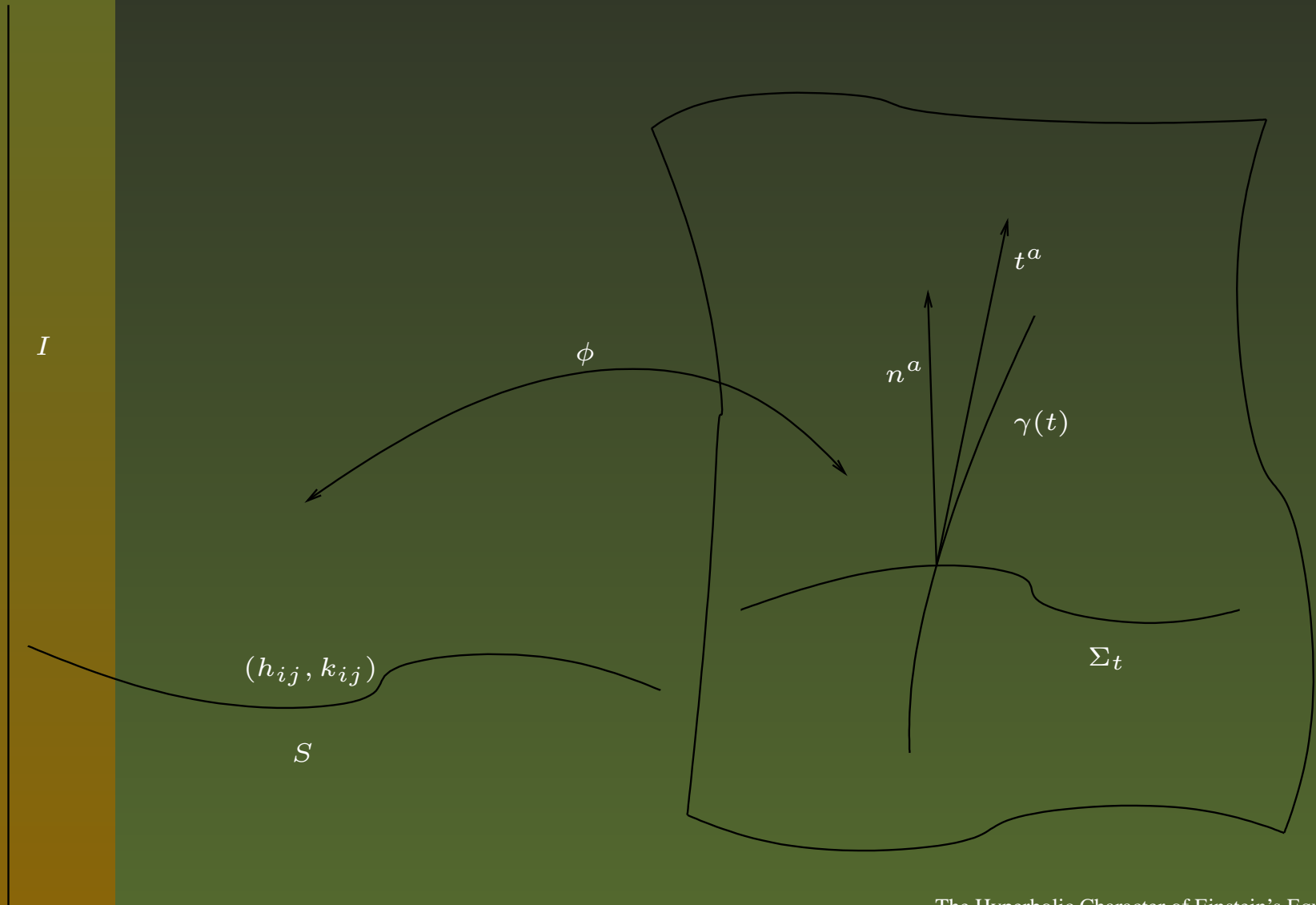


The Hyperbolic Character of Einstein's Equations

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The Setting



ADM equations I

$$G_{ab} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \mathcal{L}_n h_{ab} = -2k_{ab}, \\ \mathcal{L}_n k_{ab} = {}^{(3)}R_{ab} - 2k_a{}^c k_{bc} + k_{ab} k_c{}^c - \frac{D_a D_b N}{N}, \\ {}^{(3)}R + (k_c{}^c)^2 - k_{ab} k^{ab} = 0, \\ D^b k_{ba} - D_a k = 0, \end{array} \right.$$

First Order Systems I

$$K^{mA}{}_{\alpha}(\phi)\nabla_m\phi^{\alpha} - J^A(\phi) = 0$$

- We say that the system is symmetric hyperbolic if:
 - There exists $h_{A\beta}$ such that $h_{A\beta}K^{mA}{}_{\alpha}$ is symmetric.
 - For some n_a $h_{A\beta}K^{mA}{}_{\alpha}n_m$ is positive definite.
- We say that the system is strongly hyperbolic if:
 - For each ω_a there exists $h_{A\beta}(\omega)$ such that $h_{A\beta}(\omega)K^{mA}{}_{\alpha}\omega_m$ is symmetric.
 - For some n_a , $h_{A\beta}(n)K^{mA}{}_{\alpha}n_m := H_{\beta\alpha}$ is positive definite.
- Strong Hyperb. $\Leftrightarrow H^{-1\beta\gamma}h_{A\gamma}(\omega)K^{mA}{}_{\alpha}\omega_m$ has real eigenvalues and a complete set of eigenvectors. Weakly Hyperbolic \Leftrightarrow real eigenvalues.

First Order Systems II

A first order system has a well posed initial value formulation if there exists some norm and a smooth function $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|\phi(t)\| \leq F(\|\phi(0)\|, t)\|\phi(0)\|$$

- A first order system has a well posed initial value formulation stable under zeroth order changes iff it is strongly hyperbolic.
- In particular well posed systems have unique solutions for a given initial data.
- Is there an algebraic characterisation of systems well posed “up to gauge”?
- Note that the solution can grow very fast, in particular in an exponential way.
- The hyperbolicity definitions are invariant under change of variables (when properly formulated).
- Lot of issues about regularity, minimal Sobolev norms, etc. which we skip.

First Order Systems III

- The set of constraints equations is fixed once one chooses a foliation. (a first order system of equations have constraints if there exists C_A^n such that

$$C_A^{(n} K^m)A_{\alpha} = 0)$$

- A system can have many different hyperbolizations.
- Different hyperbolic systems might have different characteristic fields. (Propagation directions).
- Constraint propagation?
- Do two hyperbolizations differ by a constraint?
- Do the intersection of the characteristics of the evolution and constraint-evolution equations give the characteristics of the whole system?
- Finite propagation speed?

ADM equations II

$$\mathcal{L}_{(t-\beta)} h_{ij} = -2N k_{ij}$$

$$\mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} \left[-\partial_k \partial_l h_{ij} - (1+b) \partial_i \partial_j h_{kl} + 2\partial_k \partial_{(i} h_{j)l} \right] + B_{ij}$$

where

$$B_{ij} := N \left[\gamma_{ikl} \gamma_j^{kl} - \gamma_{ij}^k \gamma_{kl}^l - 2k_i^l k_{jl} + k_{ij} k_l^l - A_{ij} \right],$$

$$\gamma_{ij}^k := \frac{1}{2} h^{kl} (2\partial_{(i} h_{j)k} - \partial_k h_{ij}),$$

$$A_{ij} := a_i a_j - \gamma_{ij}^k a_k - 2\gamma_{ikl} \gamma_j^{(kl)} + \partial_i [(\partial_j N)/N],$$

$$a_i := (\partial_i N)/N.$$

ADM equations III

Hyperbolicity analysis: 1) consider only the principal part, 2) freeze coefficients, 3) substitute all derivatives by Fourier transforms ($\partial_k h_{ij} \rightarrow i\omega_k \hat{h}_{ik}$), and 4) define $\hat{\ell}_{ij} = i\omega \hat{h}_{ij}$. [Kreiss, Ortiz][Taylor]

The associated first order system is then

$$\begin{aligned}\partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2N \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right], \\ \partial_t \hat{k}_{ij} &\hat{=} i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]\end{aligned}$$

with $\tilde{\omega}_i = \omega_i / \omega$.

Result:

- ADM equations are only weakly hyperbolic (3 eigenvectors missing).

ADM equations IV

$$N = h^b Q \quad (h = \text{determinant of } h_{ij})$$

The associated first order system is then

$$\partial_t \hat{\ell}_{ij} \hat{=} i\omega \left[-2N \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right],$$

$$\partial_t \hat{k}_{ij} \hat{=} i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + (1+b) \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]$$

Result:

- Modified ADM equations for $b > 0$ still weakly hyperbolic (2 eigenvectors missing).
- Adding Hamiltonian constraint does not change hyperbolicity, but does change characteristics.

BSSN equations I

$$f^k = h^{ij} \gamma_{ij}{}^k + dh^{kl} \gamma_{lm}{}^m = h^{kl} (h^{ij} \partial_i h_{jl} + \partial_l \ln h)$$

$$\mathcal{L}_{(t-\beta)} h_{ij} = -2N k_{ij}$$

$$\mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} [-\partial_k \partial_l h_{ij} - b \partial_i \partial_j h_{kl}] + N \partial_{(i} f_{j)} + \mathcal{B}_{ij}$$

$$\mathcal{L}_{(t-\beta)} f_i = N [-(2-c) D^k k_{ki} + (1-c) D_i k^k{}_k] + \mathcal{C}_i$$

BSSN equations II

Hyperbolicity Analysis:

$$\begin{aligned}\partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2\alpha \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right] \\ \partial_t \hat{k}_{ij} &\hat{=} i\omega \left[\frac{\alpha}{2} \left(-\hat{\ell}_{ij} - b \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} + 2\tilde{\omega}_{(i} \hat{f}_{j)} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right] \\ \partial_t \hat{f}_i &\hat{=} i\omega \left[\alpha \left((-2 + c) \hat{k}_{ik} \tilde{\omega}^k + (1 - c) \tilde{\omega}_i h^{kl} \hat{k}_{kl} \right) + \tilde{\omega}_k \beta^k \hat{f}_i \right]\end{aligned}$$

Result:

- Modified BSSN equations for $b > 0$ $c > 0$ strongly hyperbolic.
- Eigenvalues: $(0, \pm 1, \pm \sqrt{b}, \pm \sqrt{c/2})$

Further

Having a strongly hyperbolic system (enlarged versions) is just the beginning:

- Look at the problem of constraint propagation.
- Choose nice gauge (Q, β_i) .
- Search for variables which are numerically convenient (less prone to errors and not growing exponentially). Study the lower order terms.
- Set proper boundary conditions. Gives a numerically stable evolution and are consistent with constraint propagation.
- Worry about a million other things.