Discretization of Maxwell's Equations

Ralf Hiptmair

Institut für Angewandte Mathematik
Universität Bonn

(e–mail: ralf@hiptmair.de)

(Homepage: http://na.uni-tuebingen.de/˜hiptmair)

Discretization of Maxwell's Equations

Ralf Hiptmair

Institut für Angewandte Mathematik
Universität Bonn

(e–mail: ralf@hiptmair.de)

(Homepage: http://na.uni-tuebingen.de/˜hiptmair)

Coordinate free

Maxwell’s Equations

Maxwell's Equations

Stated in four-dimensional space-time $(\mathcal{M}, g)$:

Given 3-form j, find 2-forms F, G on $\mathcal{M}$:

$$\begin{aligned} \mathbf{d}\,F &= 0 & \Leftrightarrow \quad \int_{\partial\Sigma} F &= 0 & \forall \text{oriented 3-dim. } \boldsymbol{\Sigma} \subset \mathcal{M}\ , \\ \mathbf{d}\,G &= j & \Leftrightarrow \quad \int_{\partial\Sigma} G &= \int_{\Sigma} j & \forall \text{oriented 3-dim. } \boldsymbol{\Sigma} \subset \mathcal{M}\ , \end{aligned}$$

$$G = \star_g F$$

Stated in flat space time $A(\mathbb{R}^3) \times A(\mathbb{R})$ (including materials):

Given 2-form $\mathbf{j}$, find time-dependent 1-forms $\mathbf{E}, \mathbf{H}$, 2-forms $\mathbf{B}, \mathbf{D}$ on $A(\mathbb{R}^3)$:

$$\begin{aligned} \mathbf{d}\,\mathbf{E} &= -\frac{d}{dt}\mathbf{B} & \Leftrightarrow \quad \int_{\partial\Sigma} \mathbf{E} &= -\frac{d}{dt}\int_{\Sigma} \mathbf{B} & \forall \text{surfaces } \boldsymbol{\Sigma} \subset A(\mathbb{R}^3)\ , \\ \mathbf{d}\,\mathbf{H} &= \frac{d}{dt}\mathbf{D} + \mathbf{j} & \Leftrightarrow \quad \int_{\partial\Sigma} \mathbf{H} &= \frac{d}{dt}\int_{\Sigma} \mathbf{D} + \int_{\Sigma} \mathbf{j} & \forall \text{surfaces } \boldsymbol{\Sigma} \subset A(\mathbb{R}^3)\ , \end{aligned}$$

$$\mathbf{B} = \star_\mu \mathbf{H} \quad , \quad \mathbf{D} = \star_\epsilon \mathbf{E} \qquad \text{(material laws)}$$

Co-chains and Differential Forms

Integral perspective:

An l-co-chain (space $\mathcal{F}^l$) on an n-manifold Σ is a continuous additive mapping of oriented piecewise smooth l-dimensional submanifolds of Σ into $\mathbb{R}$.

[Riemann summation]

Local perspective:

A (continuous) differential l-form ω on a differentiable manifold Σ is a continuous mapping $\omega : \Sigma \mapsto \bigwedge^l(T_\Sigma(\cdot))$, $\bigwedge^l(T_\Sigma(\mathbf{x}))$ = space of alternating l-multilinear forms on tangent space at Σ in $\mathbf{x} \in \Sigma$.

Integral perspective ↔ Balance laws

Local prespective ↔ Material laws

Integral Exterior Calculus

Exterior derivative (Stokes' theorem): Linear operator $d : \mathcal{F}^l \mapsto \mathcal{F}^{l+1}$

$$\int_S d\omega = \int_{\partial S} \omega \qquad \forall\, l\text{-dim. oriented sub-manifolds } S \subset \Sigma\,.$$

Pullback: Map $\mathbf{\Phi}^* : \mathcal{F}^l(\Sigma) \mapsto \mathcal{F}^l(\widehat{\Sigma})$, $\mathbf{\Phi} : \widehat{\Sigma} \mapsto \Sigma$ diffeomorphism

$$\int_{\widehat{S}} \mathbf{\Phi}^*\omega = \int_{\mathbf{\Phi}(\widehat{S})} \omega \qquad \forall\, l\text{-dim. oriented sub-manifolds } \widehat{S} \subset \widehat{\Sigma}\,.$$

Trace: Linear restriction mapping $\mathbf{t}_\Gamma : \mathcal{F}^l(\Sigma) \mapsto \mathcal{F}^l(\Gamma)$

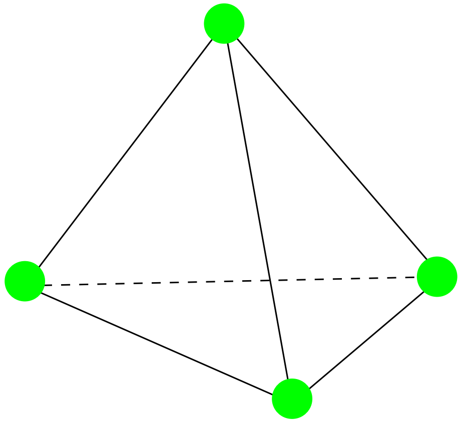
$$\int_S \mathbf{t}_\Gamma\omega = \int_S \omega \qquad \forall\, l\text{-dim. oriented sub-manifolds } S \subset \Gamma\,.$$

Discrete Fields

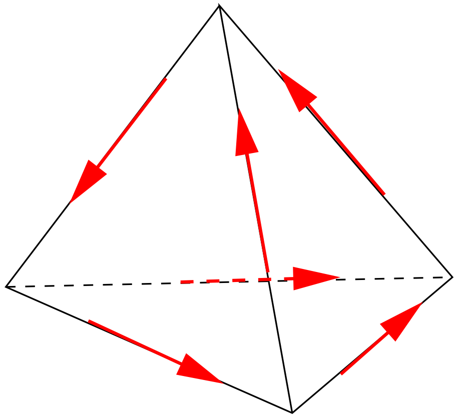
Discrete Co-Chains

Discrete l-co-chain $\in \mathcal{C}^l$: $\{$ l-faces of an oriented cellular complex$\}$ $\mapsto \mathbb{R}$

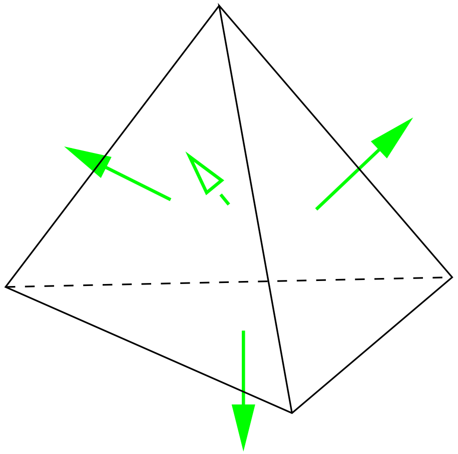
Discrete
0-co-chain:



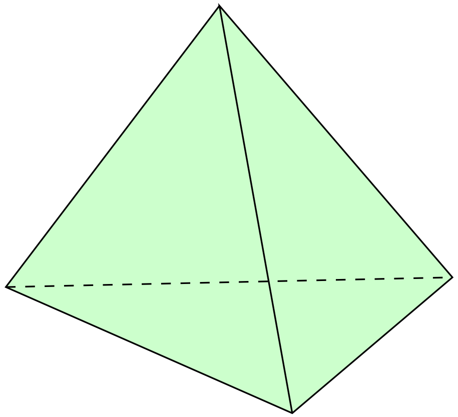
Discrete
1-co-chain:



Discrete
2-co-chain:



Discrete
3-co-chain:

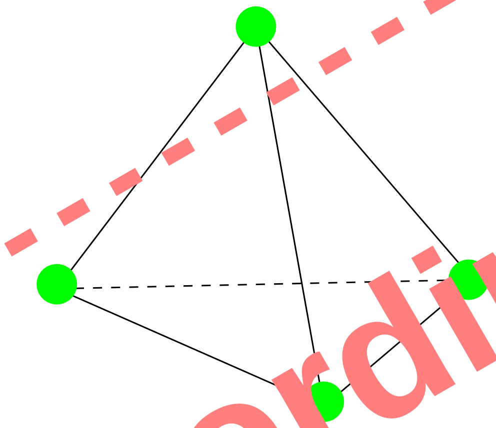


Discrete Co-Chains

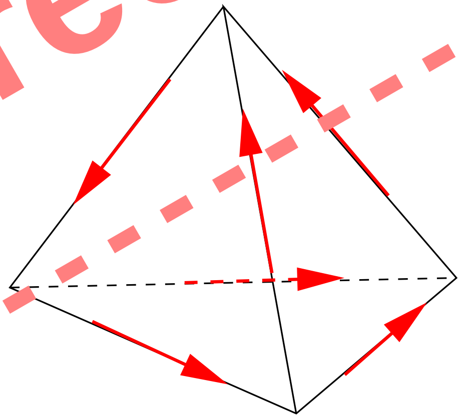
Discrete l-co-chain $\in \mathcal{C}^l$: $\{$ l-faces of an oriented cellular complex$\}$ $\mapsto \mathbb{R}$

Coordinate free

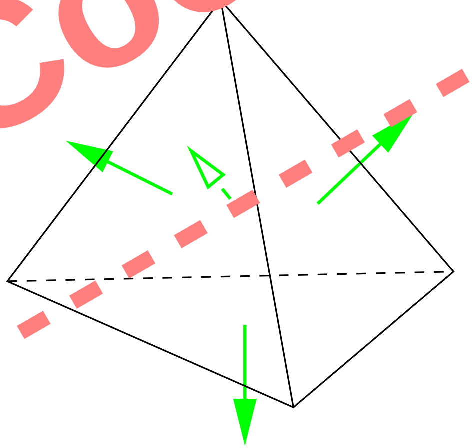
Discrete
0-co-chain:



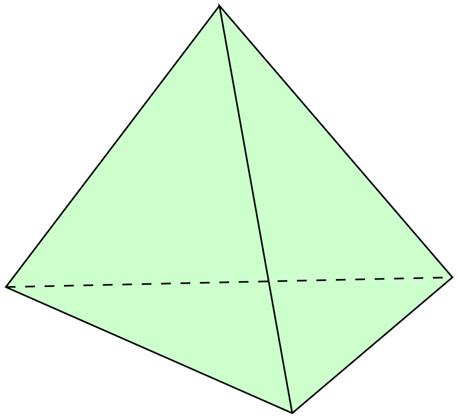
Discrete
1-co-chain:



Discrete
2-co-chain:



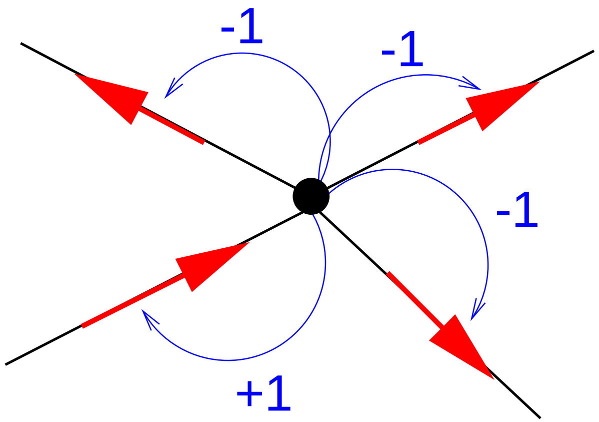
Discrete
3-co-chain:



Discrete Derivatives

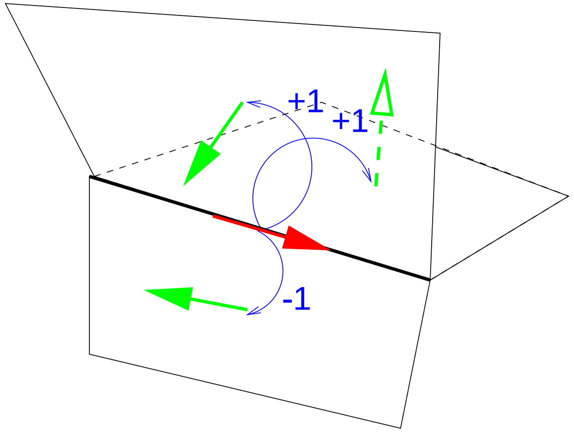
Same definition of discrete exterior derivative: $\int_{\partial\Sigma} \omega = \int_{\Sigma} \mathbf{d}\,\omega \Rightarrow$ Matrix D^l

Discrete gradient



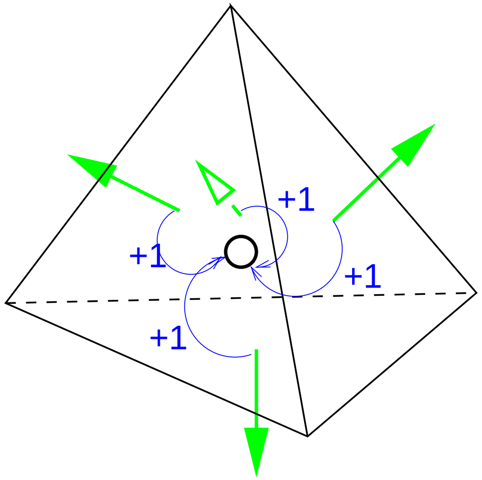
D^0: vertex-edge incidence matrix

Discrete rotation



D^1: edge-face incidence matrix

Discrete divergence



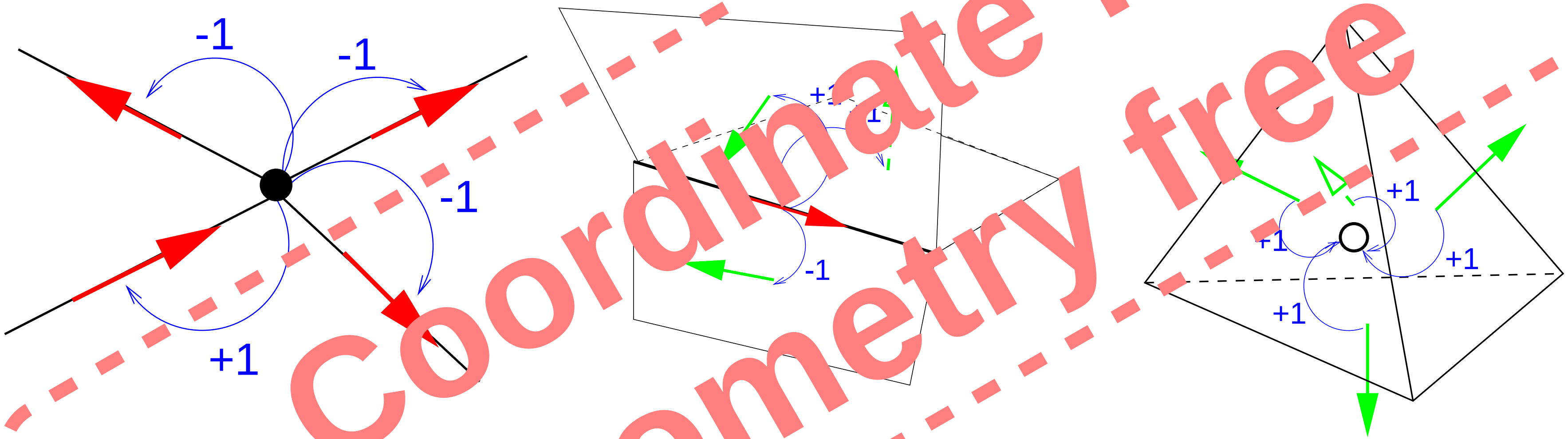
D^2: face-cell incidence matrix

For exterior derivative	$\Leftrightarrow$	For incidence matrices
$\mathbf{d} \circ \mathbf{d} = 0$		$\mathsf{D}^l \mathsf{D}^{l-1} = 0$

Discrete Derivatives

Same definition of discrete exterior derivative: $\int_{\partial\Sigma} \omega = \int_{\Sigma} \mathbf{d}\,\omega \Rightarrow$ Matrix D^l

Discrete gradient | Discrete rotation | Discrete divergence



D^0: vertex-edge incidence matrix

D^1: edge-face incidence matrix

D^2: face-cell incidence matrix

For exterior derivative $\mathbf{d} \circ \mathbf{d} = 0$ $\Leftrightarrow$ For incidence matrices $\mathrm{D}^l \mathrm{D}^{l-1} = 0$

Coordinate free

Geometry free

Discrete Potentials

Remember Poincaré's lemma (for C^1 differential forms):

On domain with trivial topology

$$d\omega = 0 \quad \Leftrightarrow \quad \exists \eta\colon \omega = d\eta$$

$$[\mathbf{curl}\,\mathbf{E} = 0 \Leftrightarrow \mathbf{E} = \mathbf{grad}\,\Phi,\ \mathrm{div}\,\mathbf{B} = 0 \Leftrightarrow \mathbf{B} = \mathbf{curl}\,\mathbf{A}]$$

Carries over to the discrete setting

If the topology of the triangulated manifold is trivial, then

$$\vec{\omega} \in \mathcal{C}^l :\ \mathsf{D}^l\vec{\omega} = 0 \quad \Leftrightarrow \quad \exists \vec{\eta} \in \mathcal{C}^{l-1} :\ \mathsf{D}^{l-1}\vec{\eta} = \vec{\omega}$$

► Discrete DeRham's theorem:

There is a discrete co-homology space $\mathcal{H}^l \subset \mathcal{C}^l$, $\dim \mathcal{H}^l = \beta_l(\Omega)$:

$$\vec{\omega} \in \mathcal{C}^l :\ \mathsf{D}^l\vec{\omega} = 0 \quad \Leftrightarrow \quad \exists \vec{\eta} \in \mathcal{C}^{l-1}, \vec{\gamma} \in \mathcal{H}^l :\ \vec{\omega} = \mathsf{D}^{l-1}\vec{\eta} + \vec{\gamma}$$

Network Equations

Discrete co-chains ⇔ Vectors of nodal values

Faraday's law: $\mathbf{d}\,F = 0$ ▶ $\mathsf{D}^2\vec{F} = 0$

Ampere's law: $\mathbf{d}\,G = j$ ▶ $\widetilde{\mathsf{D}}^2\vec{G} = \vec{j}$

Discrete conservation laws only depend on the topology of the mesh

▶ Constraint: $\widetilde{\mathsf{D}}^3\vec{j} = 0$

By the existence of discrete potentials:

$$\mathsf{D}^2\vec{F} = 0 \quad \Leftrightarrow \quad \vec{F} = \mathsf{D}^1\vec{A} \quad \text{(Four-potential)}\ .$$

The two network equations are unrelated on the topological level.

▶ They can be defined on two different triangulations $\mathcal{M}_h, \widetilde{\mathcal{M}}_h$

Consistency

$\mathcal{S}_l(\mathcal{M}_h)$: set of l-facets of triangulation $\mathcal{M}_h$

"Nodal values": $I^l : \mathcal{F}^l \mapsto \mathcal{C}^l, \quad (I^l\omega)(S) := \int_S \omega \quad \forall S \in \mathcal{S}_l(\mathcal{M}_h)$

▶ $$\mathsf{D}^l \circ I^l = I^{l+1} \circ \mathbf{d}$$

▼

If F, G are solutions of Maxwell's equations, then

$$\mathsf{D}^1 I^2 F = 0 \quad , \quad \mathsf{D}^1 I^2 G = I^3 j \ .$$

Network equations are consistent with Maxwell conservation laws.
(In a finite difference sense $(*)$)

$(*)$ Remember that I^0 is point evaluation!

Discrete Material Laws

Hodge Operators

Definition of $\star_g : \mathcal{DF}^l(\mathcal{M}) \mapsto \mathcal{DF}^{n-l}(\mathcal{M})$

Pseudo-Riemannian metric g on $\mathcal{M}$

For continuous l-form ω, $\mathbf{p} \in \mathcal{M}$, pick positively oriented $g(\mathbf{p})$-ONB $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $T_{\mathbf{p}}(\mathcal{M})$. For any partition $I \cup J = \{1, \dots, n\}$ define

$$(\star_g\omega)(\mathbf{p})(\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_{n-l}}) := \mathrm{sgn}(I)\,\omega(\mathbf{p})(\mathbf{b}_{j_1}, \dots, \mathbf{b}_{j_l}) \;.$$

Symmetric “energy bilinear form” $(\cdot,\cdot)_g : \mathcal{DF}^l(\mathcal{M}) \times \mathcal{DF}^l(\mathcal{M}) \mapsto \mathbb{R}$.

► $$\int_{\mathcal{M}} \star_g\omega \wedge \eta = (\omega, \eta)_g \;, \quad \forall \omega, \eta \in \mathcal{DF}^l(\mathcal{M}) \;.$$

Discrete Hodge-Operators

$$\omega \in \mathcal{DF}^l(\mathcal{M})$$

$$\xi = \star_g \omega \in \mathcal{DF}^m(\mathcal{M})\,\mathbf{:}$$

$$(m = n - l)$$

Continuous: $(\omega, \eta)_g \;=\; \int_{\mathcal{M}} \xi \wedge \eta \quad \forall \eta \in \mathcal{DF}^l(\mathcal{M})$

$$\Downarrow \qquad\qquad \Downarrow$$

Discrete: $\mathsf{M}_g^l \vec{\omega} \;=\; \widetilde{\mathsf{K}}_m^l \vec{\xi}$

Essential requirements:

"Mass matrices" M_g^l, $\widetilde{\mathsf{M}}_g^l$ have to be symmetric (positive definite), regular.

"Pairing matrices" K_l^m (discrete $\wedge$-product), $\widetilde{\mathsf{K}}_m^l$ satisfy

$$\mathsf{K}_l^m = -(\widetilde{\mathsf{K}}_m^l)^T \quad \Longleftrightarrow \quad \widetilde{\mathsf{K}}_m^l = -(\mathsf{K}_l^m)^T \;,$$

matching $\omega \wedge \eta = -\eta \wedge \omega$.

$$(\mathsf{D}^{l-1})^T \widetilde{\mathsf{K}}_m^l = (-1)^l \, \widetilde{\mathsf{K}}_{m+1}^{l-1} \widetilde{\mathsf{D}}^m \quad , \quad (\widetilde{\mathsf{D}}^m)^T \mathsf{K}_{l-1}^{m+1} = (-1)^l \mathsf{K}_l^m \mathsf{D}^{l-1}$$

corresponding to integration by parts.

Discrete Equations

Network equations: $\mathsf{D}^2\vec{F} = 0 \quad , \quad \widetilde{\mathsf{D}}^2\vec{G} = j \, .$

Discrete constitutive equations: $\mathsf{M}_g^2\vec{F} = \widetilde{\mathsf{K}}_2^2\vec{G} \quad , \quad \widetilde{\mathsf{M}}_g^2\vec{G} = \mathsf{K}_2^2\vec{F} \, .$

$$\mathsf{D}^2\vec{F} = 0 \, , \qquad (\mathsf{D}^1)^T\mathsf{M}_g^2\vec{F} = \widetilde{\mathsf{K}}_3^2\vec{j} \, .$$

$$\widetilde{\mathsf{D}}^2\vec{G} = \vec{j} \, , \qquad (\widetilde{\mathsf{D}}^1)^T\widetilde{\mathsf{M}}_g^2\vec{G} = 0 \, .$$

$(\mathsf{D}^1)^T\mathsf{M}_g^2\mathsf{D}^1\vec{A} = \widetilde{\mathsf{K}}_3^2\vec{j}$ with discrete 4-potential $\vec{A}$

- Flat space-time, 3+1-formulation on $(\mathbf{x}, t)$-aligned grid, partial elimination:

$$\mathsf{M}_\epsilon^1(\vec{E}^{n+1} - 2\vec{E}^n + \vec{E}^{n-1})/\Delta t^2 = (\mathsf{D}^1)^T\mathsf{M}_\mu^2\mathsf{D}^1\vec{E}^n$$

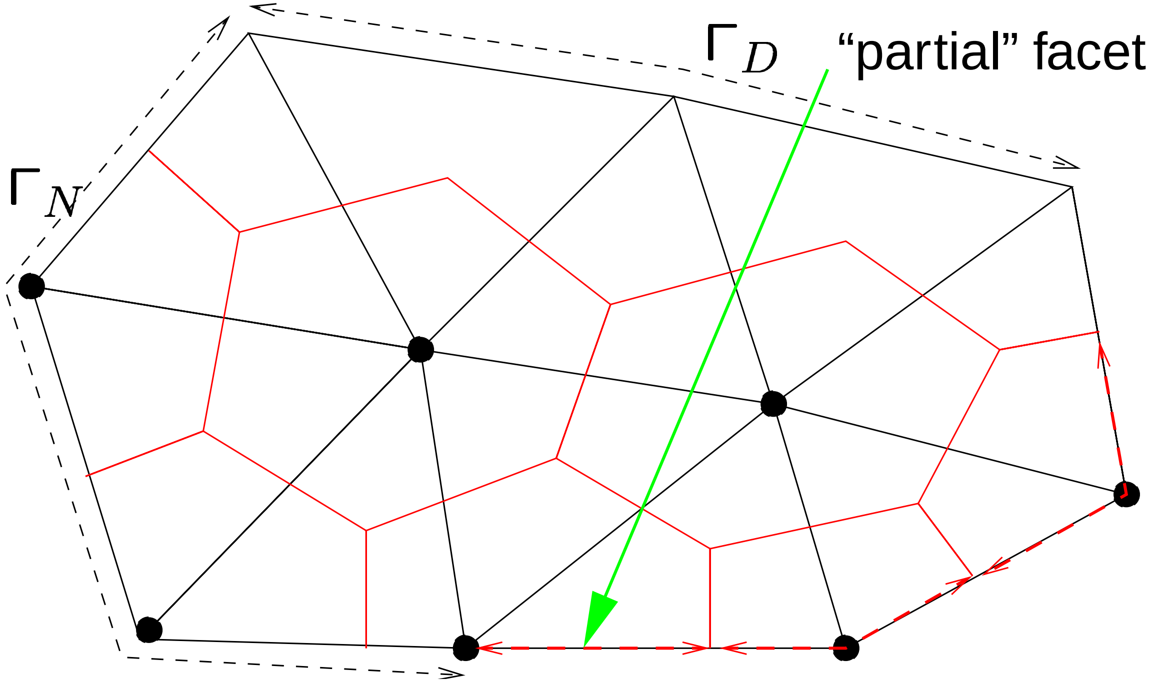
(Explicit trapezoidal rule/leap-frog)

Finite Volume Approach (I)

Idea: Choose $\tilde{\mathcal{M}}_h$ as dual mesh of $\mathcal{M}_h$

i.e. $\widetilde{\mathsf{L}}_l^T = (-1)^l \mathsf{L}_{n-l}$, ($\mathsf{L}_l$, $\widetilde{\mathsf{L}}_l$ incidence matrices of l- and $l+1$-facets)

Example:
Dual mesh in two dimensions



Since $\mathsf{D}^l, \widetilde{\mathsf{D}}^l$ agree with incidence matrices ► Identity matrices as pairing matrices $\mathsf{K}, \widetilde{\mathsf{K}}$

Finite Volume Approach (II)

We get diagonal M_g^l ($\xi = \star_g \omega \Leftrightarrow \vec{\xi} = \mathsf{M}_g^l \vec{\omega}$) for orthogonal dual mesh:
If F = l-facet of $\mathcal{T}_h$, $\tilde{F}$ associated $n-l$-facet of $\tilde{\mathcal{T}}_h$, then

$$T_{\mathbf{p}}(F) \perp_g T_{\mathbf{p}}(\tilde{F}), \quad \mathbf{p} = F \cap \tilde{F}\ .$$

▼

Choose $a(\mathbf{p})$-ONB $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ such that

$$T_{\mathbf{p}}(F) = \mathsf{Span}\,\{\mathbf{b}_1, \dots, \mathbf{b}_l\}\ ,\ T_{\mathbf{p}}(\tilde{F}) = \mathsf{Span}\left\{\mathbf{b}_{l+1}, \dots, \mathbf{b}_n\right\}\ .$$

▶ $$\star_g \omega(\mathbf{p})(\mathbf{b}_{l+1}, \dots, \mathbf{b}_n) = \omega(\mathbf{p})(\mathbf{b}_1, \dots, \mathbf{b}_l)$$

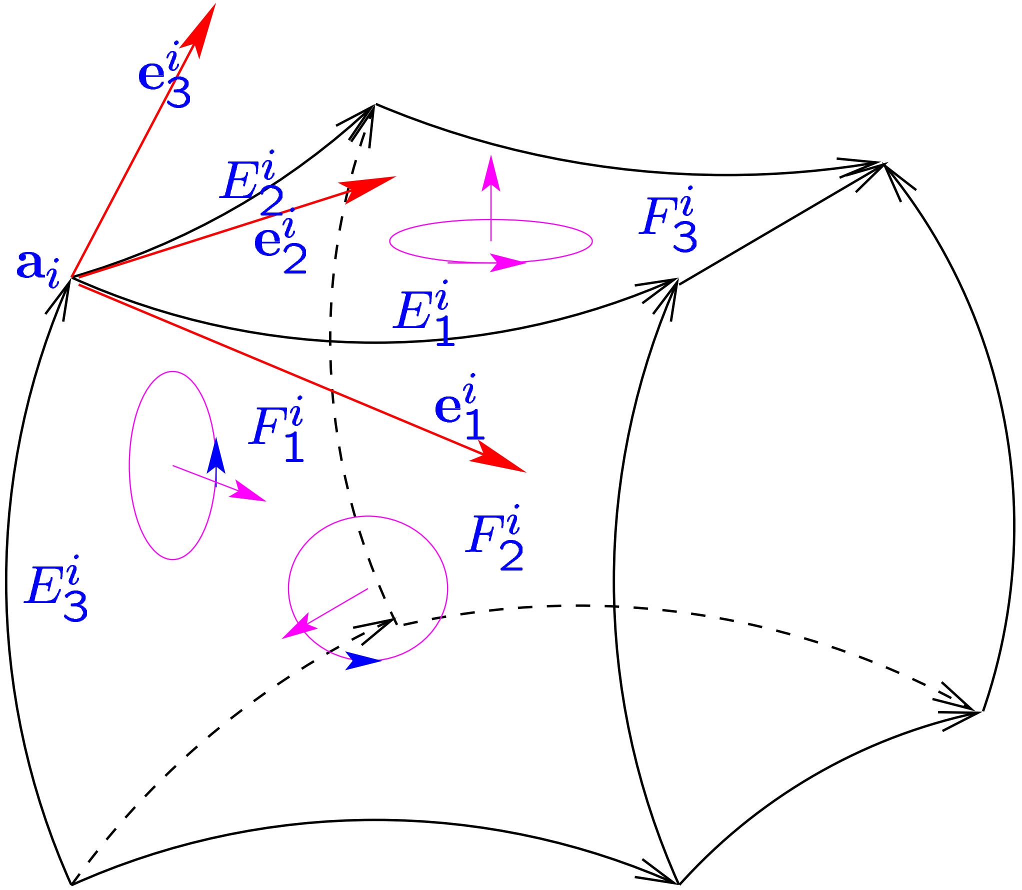
Assumption for approximation: $\mathbf{t}_F \omega \equiv \mathsf{const}$, $\mathbf{t}_{\tilde{F}} \xi \equiv \mathsf{const}$

$$(\mathbf{t}_F \omega)(\mathbf{p}) = \frac{1}{|F|} \int_F \omega\ ,\ \int_{\tilde{F}} \xi = |\tilde{F}|\, \mathbf{t}_{\tilde{F}} \xi(\mathbf{p}) \quad \blacktriangleright \quad \mathsf{M}_g^l(F, \tilde{F}) = a(\mathbf{p}) \cdot |\tilde{F}| : |F|$$

▶ Yields Yee's scheme when applied to Maxwell's equations in flat space-time on coordinate-aligned Cartesian grid.

Generalized Finite Differences (I)

Use same mesh $\mathcal{M}_h$ as primal/dual mesh



In general, evaluation of $\star_g\omega$ and $\omega \wedge \eta$ at point $\mathbf{p}$ requires full knowledge about $\omega(\mathbf{p})$, $\eta(\mathbf{p})$, i.e. the action of the multilinear forms on all subsets of a basis.

Approximations:
- 1-forms: $\omega(\mathbf{a}_i)(\mathbf{e}_i) \approx \int_{E_i} \omega$
- 2-forms: $\omega(\mathbf{a}_i)(\mathbf{e}_1, \mathbf{e}_2) \approx \int_{F_3} \omega$

Generalized Finite Differences (II)

▶ Evaluation of $\star_g\omega$ at vertices is possible. (Ex: 1-forms ω, η in 3D)

① Construct $a(\mathbf{a}_i)(\cdot,\cdot)$-ONB $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ (e.g. by Gram-Schmidt-orthogonalization): $\mathbf{b}_j = s_{j1}\mathbf{e}_1 + s_{j3}\mathbf{e}_2 + s_{j3}\mathbf{e}_3$, $\mathbf{S} = (s_{ij}) \in \mathbb{R}^{3,3}$

② From definition of Hodge-operator:

$$\begin{aligned} \star_g\omega(\mathbf{b}_2, \mathbf{b}_3) &= \omega(\mathbf{b}_1)\ , \\ \star_g\omega(\mathbf{b}_1, \mathbf{b}_2) &= \omega(\mathbf{b}_1)\ , \\ \star_g\omega(\mathbf{b}_1, \mathbf{b}_3) &= -\omega(\mathbf{b}_2) \end{aligned}$$

③ Evaluation of 3-form $\star_g\omega \wedge \eta$ at $\mathbf{a}_i$:

$$(\star_g\omega \wedge \eta)(\mathbf{a}_i)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) =$$

$$(\star_g a\omega)(\mathbf{e}_1, \mathbf{e}_2)\cdot\eta(\mathbf{e}_3) - (\star_g\omega)(\mathbf{e}_1, \mathbf{e}_3)\cdot\eta(\mathbf{e}_2) + (\star_g\omega)(\mathbf{e}_2, \mathbf{e}_3)\cdot\eta(\mathbf{e}_1) =$$

$$= \frac{1}{\det(S)} \cdot \begin{pmatrix} \eta(\mathbf{e}_1) \\ \eta(\mathbf{e}_2) \\ \eta(\mathbf{e}_3) \end{pmatrix}^T \mathbf{S}^T\mathbf{S} \begin{pmatrix} \omega(\mathbf{e}_1) \\ \omega(\mathbf{e}_2) \\ \omega(\mathbf{e}_3) \end{pmatrix}$$

Generalized Finite Differences (II)

- Evaluation of $\star_g\omega$ at vertices is possible. (Ex: 1-forms ω, η in 3D)

① Construct $a(\mathbf{a}_i)(\cdot,\cdot)$-ONB $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ (e.g. by Gram-Schmidt-orthogonalization): $\mathbf{b}_j = s_{j1}\mathbf{e}_1 + s_{j3}\mathbf{e}_2 + s_{j3}\mathbf{e}_3$, $S = (s_{ij}) \in \mathbb{R}^{3,3}$

② From definition of Hodge-operator:

$$\begin{aligned} \star_g\omega(\mathbf{b}_2,\mathbf{b}_3) &= \omega(\mathbf{b}_1)\ , \\ \star_g\omega(\mathbf{b}_1,\mathbf{b}_2) &= \omega(\mathbf{b}_1)\ , \\ \star_g\omega(\mathbf{b}_1,\mathbf{b}_3) &= -\omega(\mathbf{b}_2) \end{aligned}$$

③ Evaluation of 3-form $\star_g\omega \wedge \eta$ at $\mathbf{a}_i$:

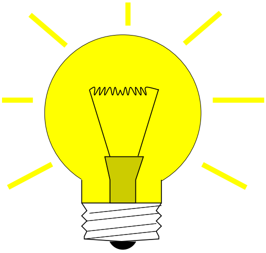
$$(\star_g\omega \wedge \eta)(\mathbf{a}_i)(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) =$$

$$(\star_g a\omega)(\mathbf{e}_1,\mathbf{e}_2)\cdot\eta(\mathbf{e}_3) - (\star_g\omega)(\mathbf{e}_1,\mathbf{e}_3)\cdot\eta(\mathbf{e}_2) + (\star_g\omega)(\mathbf{e}_2,\mathbf{e}_3)\cdot\eta(\mathbf{e}_1) =$$

$$= \frac{1}{\det(S)} \cdot \begin{pmatrix} \eta(\mathbf{e}_1) \\ \eta(\mathbf{e}_2) \\ \eta(\mathbf{e}_3) \end{pmatrix}^T \mathbf{S}^T\mathbf{S} \begin{pmatrix} \omega(\mathbf{e}_1) \\ \omega(\mathbf{e}_2) \\ \omega(\mathbf{e}_3) \end{pmatrix}$$

Coordinate free

Generalized Finite Differences (III)



Approximate integrals $\int_\Omega$ by vertex-based quadrature formulas, e.g., for a cell T and 3-form ϕ

$$\int_T \phi \approx \sum_i w_i \phi(\mathbf{a}_i)(\mathbf{e}_1^i, \mathbf{e}_2^i, \mathbf{e}_3^i)$$

($w_i = \frac{1}{8}$ for hexahedron, $w_i = \frac{1}{4}$ for tetrahedron)

Mass matrix:

$$\vec{\omega}^T \mathsf{M}_g^1 \vec{\eta} = \sum_T \sum_i w_i \cdot \Big((\star_g \omega \wedge \eta)(\mathbf{a}_i)(\mathbf{e}_1^i, \mathbf{e}_2^i, \mathbf{e}_3^i) \Big) \approx \int_\Omega \star_g a \omega \wedge \eta$$

(where ω, η are “virtual differential forms” associated with $\vec{\omega}, \vec{\eta}$)

(Same approach to coupling matrices)

Finite Elements

Use integral (weak) form of constitutive law:

$$\omega \in \mathcal{DF}^{l(\mathcal{M})} : \xi = \star_g \omega \quad \Rightarrow \quad \int_{\mathcal{M}} \xi \wedge \eta = \int_{\mathcal{M}} \star_g \omega \wedge \eta \quad \forall \eta \in \mathcal{DF}^l(\mathcal{M}) \,.$$

▶ Linear extension $W^l : \mathcal{C}^l \mapsto \mathcal{F}^l$ required such that

- $W^l \vec{\omega}$ is a continuous differential form almost everywhere,
- $\int_F W^l \vec{\omega} = \vec{\omega}(F) \quad \forall\, l$-facets F of the mesh.

▶ Galerkin-Hodge:

$$\vec{\omega}^T \mathsf{M}_g^l \vec{\eta} := \int_{\mathcal{M}} \star_g (W^l \vec{\omega}) \wedge W^l \vec{\eta} \,.$$

Range $\mathcal{W}^l(\mathcal{M}_h)$ of W^l = discrete differential forms
(Finite elements for differential forms)

Whitney Forms

Extension of Co-Chains

Considerations for tetrahedral triangulations

Example: Local extension of a 0-co-chain $\vec{\omega}$

▶ Natural: Linear interpolation, on tetrahedron with vertices $\mathbf{a}_1, \ldots, \mathbf{a}_4$

$$(W_T^0 \vec{\omega})(\mathbf{x}) = \sum_{i=1} \lambda_i(\mathbf{x})\, \vec{\omega}(\mathbf{a}_i)$$

λ_k: barycentric coordinate functions

◆

“Writing a point $\mathbf{x} \in T$ as a combination of vertices”

$$\mathbf{x} = \sum_{i=1}^{4} \lambda(\mathbf{x})\, \mathbf{a}_i \ .$$

Extension of 1-Co-Chains

Idea: Represent line segements in T as combination of edges

$\mathbf{x}, \mathbf{y} \in T$, $\mathbf{x} = \sum_i \lambda_i(\mathbf{x})\mathbf{a}_i$, $\mathbf{y} = \sum_i \lambda_i(\mathbf{y})\mathbf{a}_i$, λ_i barycentric coordinates

$$\begin{aligned}
[\mathbf{x}, \mathbf{y}] &= \{t\mathbf{x} + (1-t)\mathbf{y}\ ;\ 0 \leq t \leq 1\} \\
&= \{\textstyle\sum_i [t\lambda_i(\mathbf{x}) + (1-t)\lambda_i(\mathbf{y})]\mathbf{a}_i\ ;\ 0 \leq t \leq 1\} \\
&= \left\{\textstyle\sum_i [t \sum_j \lambda_j(\mathbf{y})\lambda_i(\mathbf{x}) + (1-t)\sum_j \lambda_j(\mathbf{x})\lambda_i(\mathbf{y})]\mathbf{a}_i\right\} \\
&= \left\{\textstyle\sum_i \sum_j \lambda_i(\mathbf{x})\lambda_j(\mathbf{y})[t\mathbf{a}_i + (1-t)\mathbf{a}_j]\ ;\ 0 \leq t \leq 1\right\}\ .
\end{aligned}$$

- Define
$$\int_{[\mathbf{x},\mathbf{y}]} W_T^1 \vec{\omega} := \sum_{i<j} [\lambda_i(\mathbf{x})\lambda_j(\mathbf{y}) - \lambda_i(\mathbf{y})\lambda_j(\mathbf{x})] \cdot \int_{[\mathbf{a}_i,\mathbf{a}_j]} \vec{\omega}$$

$$W_T^1 \vec{\omega}(\mathbf{x}) = \sum_{i<j} \Big(d\lambda_i(\mathbf{x}) \wedge \lambda_j(\mathbf{x}) - d\lambda_j(\mathbf{x}) \wedge \lambda_i(\mathbf{x}) \Big) \cdot \int_{[\mathbf{a}_i,\mathbf{a}_j]} \vec{\omega}\ .$$

Whitney 1-Forms

Locality: For face F of T extension $W^1\vec{\omega}_{|F}$ only depends on values of $\vec{\omega}$ on edges belonging to F

▶ Define global extension: $W^1\vec{\omega}(\mathbf{x}) := W^1_T\vec{\omega}(\mathbf{x})$, if $\mathbf{x} \in T \Rightarrow W^1\vec{\omega} \in \mathcal{F}^l$.

Whitney 1-forms: $\mathcal{W}^1(\mathcal{M}_h) := W^1(\mathcal{C}^1) \in \mathcal{F}^1(\mathcal{M})$

▶ **Commuting diagram property**:

$$\int_{[\mathbf{x},\mathbf{y}]} \mathsf{D}^0\vec{\phi} = \sum_{i=0}^{3}\sum_{j=0}^{3} \lambda_i(\mathbf{x})\lambda_j(\mathbf{y})(\mathsf{D}^0\vec{\phi})([\mathbf{a}_i,\mathbf{a}_j]) = \sum_{i=0}^{3}\sum_{j=0}^{3} \lambda_i(\mathbf{x})\lambda_j(\mathbf{y})(\vec{\phi}(\mathbf{a}_j) - \vec{\phi}(\mathbf{a}_i))$$

$$= \sum_{j=0}^{3} \lambda_j(\mathbf{y})\vec{\phi}(\mathbf{a}_j) - \sum_{i=0}^{3} \lambda_i(\mathbf{x})\vec{\phi}(\mathbf{a}_i) = (W^0_{|T}\vec{\phi})(\mathbf{y}) - (W^0_{|T}\vec{\phi})(\mathbf{x}) = \int_{\partial[\mathbf{x},\mathbf{y}]} W^0_T\vec{\phi}\,.$$

▶ $W^1 \circ \mathsf{D}^0 = d \circ W^0$

Simplicial Whitney l-Forms

n-Simplex $T = [\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}]$, $\mathbf{x}_k = \sum_{i=1}^{n+1} \lambda_i(\mathbf{x}_k)\mathbf{a}_i \in T$

$$[\mathbf{x}_1, \ldots, \mathbf{x}_{l+1}] = \left\{ \sum_{i_1=1}^{n+1} \cdots \sum_{i_{l+1}=1}^{n+1} \lambda_{i_1}(\mathbf{x}_1) \cdots \lambda_{i_{l+1}}(\mathbf{x}_{l+1}) [\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{l+1}}] \right\}$$

▼

$$\int_{[\mathbf{x}_1,\ldots,\mathbf{x}_{l+1}]} W^l \vec{\omega} := \sum_{i_1=1}^{4} \cdots \sum_{i_{l+1}=1}^{4} \lambda_{i_1}(\mathbf{x}_1) \cdots \lambda_{i_{l+1}}(\mathbf{x}_{l+1}) \cdot \int_{[\mathbf{a}_{i_1},\ldots,\mathbf{a}_{i_{l+1}}]}$$

▼

$$W^l \vec{\omega} = \sum_{i_1 < \cdots < i_{l+1}} \sum_{k=1}^{l} (-1)^k \lambda_{i_1} \wedge \ldots \wedge \check{\lambda}_{i_k} \wedge \ldots \wedge \lambda_{i_{l+1}} \cdot \vec{\omega}([\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{l+1}}])$$

Algebraic Properties

Nodal projectors: $\Pi^l := W^l \circ I^l : \mathcal{F}^l(\mathcal{M}) \mapsto \mathcal{W}^l(\mathcal{M}_h)$

$$\begin{tikzcd}
\mathcal{F}^l(\mathcal{M}) \arrow[r, "d"] \arrow[d, "\Pi^l"'] & \mathcal{F}^{l+1}(\mathcal{M}) \arrow[d, "\Pi^{l+1}"] \\
\mathcal{W}^l(\mathcal{M}_h) \arrow[r, "d"] & \mathcal{W}^{l+1}(\mathcal{M}_h)
\end{tikzcd}$$

De Rham's theorem for Whitney-forms:

There is $\mathcal{H}^l_h \subset \mathcal{W}^l(\mathcal{M}_h)$, $\dim \mathcal{H}^l_h = \beta_l(\mathcal{M})$, such that

$$\omega_h \in \mathcal{W}^l(\mathcal{M}_h), d\omega_h = 0 \Rightarrow \exists \eta_h \in \mathcal{W}^{l-1}(\mathcal{M}_h), \gamma_h \in \mathcal{H}^l_h : \omega_h = d\eta_h + \gamma_h$$

Non-Simplicial Discrete Differential Forms

Example: Whitney forms on hypercube $C = [0;1]^n$, Cartesian coordinates

$$\mathcal{W}^l(C) = \mathrm{Span}\,\{Q_J(\mathbf{x})\,dx_I,\ I \cup J = \{1,\dots,n\}, I \cap J = \emptyset,\ |I| = l\}\ ,$$

where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_l}$,

$$Q_J = \mathrm{Span}\left\{\prod_{j\in J} x_j^{\epsilon_j},\ \epsilon_j \in \{0,1\}\right\}.$$

- Possible generalization to Whitney forms on polyhedra $T := \mathrm{convex}\{\mathbf{a}_1,\dots,\mathbf{a}_k\}$, provided that "barycentric coordinates" λ_i with $\lambda_i(\mathbf{a}_j) = \delta_{ij}$, $\sum_i \lambda_i \equiv 1$, are available.

Higher Order Whitney Forms

$\mathcal{PF}_p^l(T)$: Polynomial differential forms of degree p on simplex T

For $l > 0$: $\mathcal{PF}_0^l(T) \subset \mathcal{W}^l(T)$, $\mathcal{PF}_1^l(T) \not\subset \mathcal{W}^l(T)$

► First order convergence $O(h)$ (h meshwidth)
(In norms induced by Riemannian metrices and Hodge operators)

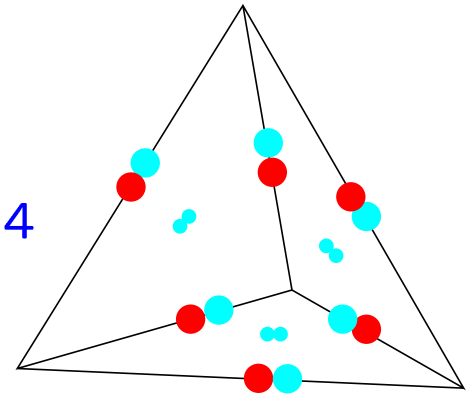
For any p it is possible to construct $W_p^l(T)$ such that

- $\mathcal{PF}_p(T) \subset W_p^l(T)$,
- suitable local degrees of freedom are available

Local basis for second order tetrahedral discrete 1-forms:

$$\begin{aligned}
\beta_{ij} &= \lambda_i \,\mathbf{grad}\, \lambda_j - \lambda_j \,\mathbf{grad}\, \lambda_i \ , \\
\widehat{\beta}_{ij} &= \lambda_i \,\mathbf{grad}\, \lambda_j + \lambda_j \,\mathbf{grad}\, \lambda_i \ , \\
\widehat{\beta}^a_{ijk} &= \lambda_i \lambda_j \,\mathbf{grad}\, \lambda_k - \lambda_i \lambda_k \,\mathbf{grad}\, \lambda_j \ , \\
\widehat{\beta}^b_{ijk} &= \lambda_i \lambda_j \,\mathbf{grad}\, \lambda_k - \lambda_j \lambda_k \,\mathbf{grad}\, \lambda_i \ .
\end{aligned}$$

$1 \le i < j < k \le 4$



Summary

Discrete co-chains

+

Discrete Hodge operators

Coordinate free discrete models for Maxwell’s equations

Guideline for numerical relativity ?

(Regge calculus & related ideas ?)