

# The State of the Art in Numerical Relativity

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## This talk ...

I will focus attention on 2D and 3D numerical evolutions of:

- Single BHs
- BH/BH, NS/NS and BH/NS binaries
- Gravitational collapse

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Not included:

- Bar mode instabilities of rotating stars
- Linear and non-linear waves
- Initial data
- Quasi-equilibrium evolutions
- Numerical quantum gravity

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**State of the Art:** Numerical results with **quality** in the engineering, perhaps not German engineering, sense (e.g. 2 or 3 level of refinements in convergence tests, ICN integrators, non-constraint preserving boundary conditions, etc).

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## Webster Dictionary, 1913

**Hyperbolic, Hyperbolical:** Hy'per\*bol"ic (?), Hy'per\*bol"ic\*al (?), a. [L. hyperbolicus, Gr. : cf. F. hyper bolique.] Relating to, containing, or of the nature of, hyperbole; exaggerating or diminishing beyond the fact; exceeding the truth; as, an hyperbolical expression.

# Formulations of the Einstein Equation

Relevant to this talk

- ADM (Arnowitt, Deser and Misner)
- BSSN (Baumgarte, Shapiro, Shibata and Nakamura)
- WMM (Wilson, Mathews and Maronetti)
- KST (Kidder, Scheel and Teukolsky)
- 2+2 (Winicour et al)

# ADM Formulation

Arnowitt, Deser and Misner (1962)

Primary variables:

$g_{ij}$  : spatial metric

$K_{ij}$  : extrinsic curvature

$\alpha$  : lapse function

$\beta^i$  : shift vector

Evolution equations:

$$\partial_o g_{ij} = -2 \alpha K_{ij}$$

$$\partial_o K_{ij} = -\nabla_i \nabla_j \alpha + \alpha R_{ij} + \alpha K K_{ij} - 2 \alpha K_{ik} K^k_j$$

Constraints:

$$R + K^2 - K_{ij} K^{ij} = 0$$

$$\nabla_j K^{ij} - \nabla^i K = 0$$

Note:  $\partial_o \equiv \partial_t - \mathcal{L}_\beta$

# BSSN Formulations

Baumgarte and Shapiro, PRD 59, 024009 (1999); Shibata and Nakamura, PRD 52, 5428 (1995)

Primary variables:

$$\hat{g}_{ij} = e^{-4\Phi} g_{ij}$$

$$\Phi = \frac{1}{12} \ln g$$

$$\hat{A}_{ij} = e^{-4\Phi} A_{ij}$$

$$K = g^{ij} K_{ij}$$

$$\hat{\Gamma}^i = \hat{g}^{jk} \hat{\Gamma}_{jk}^i$$

$\alpha$  : lapse function

$\beta^i$  : shift vector

Where:

$$\begin{aligned}
 g &= \det(g_{ij}) \\
 K_{ij} &= A_{ij} + \frac{1}{3} g_{ij} K \\
 \Gamma_{jk}^i &= \frac{1}{2} g^{il} (\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk})
 \end{aligned}$$

Constraints:

$$\begin{aligned}
 R + K^2 - K_{ij}K^{ij} &= 0 \\
 \nabla_j K^{ij} - \nabla^i K &= 0 \\
 \hat{A} &= 0 \\
 \hat{g} &= 1 \\
 \hat{\Gamma}^i &= -\partial_j \hat{g}^{ij}
 \end{aligned}$$

Evolution equations:

$$\begin{aligned}
\partial_o \Phi &= -\frac{1}{6} \alpha K \\
\partial_o \hat{g}_{ij} &= -2 \alpha \hat{A}_{ij} \\
\partial_o K &= -\nabla_i \nabla^i \alpha + \alpha \hat{A}_{ij} \hat{A}^{ij} + \alpha K^2 / 3 \\
\partial_o \hat{A}_{ij} &= e^{-4\Phi} (-\nabla_i \nabla_j \alpha + \alpha R_{ij})^{TF} \\
&\quad + \alpha K \hat{A}_{ij} - 2 \alpha \hat{A}_{il} \hat{A}^l_j \\
\partial_o \hat{\Gamma}^i &= \hat{g}^{jk} \partial_{jk} \beta^i + \frac{1}{3} \hat{g}^{ij} \partial_{jk} \beta^k - 2 \hat{A}^{ij} \partial_j \alpha \\
&\quad + 2 \alpha \hat{\Gamma}^i_{jk} \hat{A}^{jk} + 12 \alpha \hat{A}^{ij} \partial_j \Phi - \frac{4}{3} \alpha \hat{\nabla}^i K
\end{aligned}$$

**Note:** The Hamiltonian constraint is used to eliminate  $R$  in the  $\partial_o K$  equation and similarly the momentum constraint to eliminate  $\partial A$  in  $\partial_o \hat{\Gamma}^i$ .

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**Question:** How important are the lower terms?

# Conformal-Traceless-Mixed Formulation

Laguna and Shoemaker, CQG in press

Primary variables:

$$\begin{aligned}
 \hat{g}_{ij} &= e^{-4\Phi} g_{ij} \\
 \Phi &= \frac{1}{12} \ln g \\
 \hat{\Gamma}^i &= \hat{g}^{jk} \hat{\Gamma}_{jk}^i \\
 \hat{A}^i_j &= e^{6n\Phi} A^i_j \\
 \hat{K} &= e^{6n\Phi} K \\
 N &= e^{-6n\Phi} \alpha \\
 \beta^i &: \text{shift vector}
 \end{aligned}$$

Constraints:

Same as BSSN

Evolution equations:

$$\begin{aligned}
 \partial_o \Phi &= -\frac{1}{6} N \hat{K} \\
 \partial_o \hat{g}_{ij} &= -2 N \hat{A}_{ij} \\
 \partial_o \hat{K} &= -e^{6n\Phi} \nabla_i \nabla^i \alpha + N \hat{A}^i_j \hat{A}^j_i \\
 &\quad + (1 - 3n) N \hat{K}^2 / 3 \\
 \partial_o \hat{A}^i_j &= e^{6n\Phi} \left( -\nabla^i \nabla_j \alpha + \alpha R^i_j \right)^{TF} \\
 &\quad + (1 - n) N \hat{K} \hat{A}^i_j \\
 \partial_o \hat{\Gamma}^i &= \hat{g}^{jk} \partial_{jk} \beta^i + \frac{1}{3} \hat{g}^{ij} \partial_{jk} \beta^k - 2 \hat{A}^{ij} \partial_j N + \dots
 \end{aligned}$$

Note:

- $\hat{A}^i_j$  is not symmetric
- Principal part same as BSSN

# WMM Formulation

Wilson, Mathews and Maronetti, PRD **54**, 1317 (1996)

Recall:

$$K_{ij} = A_{ij} + \frac{1}{3} g_{ij} K$$

$$A_{ij} = A_{ij}^{TT} + A_{ij}^{TL}$$

$$\nabla_i A_{TT}^{ij} = 0$$

$$A_{TL}^{ij} = \nabla^i W^j + \nabla^j W^i - \frac{2}{3} g^{ij} \nabla_k W^k$$

Assumptions:

$$g_{ij} = \Phi^4 \delta_{ij}$$

$$A_{ij}^{TT} = 0$$

$$K = 0$$

ADM,

$$\partial_o g_{ij} = -2\alpha K_{ij}$$

$$\begin{aligned} \partial_o K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k_j \\ &\quad - \alpha S_{ij} - \frac{1}{2} \alpha g_{ij} (\rho - S) \end{aligned}$$

$$R + K^2 - K_{ij} K^{ij} = \rho$$

$$\nabla_j K^{ij} - \nabla^i K = J^i$$

WMM,

$$A_{TL}^{ij} = \frac{1}{2\alpha} (\nabla^i \beta^j + \nabla^j \beta^i - \frac{2}{3} g^{ij} \nabla_k \beta^k) \quad \partial_o g_{ij}$$

$$\hat{\Delta}(\alpha \Phi) = \dots + \rho + \dots \quad \partial_o K$$

$$\hat{\Delta} \Phi = \dots + \rho + \dots \quad \text{Hamiltonian Constraint}$$

$$\hat{\Delta} \beta^i = \dots + J^i + \dots \quad \text{Momentum Constraint}$$

**Note:** Evolution is obtained via the evolution equations of the matter fields.

# KST Formulation

Kidder, Scheel and Teukolsky, PRD **62**, 084032 (2000)

Field variables:

$$\begin{aligned}
 g_{ij} & : \text{spatial metric} \\
 K_{ij} & : \text{extrinsic curvature} \\
 d_{kij} & \equiv \partial_k g_{ij} \\
 Q & \equiv \ln(\alpha g^{-\sigma}) \\
 \beta^i & : \text{shift vector}
 \end{aligned}$$

redefine

$$\begin{aligned}
 P_{ij} & \equiv K_{ij} + \hat{z} g_{ij} K \\
 M_{kij} & \equiv \frac{1}{2} \hat{k} d_{kij} + \frac{1}{2} \hat{e} d_{(ij)k} \dots
 \end{aligned}$$

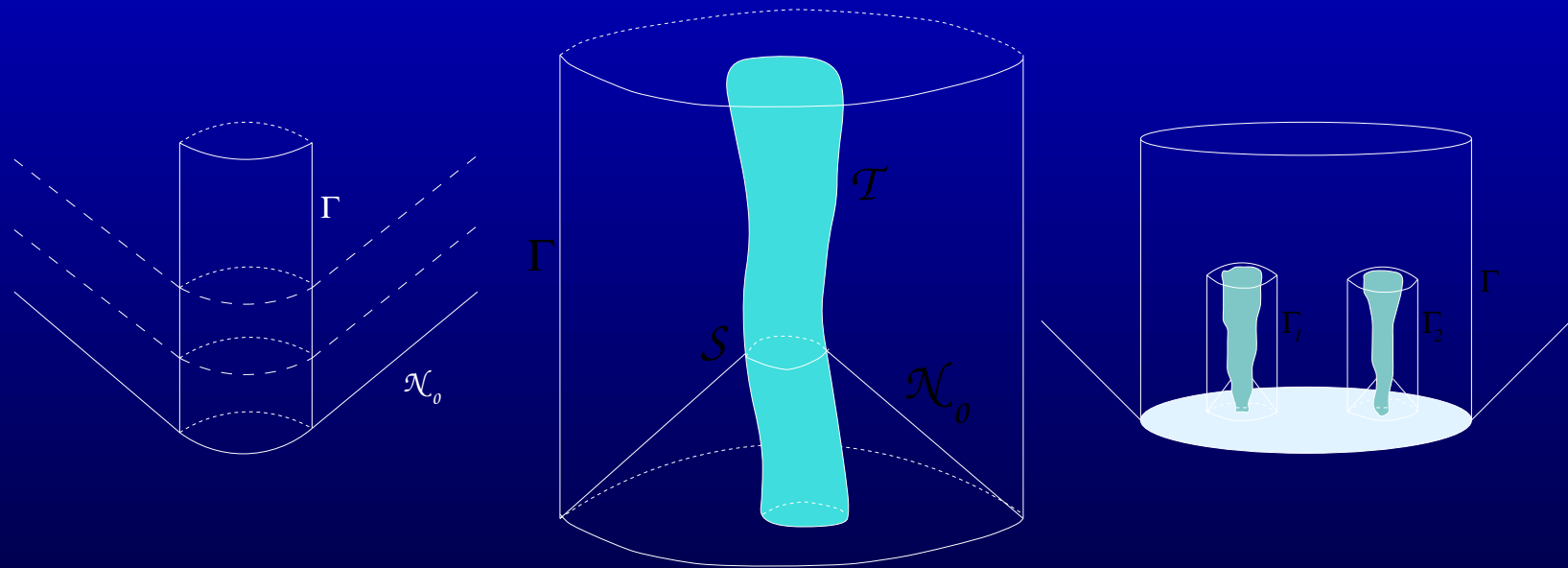
Evolution equations:

$$\begin{aligned} \partial_o g_{ij} &= \dots \\ \partial_o P_{ij} + \alpha g^{kl} (\mu_1 \partial_k M_{lij} + \dots) &= \dots \\ \partial_o M_{kij} + \alpha (\nu_1 \partial_k K_{ij} + \dots) &= \dots \end{aligned}$$

**Note:** Parametrized hyperbolic formulation.  $\{\mu_i, \nu_i\}_{i=1\dots 6}$

# 2+2 or Characteristic Formulation

Winicour et al



# Single BH evolutions

- Partially<sup>1</sup> useful for testing and developing:
  - ★ Black hole excision
  - ★ Gauge conditions
  - ★ Formulations of Einstein equations
  - ★ Convergence
  - ★ Comparison with 1D results
- Limited use for testing outer boundary conditions

Ingoing Eddington Finkelstein:

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \frac{4m}{r} dt dr + \left( 1 + \frac{2m}{r} \right) dr^2 + r^2 d\Omega^2$$

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<sup>1</sup>If you agree with Matt, please substitute “Partially” with “Not”

## An approach that does not work:

- Initial data:  $\{g_{ij}, K_{ij}\}$  from exact solution.
- Outer boundary conditions: from exact solution, unless small domains.
- Gauge conditions:  $\{\alpha, \beta^i\}$  from exact solution
- Unconstrained evolution of  $g_{ij}$  and  $K_{ij}$

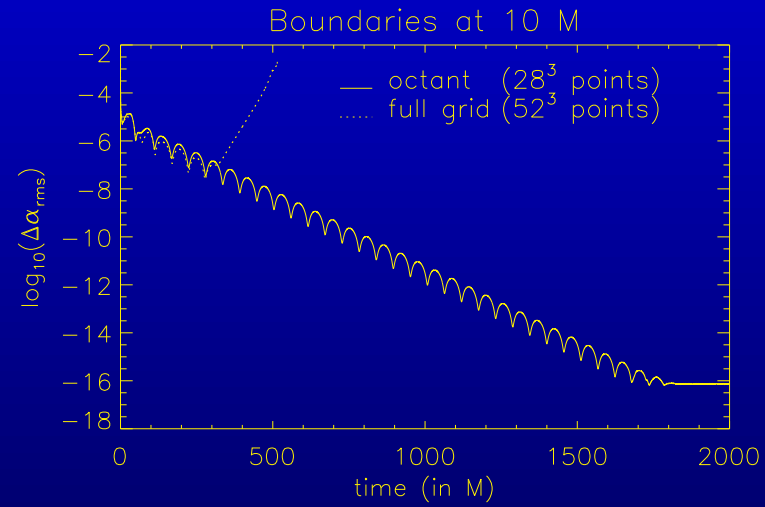
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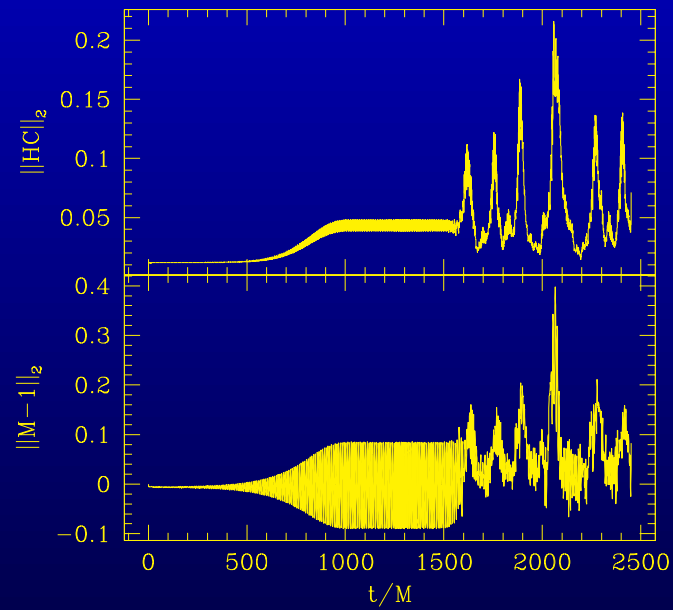
### A better approach:

- Initial data:  $\{g_{ij}, K_{ij}\}$  from exact solution.
- Outer boundary conditions:  $u_n = u_e + \phi(t - r)/r$
- Gauge conditions:  $\{\alpha, \beta^i\}$  from  $t$ -dependent equations that have as  $t$ -independent solutions the exact solutions.
- Unconstrained evolution of  $g_{ij}$  and  $K_{ij}$

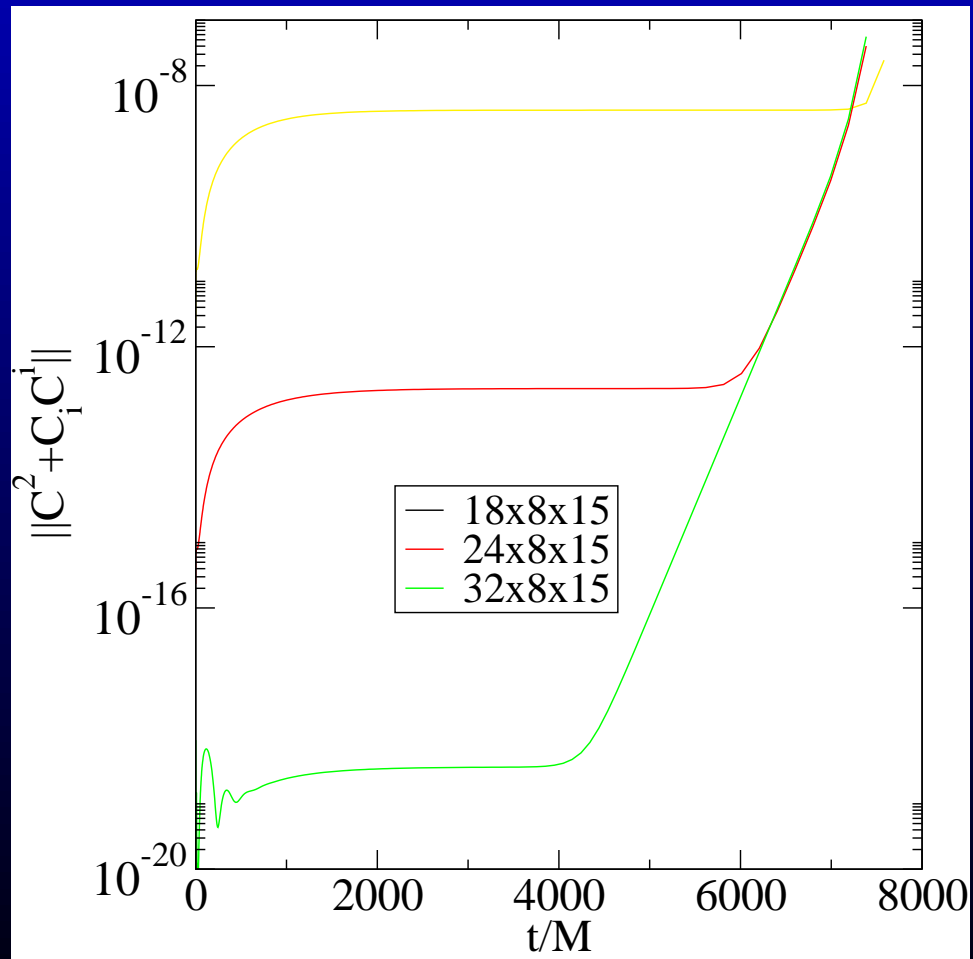
# BSSN: Alcubierre and Bruegmann (2001)



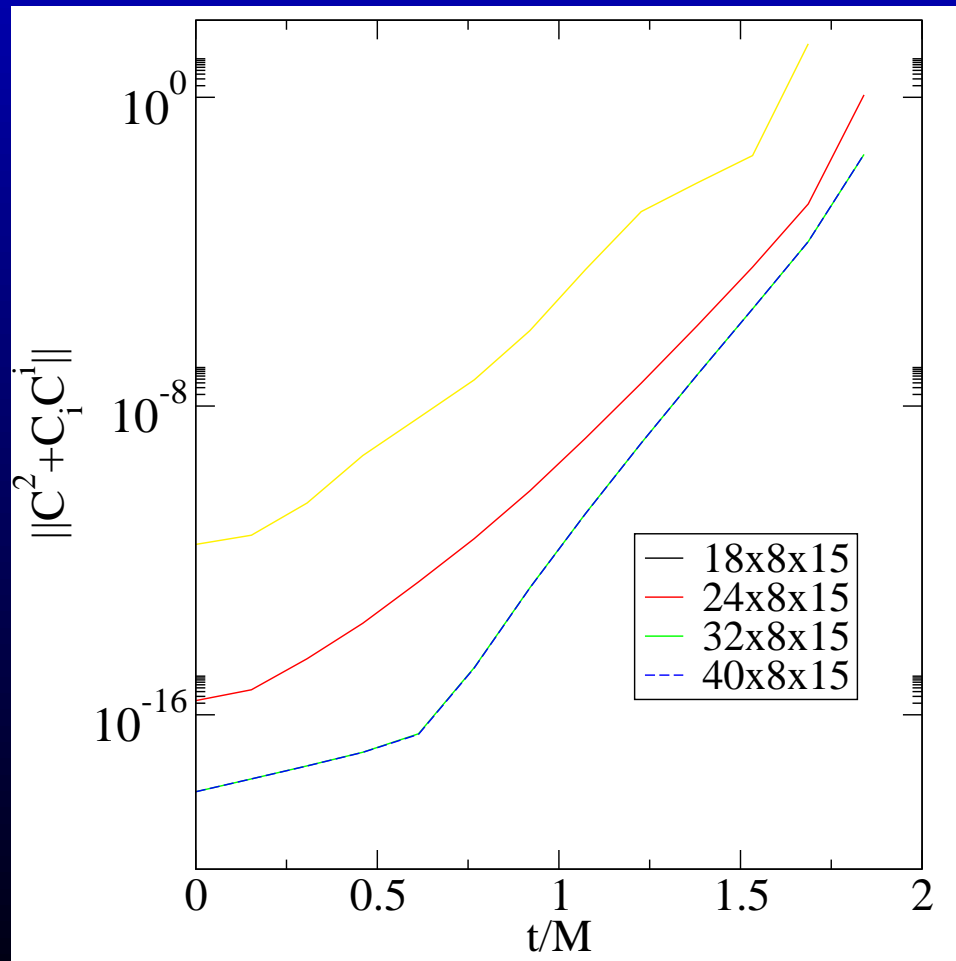
# BSSN: Shoemaker, Smith and Laguna (2002)



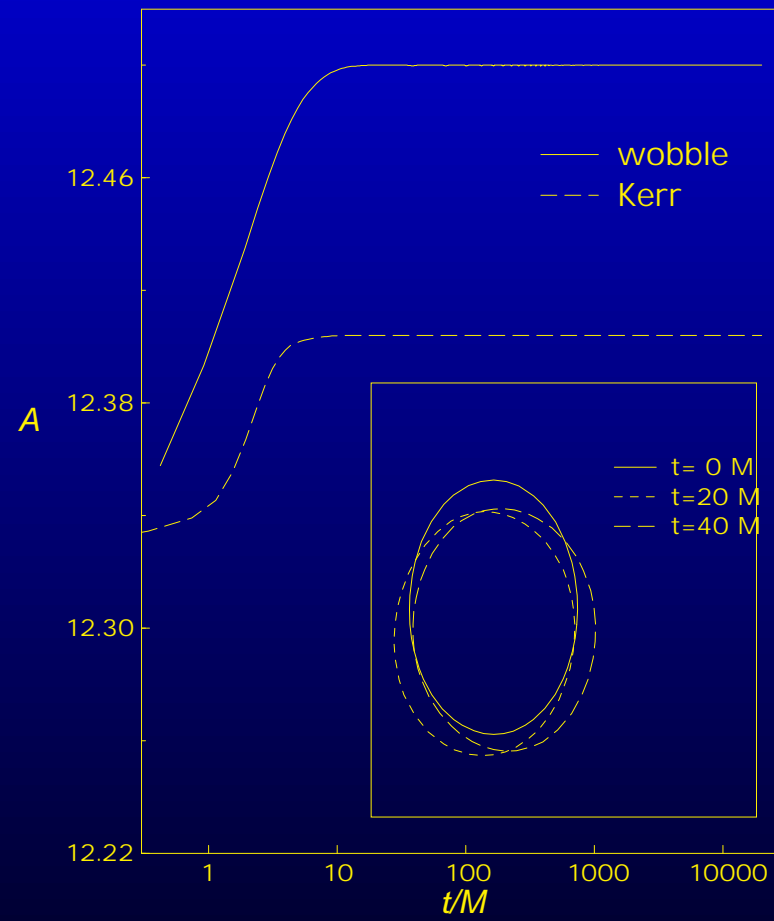
# KST: Scheel et al (2002)



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## 2+2: Winicour et al (2001)



# BH/BH Binaries

## Grazing Collisions:

- Texas-PSU-Pitt
- AEI

## Binary Orbits:

- Cornell-Caltech
- AEI

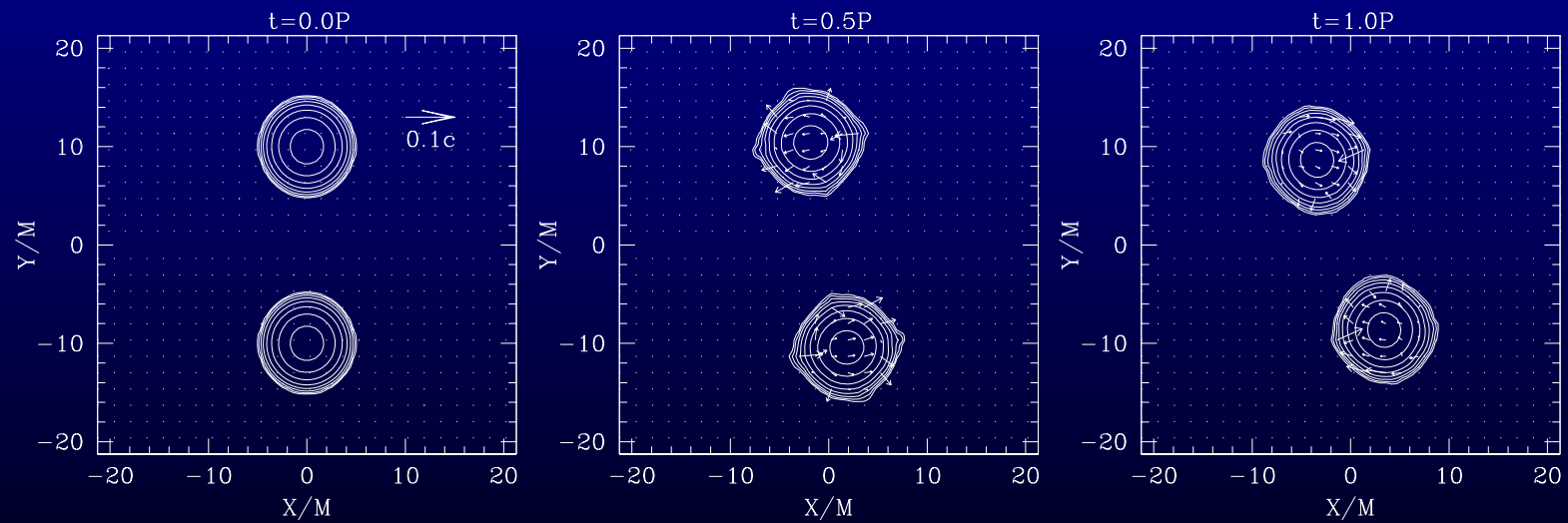
## Head-on Collisions:

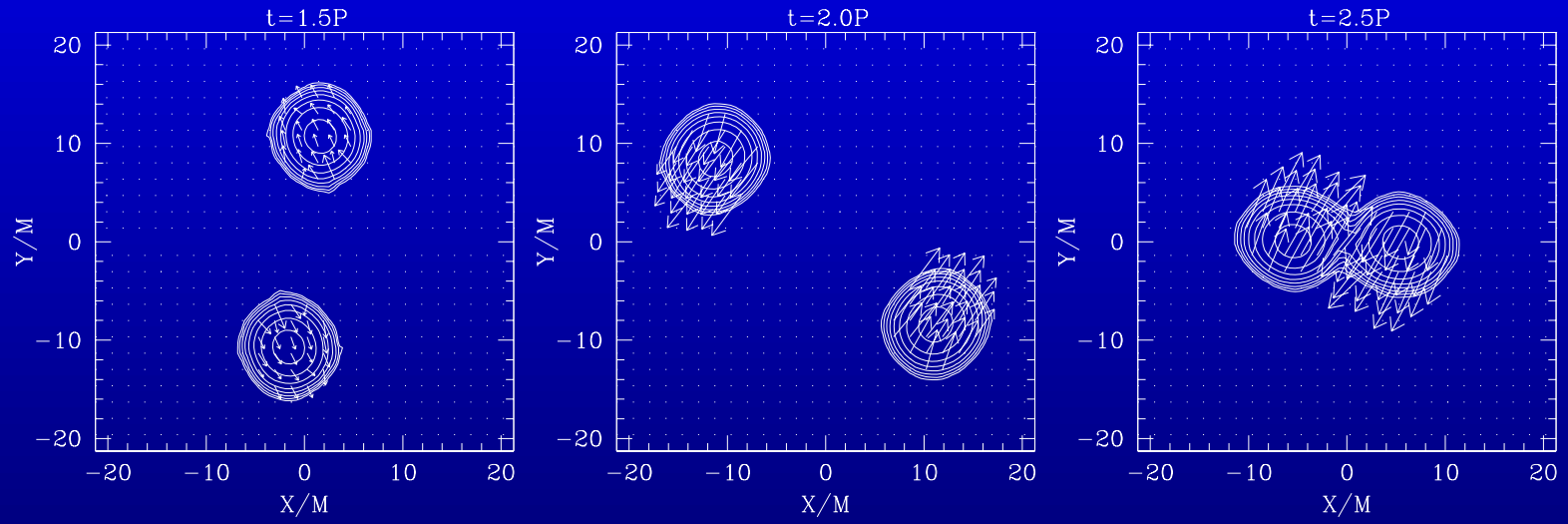
- AEI

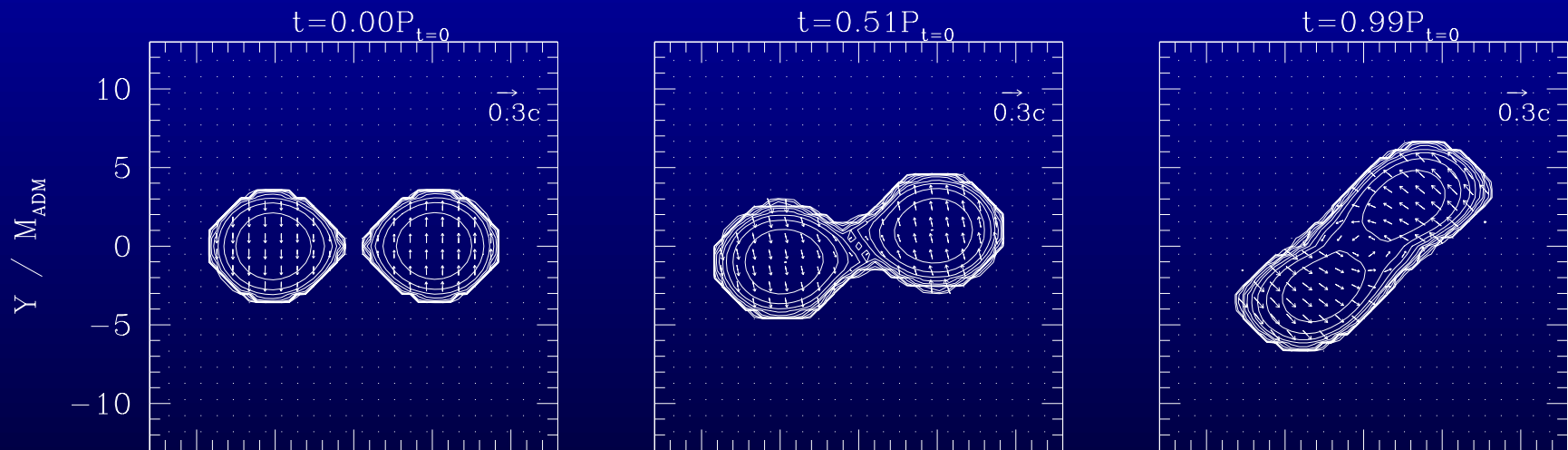
# NS/NS Binaries

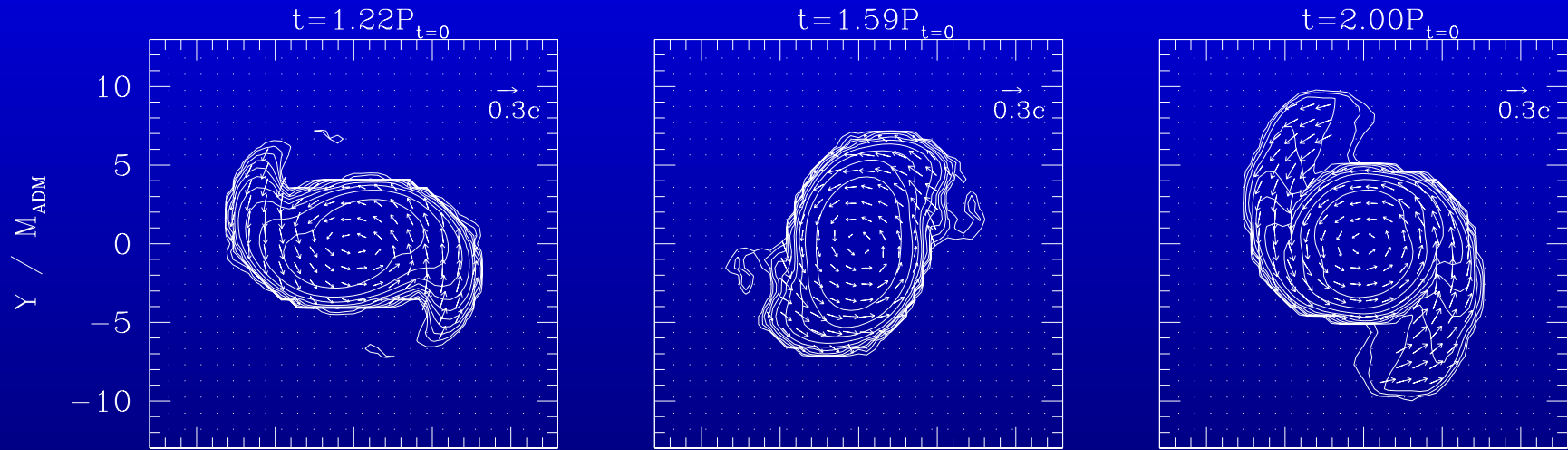
BSSN: Miller and Suen (2002)

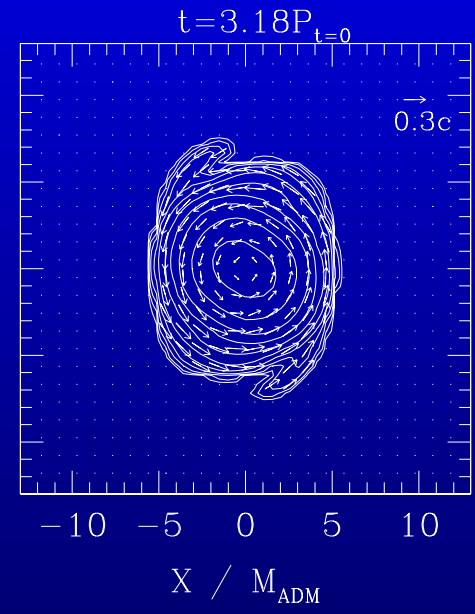
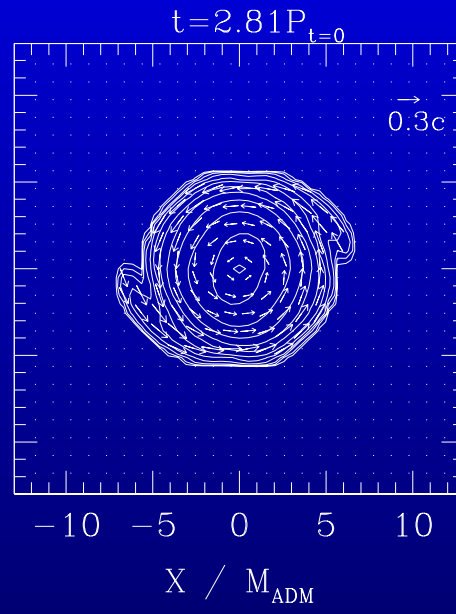
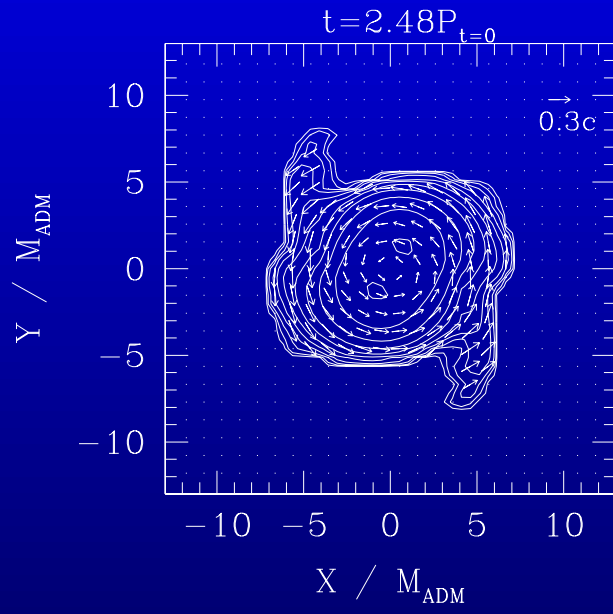
BSSN: Shapiro, et al (2002)

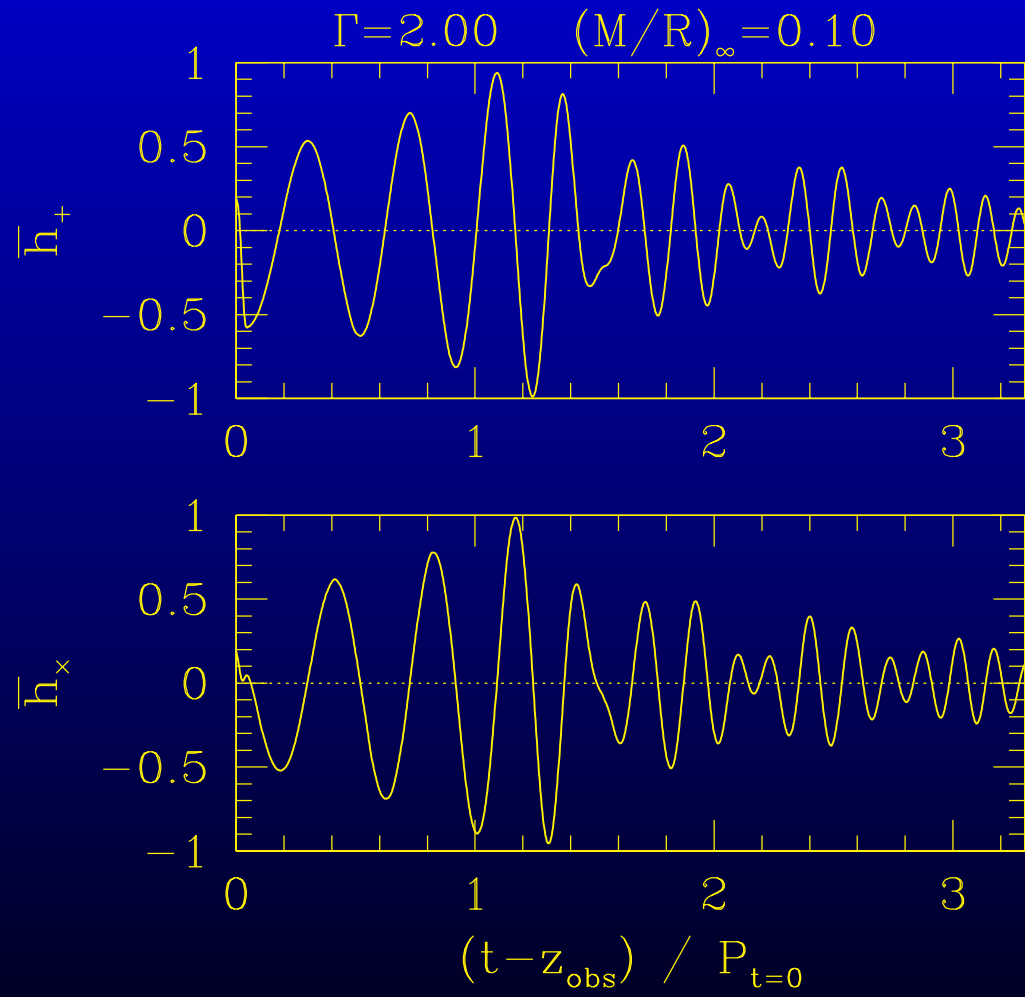




**BSSN:** Shibata and Uryu (2002)

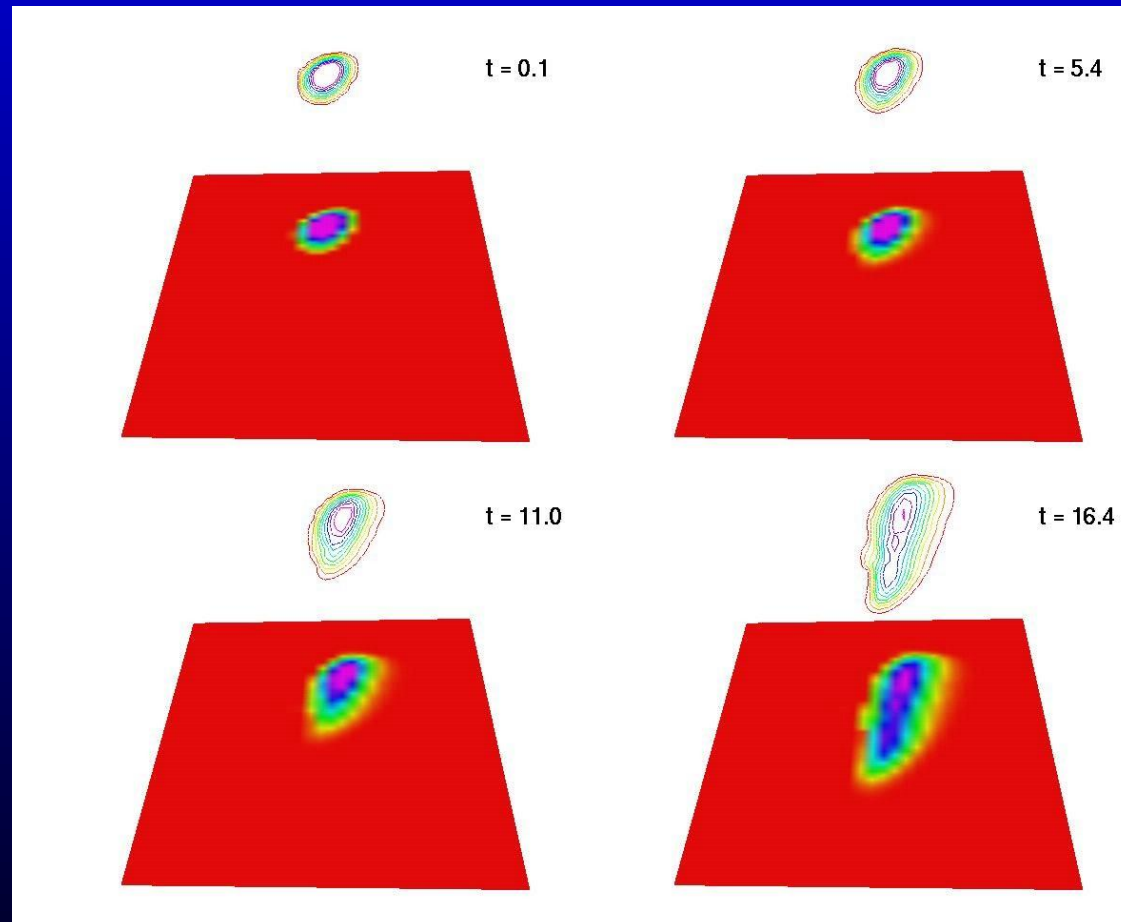






# BH/NS Binaries

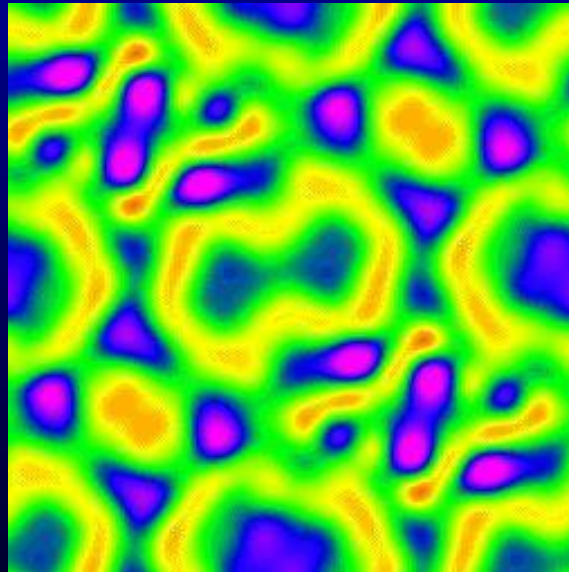
2+2: Lehner, Winicour, et al (2002)



# Gravitational Collapse

Collapsing Cosmological Spacetimes: Berger et al (2002)

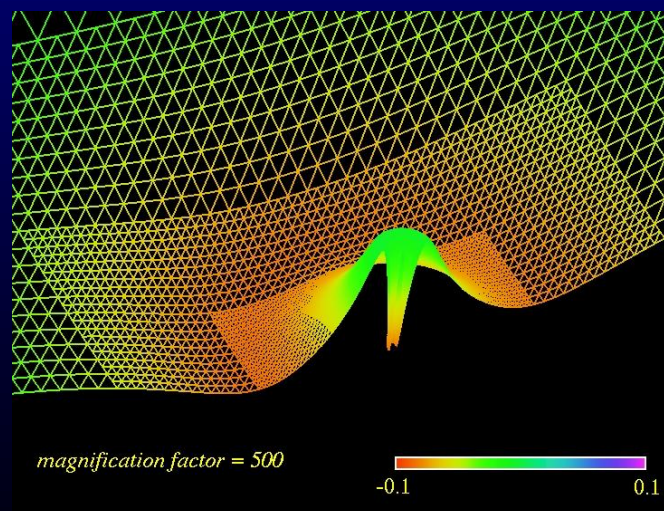
- Spatially compact spacetimes with vacuum or scalar field matter.
- Asymptotic behavior in the approach to the big crunch singularity.
- Interplay of numerical simulations and mathematical analysis.
- Test of the evolution algorithms with trivial boundary conditions.

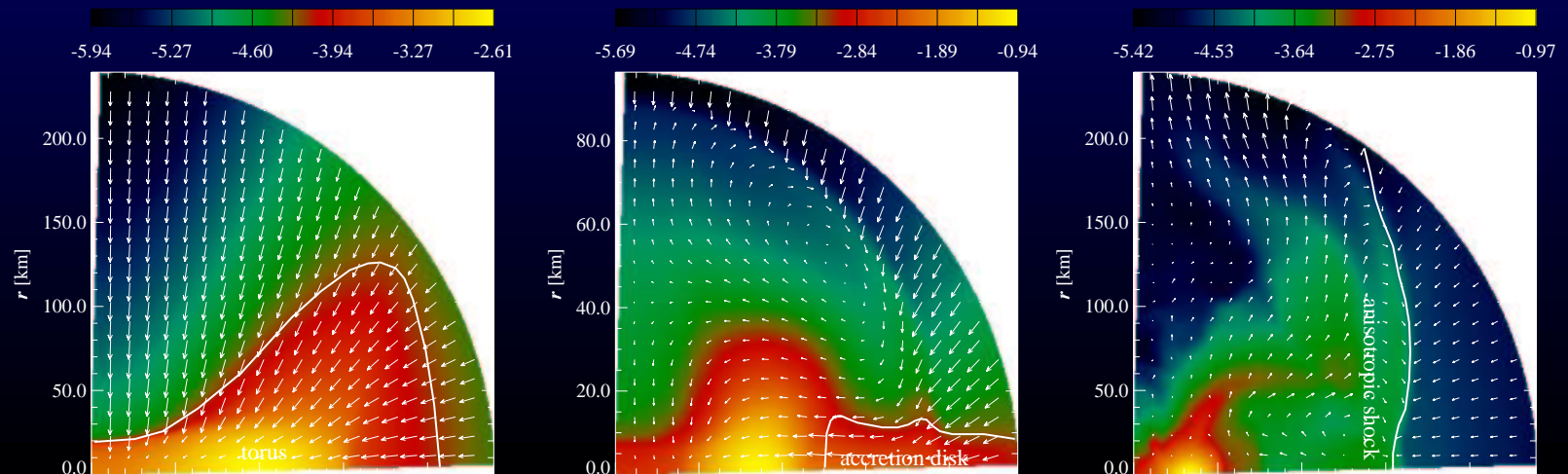


Gravitational wave amplitude on a 2-torus in a  $U(1)$  symmetric collapsing vacuum spacetime.

## 2D Scalar Field Critical Collapse with AMR: Choptuik, Hirschmann, and Liebling (2002)

- Berger and Oligier adaptive mesh refinement.
- Truncation error estimation using a self-shadow hierarchy.
- Equations derived using the 2+1+1 formalism, in cylindrical  $(\rho, z)$  coordinates.
- Axis instabilities eliminated via regularity conditions and Kreiss-Oliger dissipation.
- Partially-constrained evolution.
- Current matter source is a minimally-coupled, massless scalar field.





# Conclusions

- There has been substantial progress in numerical relativity.
- Physics content of numerical results is increasing.
- Few orbits in binary evolutions is around the corner.
- Formal mathematical input has become an important tool.

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- There has been substantial progress in numerical relativity.
- Physics content of numerical results is increasing.
- Few orbits in binary evolutions is around the corner.
- Formal mathematical input has become an important tool.

**But**, codes exhibit limited convergence and stability properties.

**We need:**

- To be more open to consider alternatives to finite differences.
- Better outer boundary conditions.
- Efficient elliptic solvers.
- AMR
- Larger and faster computers.



# RISKS

IF YOU NEVER TRY ANYTHING NEW,  
YOU'LL MISS OUT ON MANY OF LIFE'S GREAT DISAPPOINTMENTS.

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