d_0}{2},\frac{h_1}{\sqrt{n}}\}$, where $\bar{\theta}$ is the same which appears in Proposition 4.1. Let us cover D_{h_1} with internally non overlapping closed cubes Q_l of side ϵ, for $l=1,...,L$. By the choice of ϵ the cubes Q_l are contained in D. Hence

$$\int_D\left|\widehat{\nabla}u_0\right|^2\geq\int_{\bigcup_{l=1}^L Q_l}\left|\widehat{\nabla}u_0\right|^2\geq\frac{|D_{h_1}|}{\epsilon^n}\int_{Q_{\bar{l}}}\left|\widehat{\nabla}u_0\right|^2, \tag{5.26}$$

where $\bar{l}$ is such that $\int_{Q_{\bar{l}}}\left|\widehat{\nabla}u_0\right|^2=\min_l\int_{Q_l}\left|\widehat{\nabla}u_0\right|^2$. Let $\bar{x}$ be the center of $Q_{\bar{l}}$. From (5.26), estimate (4.1) with $x=\bar{x}$ and $\rho=\epsilon/2$, (5.13) and from the weak formulation of (2.27)-(2.28) we have

$$\int_D\left|\widehat{\nabla}u_0\right|^2\geq K|D|\int_{\partial\Omega}g_0\cdot\varphi, \tag{5.27}$$

where K depends on α_0, β_0, d_0, $|\Omega|$, r_0, M_0, M, h_1, $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$ only. The upper bound for D in (2.34),(2.35) follows from the left hand side of (5.14),(5.15) and from (5.27). □

Proof of Theorem 2.5. Let $\bar{r}=\frac{d_0}{2}$ and $\epsilon=\min\{\frac{\theta^* d_0}{4\sqrt{n}},\frac{\bar{\theta}d_0}{4}\}$, where θ^* is the same which appears in Proposition 4.4. Let us cover D with internally non overlapping closed cubes Q_j, $j = 1, ..., J$, with side ϵ. By Hölder inequality we have

$$|D| \leq \left(\int_{\bigcup_{j=1}^{J} Q_j} |\widehat{\nabla} u_0|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_D |\widehat{\nabla} u_0|^2 \right)^{\frac{1}{p}}, \tag{5.28}$$

where p is chosen as in Proposition 4.4. By applying Proposition 4.4 to the balls B_j circumscribing each Q_j, $j = 1, ..., J$, we have

$$\left(\int_{\bigcup_{j=1}^{J} Q_j} |\widehat{\nabla} u_0|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}} = \left(\epsilon^n \sum_{j=1}^{J} \frac{1}{|Q_j|} \int_{Q_j} |\widehat{\nabla} u_0|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}} \leq$$

$$\leq \left(\epsilon^n \sum_{j=1}^{J} \left(\frac{B[C(n)]^p}{\frac{1}{|Q_j|} \int_{Q_j} |\widehat{\nabla} u_0|^2} \right)^{\frac{1}{p-1}} \right)^{\frac{p-1}{p}} \leq \frac{(J\epsilon^n)^{\frac{p-1}{p}} B^{\frac{1}{p}} C(n)}{\min_j \left(\frac{1}{|Q_j|} \int_{Q_j} |\widehat{\nabla} u_0|^2 \right)^{\frac{1}{p}}}, \tag{5.29}$$

where $C(n) = \omega_n \left(\frac{\sqrt{n}}{2} \right)^n$ and B is as in Proposition 4.4. Now $J\epsilon^n = \sum_{j=1}^{J} |Q_j| \leq |\Omega|$ and hence, from (5.28), we have

$$|D| \leq |\Omega|^{\frac{p-1}{p}} B^{\frac{1}{p}} C(n) \left(\frac{\epsilon^n \int_D |\widehat{\nabla} u_0|^2}{\min_j \int_{Q_j} |\widehat{\nabla} u_0|^2} \right)^{\frac{1}{p}}. \tag{5.30}$$

By Proposition 4.1, (5.30), (5.13) and the weak formulation of (2.27)-(2.28), we have

$$\int_D |\widehat{\nabla} u_0|^2 \geq \left(K \int_{\partial\Omega} g_0 \cdot \varphi \right) |D|^p, \tag{5.31}$$

where K depends on α_0, β_0, d_0, $|\Omega|$, r_0, M_0, M, h_1, $\|\varphi\|_{L^2(\partial\Omega)}/\|\varphi\|_{H^{-1/2}(\partial\Omega)}$ only. The right hand side of (2.36),(2.37) follow from the left hand side of (5.14),(5.15) and (5.31). $\square$

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