

Numerical relativity as an initial–boundary value problem: Well posedness, constraint preservation and numerical stability

Manuel Tiglio (Louisiana State University)

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Outline

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- Some problems in numerical relativity

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 - Initial value problem in NR → **lot of work done**
 - Initial-boundary value problem in NR → **Little but promising work done**
 - Numerical stability in NR → **We should sit down and do some work...**

Numerical relativity 101

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 - Equivalent to **convergence** (Lax theorem)
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Discretize them with the iterative Crank–Nicholson with p iterations (pICN) method :

$$u_k^{n+1} = \left[1 + 2 \sum_{j=1}^{p+1} (\lambda AD_0 4^{-1})^j \right] u_k^n \quad , \quad \lambda = \frac{\Delta t}{\Delta x} \quad ,$$

pICN is an **explicit** method for which → CN when $p \rightarrow \infty$.

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- **Completely ill posed case**: The scheme is **unconditionally unstable** for all p . The numerical solution at fixed time grows as $u \propto \exp(t/\Delta x)$.
- **Weakly hyperbolic case**: The scheme is **unconditionally unstable** for all p . Two subcases:
 - The VN condition is violated: then $u \propto \exp(t/\Delta x)$.
 - The VN condition is satisfied: then $u \propto (t/\Delta x)^a$. → **Careful!**

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- Many of the problems in numerical relativity can be traced down to a weakly hyperbolic formulation, or a formulation whose level of hyperbolicity is unknown.
- Example: the standard ADM equations.
- Completely ill posed formulations are rarely used in NR. They are easily spotted in simulations and discarded.

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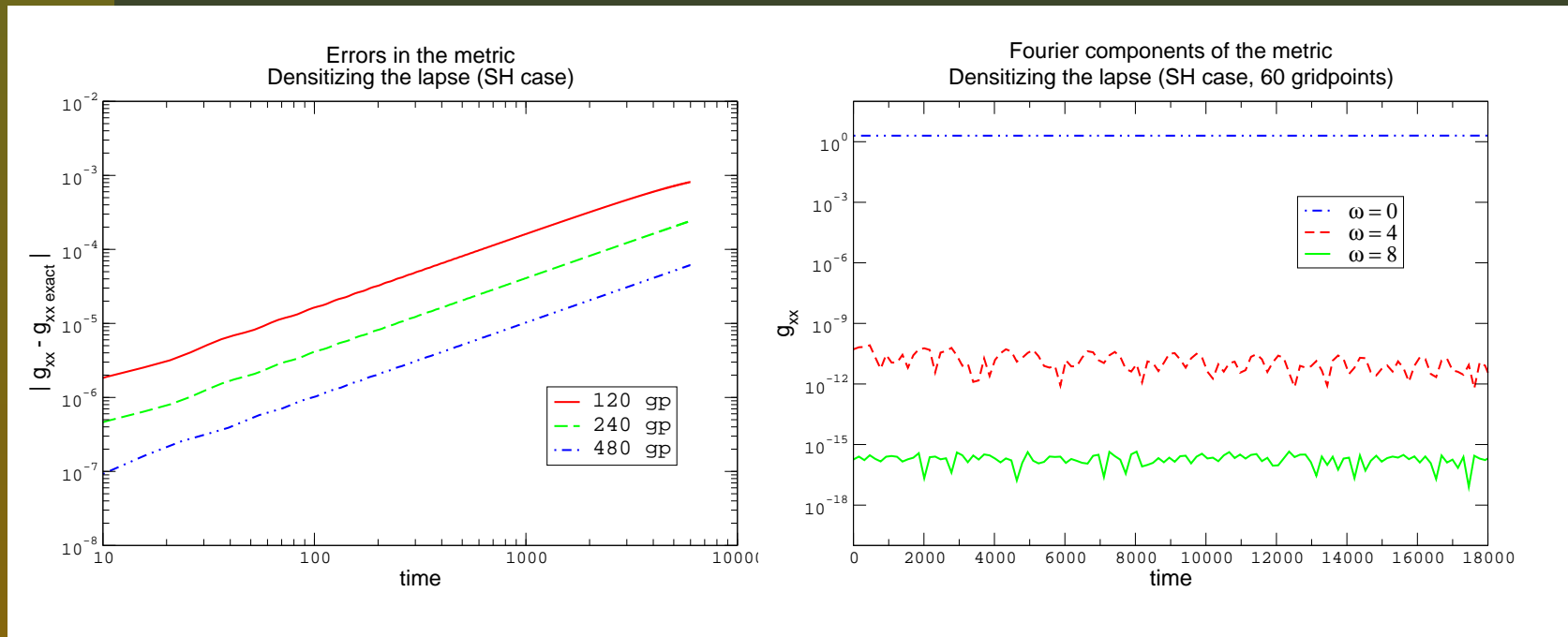
Example: evolve

$$ds^2 = e^{A \sin(\omega t + \omega x)} (-dt^2 + dx^2) + dz^2 + dy^2$$

Choose $A = 0.01 \rightarrow$. The only nontrivial component is $g \approx 1 + A \sin(\omega t + \omega x)$.

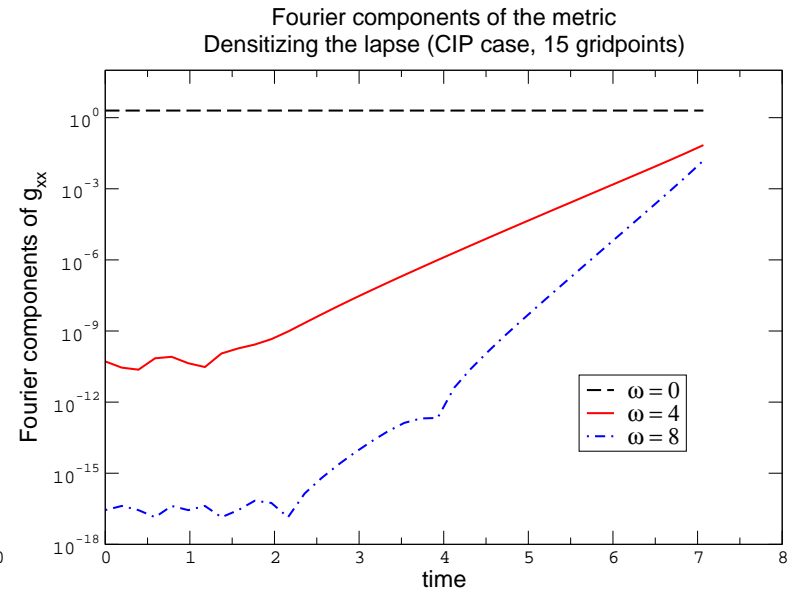
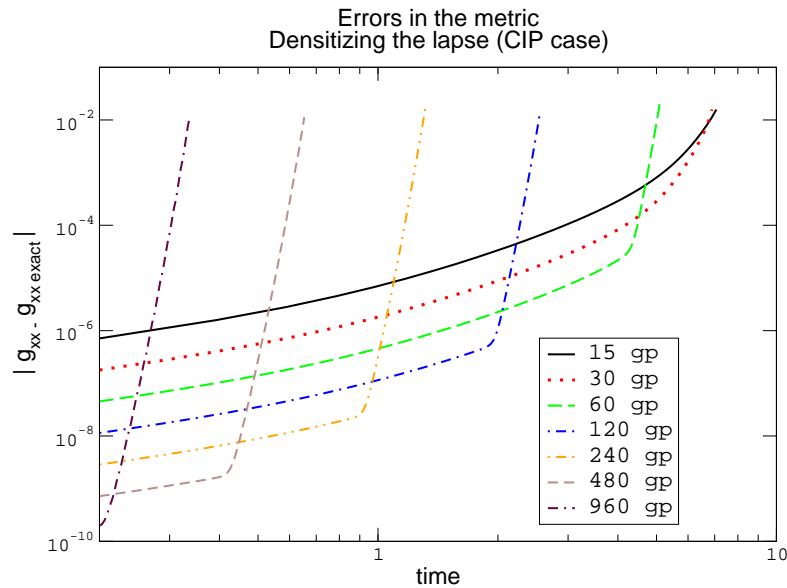
Choose small frequencies in the initial data: $\omega = 1$

The strongly hyperbolic case



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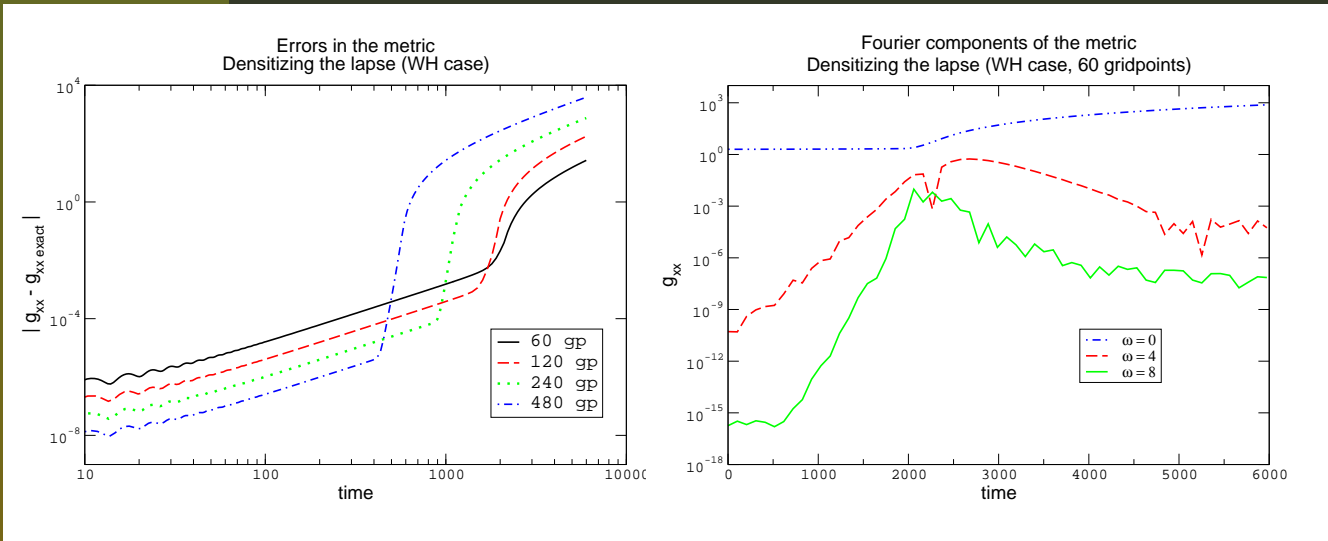
The completely ill posed case



Instabilities become obvious in a very short time scale.

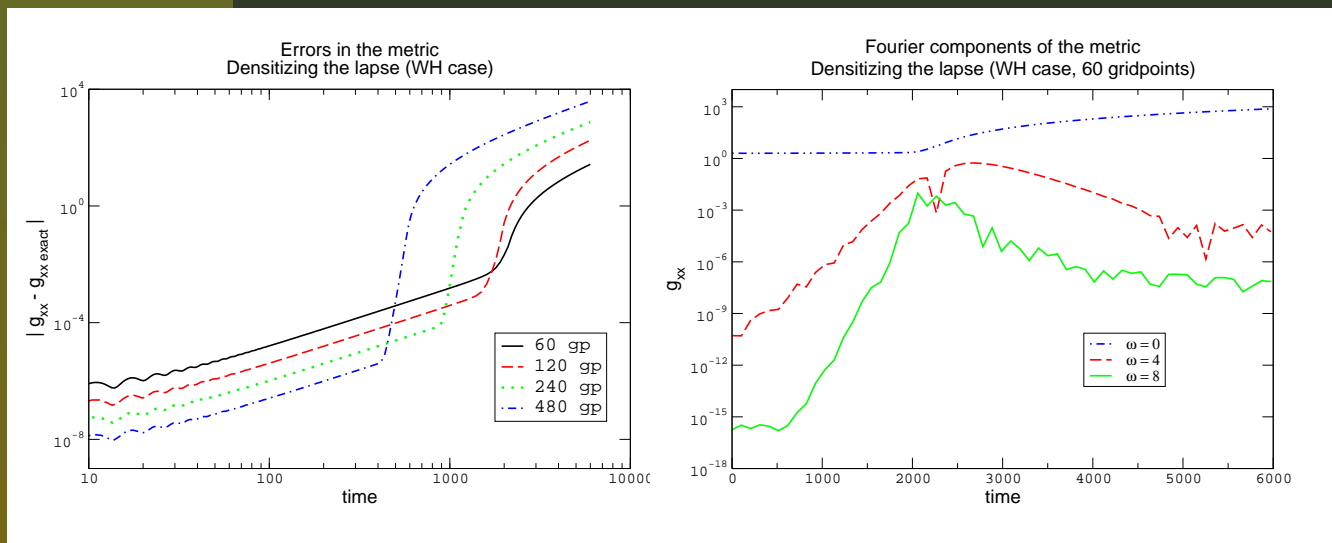
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The weakly hyperbolic hyperbolic case



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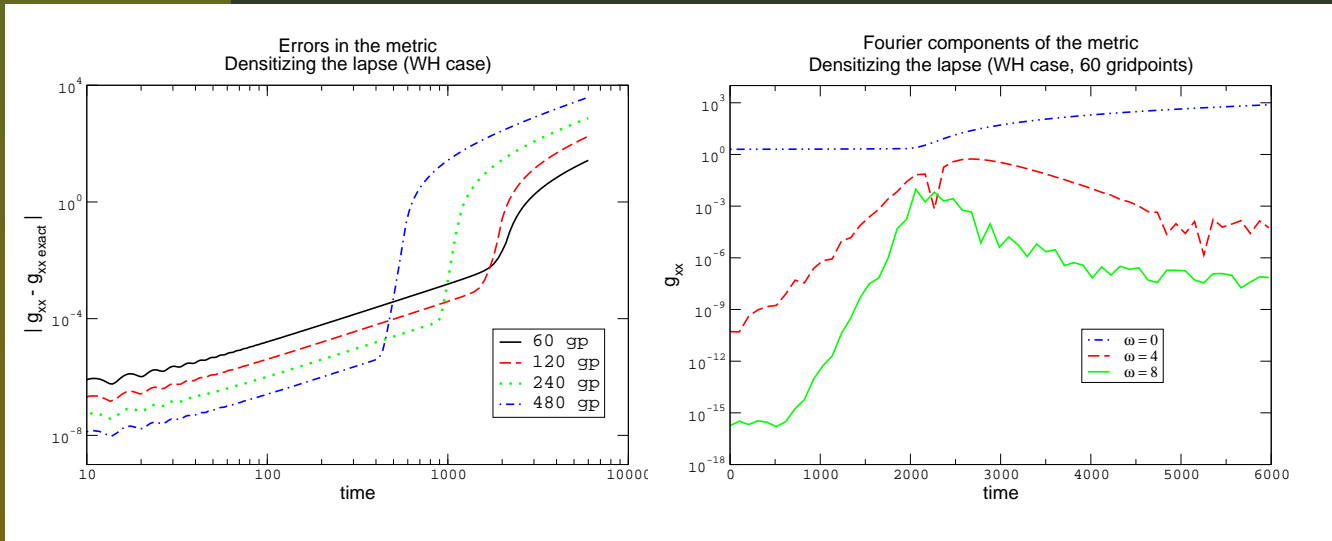
The weakly hyperbolic hyperbolic case



- All frequencies grow exponentially, $u(t) \propto \exp(t/\Delta x)$, some of them (in this example) are not excited initially and start at the order of truncation error.

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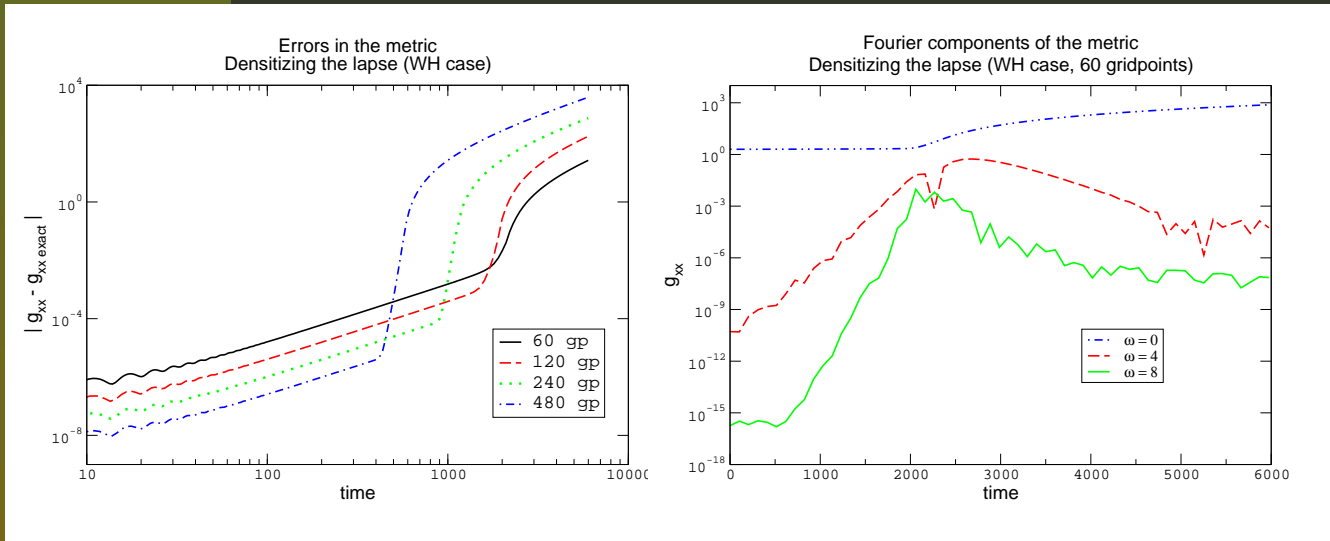
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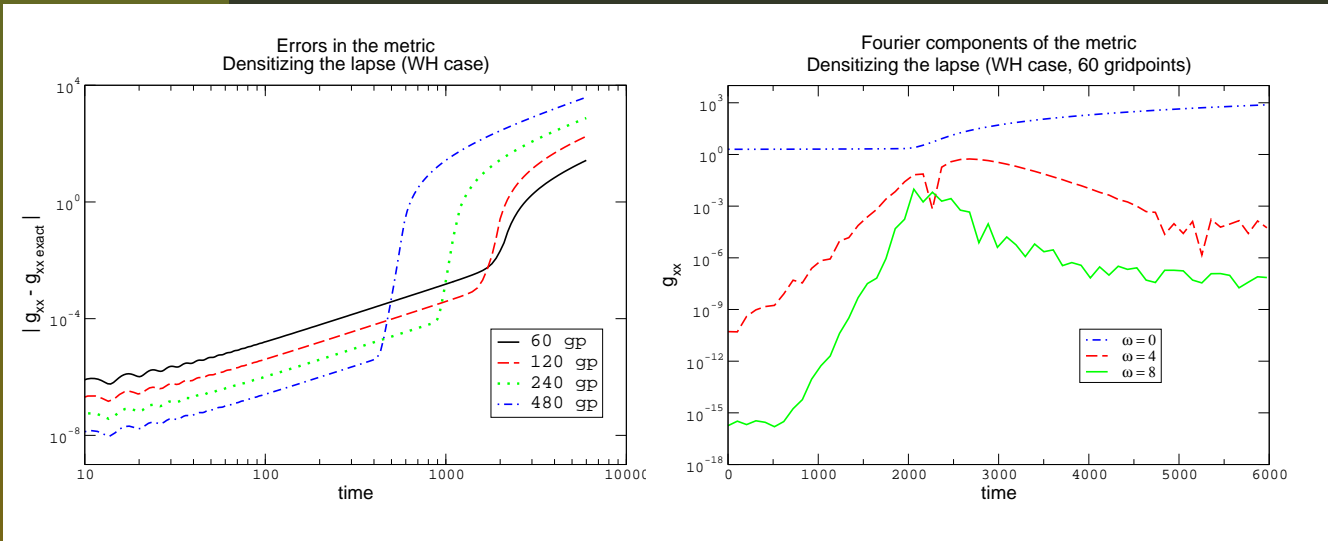
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- Doing a convergence test with a low frequency initial data and couple of coarse resolutions can be misleading.

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- All frequencies grow **exponentially**, $u(t) \propto \exp(t/\Delta x)$, **some** of them (in this example) are not excited initially and **start at the order of truncation error**.
- You don't **"see"** the instability until everything goes to hell.
- Doing a convergence test with a **low frequency initial data** and **couple of coarse resolutions** can be **misleading**.
- Adding more dissipation might help, but exceeding the von-Neumann condition $\sigma \leq 1/(8\lambda)$ is even worse.

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More problems in NR

- Coordinate singularities: the gauge becomes ill defined after a finite time
- The equations are well posed but there are frequency independent instabilities.
- The initial-boundary value problem is well posed but one does not control the incoming radiation.

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- “Gauge conditions” relaxed → Sarbach and Tiglio '02
 - Formulations with algebraic lapse of the form $N = N(g, x^\mu)$ or “live gauges of the form $\partial_t N - \beta^i \partial_i N = F(N, K, x^\mu)$ ”
 - There are **gauge** characteristic modes whose **speed** depend on the solution, **except in the time harmonic or densitized lapse case**.
 - One cannot assume a priori physical characteristic speeds → **Careful with excision !**
 - Does not seem possible to show well posedness with only strong hyperbolicity → Is the Bona-Masso formulation well posed?.

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- The “hyperbolicity” of BSSN → Rendall and Friedrich '01, Sarbach, Calabrese, Pullin and Tiglio '02, Nagy, Ortiz and Reula '02.

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Why is it so difficult?: Consider a linear symmetric hyperbolic system:

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and the energy $E(t) = \int_{\Omega} (u, u) d^3 x$. Imposing boundary conditions for the ingoing characteristic modes of the form

$$u^{(+)} = Ru^{(-)} + g(t, x^A)$$

with R small enough such that $R^T R \leq 1$, one can show that $E(t) \leq E(0) \rightarrow$ **Basic energy estimate for well posedness**

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 - **No numerical** implementation up to now.
- **Iriondo and Reula '01** → 1D nonlinear
 - Non linear spherical black holes + matter.
 - Formulation and gauge **adapted to spherical symmetry**.
 - **Numerical “stability”** of the semidiscrete problem through Strand’s ’94 symmetric operators + complete stability through 4th order Runge-Kutta (Kreiss and Wu ’93)
 - Nice and clean physics obtained in a very simple way.

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- Calabrese, Lehner and Tiglio '01 → 1D nonlinear
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 - Relaxed some of Iriondo–Reula's conditions:
 - Einstein–Christoffel symmetric hyperbolic formulation.
 - Moved to lower (second) order scheme.
 - Similar results → **Remark for physicists**: Good tail decay obtained with an observer in the **last grid point**

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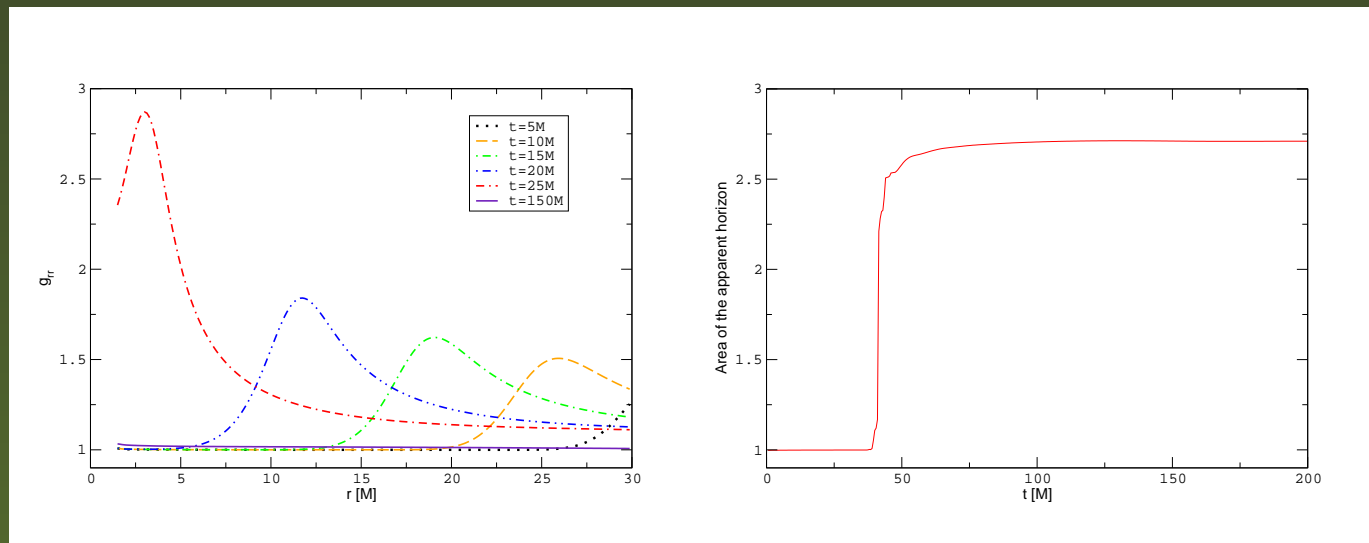
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Pulse falling into a black hole, Mass change of the black hole



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 - Numerical implementation showing “robust stability” (no exponential growth in time)

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 - Robust stability
 - At the linear level matched with Characteristic evolution (be patient, LL will talk about Cauchy-Characteristic matching in three days)

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- Calabrese, Pullin, Reula, Sarbach and Tiglio '02 → 3D linear
 - Generalized symmetric hyperbolic Einstein-Christoffel formulation
 - Cubic box, details of **corners and edges** taken into account
 - **Stability** of the semidiscrete problem through Strand's '94 symmetric operators + Olsson's '95 generalization to 3D, complete stability through 3rd-4th order Runge-Kutta (Kreiss and Wu '93)? Levy and Tadmor '98, Tadmor '01?

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- Well posed, constraint-preserving boundary conditions with **no incoming radiation** (control on the Weyl)
 - Friedrich and Nagy '98 → seminal work, but not with NR in mind
 - Nagy and Reula, work in progress. Need to consider higher order systems