

OVERVIEW OF  
FINITE ELEMENT METHODS  
FOR  
LINEAR HYPERBOLIC PROBLEMS

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## **Outline of Talk**

1. Brief review of finite elements for elliptic problems
2. Finite element methods for the wave equation
3. Finite element methods for first order hyperbolic systems

## Finite Elements for Elliptic Problems

Consider model problem:

$$-\Delta u = f, \quad \text{in } \Omega$$

with boundary conditions

- (1)  $u = 0$  on  $\Gamma_1$       essential BC
- (2)  $\partial u / \partial n = 0$  on  $\Gamma_2$       natural BC

Finite element method based on variational formulation of BVP.

Find  $u \in H_*^1(\Omega)$  such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_*^1(\Omega).$$

where

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega)\},$$
$$H_*^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\}.$$

By Green's formula:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds - \int_{\Omega} \Delta u v \, dx.$$

So

$$- \int_{\Omega} \Delta u v \, dx + \int_{\Gamma_2} \frac{\partial u}{\partial n} v \, ds = \int_{\Omega} f v \, dx.$$

Recover PDE and natural BC.

Galerkin method: Let  $V_h$  be finite dimensional subspace of  $H_*^1(\Omega)$ . Find  $u_h \in V_h$  such that

$$\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx = \int_{\Omega} f v_h \, dx, \quad \forall v_h \in V_h.$$

Example:  $\Omega$  polygonal domain in  $\mathbb{R}^2$  and  $\tau_h$  triangulation of  $\Omega$ . Define:

$$V_h = \{v \in H^1(\Omega) | v|_T \in P_1\}.$$

Degrees of freedom are values of  $v$  at vertices of  $\tau_h$ . Write  $u_h = \sum_{i=1}^N \alpha_i \phi_i(x)$ , where  $\phi_i$  piecewise linear with value 1 at vertex  $i$  and 0 at all other vertices. Then

$$\sum_{i=1}^N \alpha_i \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dx = \int_{\Omega} f \phi_j dx.$$

Linear system  $A\alpha = F$ ,

where  $A_{ij} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dx$ ,

$F_j = \int_{\Omega} f \phi_j dx$ ,  $\alpha = (\alpha_1, \dots, \alpha_N)$ .

Get stability very easily, since:

$$\int_{\Omega} |\nabla u_h|^2 = \int_{\Omega} f u_h = \int_{\Omega} \nabla u \cdot \nabla u_h.$$

so

$$\|\nabla u_h\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \leq \|f\|_{H_*^{-1}(\Omega)}.$$

Consider IBVP for wave equation:

$$\begin{aligned}u_{tt} &= \Delta u \text{ in } \Omega \times (0, T), & u &= 0 \text{ on } \partial\Omega \times (0, T) \\u(x, 0) &= r(x), & u_t(x, 0) &= s(x).\end{aligned}$$

Semi-discrete (continuous time) Galerkin: Based on variation:

$$\int_{\Omega} u_{tt}v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx = 0$$

$v(\cdot, t) \in H_0^1(\Omega)$ . Write:  $u_h = \sum \alpha_i(t)\phi_i(x)$ . Then

$$\sum \alpha_i''(t) \int_{\Omega} \phi_i\phi_j + \sum \alpha_i(t) \int_{\Omega} \nabla\phi_i \cdot \nabla\phi_j = 0.$$

System of second order odes:  $M\alpha'' + A\alpha = 0$  with  $\alpha'(0)$  and  $\alpha(0)$  given.

Apply ode solver (can convert to first order system) or do finite differences in time.

Can also consider finite element approach to time discretization.

Consider simple ode model problem:

$$u' + u = f, \quad 0 < t < 1, \quad u(0) = g.$$

Then  $u$  satisfies over any time interval  $(t_n, t_{n+1})$

$$\int_{t_n}^{t_{n+1}} [u'v + uv] dt = \int_{t_n}^{t_{n+1}} fv dt.$$

Note continuous problem has a natural stability estimate by choosing  $v = u$ .

$$\frac{1}{2}u^2(t_{n+1}) + \int_{t_n}^{t_{n+1}} u^2 dt = \frac{1}{2}u^2(t_n) + \int_{t_n}^{t_{n+1}} fu dt.$$

Applying Schwarz inequality to last term, get

$$u^2(t_{n+1}) + \int_{t_n}^{t_{n+1}} u^2 dt \leq u^2(t_n) + \int_{t_n}^{t_{n+1}} f^2 dt.$$

Can sum this result to get:

$$u^2(t_n) + \int_{t_0}^{t_n} u^2 dt \leq g^2 + \int_{t_0}^{t_n} f^2 dt.$$

Consider approximations which mimic this result to get stability.

Take  $u_h$  to be a piecewise polynomial of degree  $\leq n$  over  $[0, 1]$ , i.e., a polynomial of degree  $\leq n$  on each subinterval  $(t_n, t_{n+1})$ .

Suppose  $u_h$  continuous on  $[0, 1]$ . Then  $u_h(t_n)$  known by continuity (on the first subinterval,  $u_h(0) = g$ ). If  $u_h \in P_n$  on  $(t_n, t_{n+1})$ , need  $n$  equations to determine  $u_h$ . Take variation over  $v \in P_{n-1}$ . On discrete level, can't choose  $v_h = u_h$ , but can choose  $v_h = \Pi_{n-1} u_h$ . Now

$$\begin{aligned} \int_{t_n}^{t_{n+1}} [u_h' \Pi_{n-1} u_h + u_h \Pi_{n-1} u_h] dt \\ = \int_{t_n}^{t_{n+1}} [u_h' u_h + \Pi_{n-1} u_h \Pi_{n-1} u_h] dt \end{aligned}$$

Gives stability result:

$$\begin{aligned} u_h^2(t_{n+1}) + \int_{t_n}^{t_{n+1}} [\Pi_{n-1} u_h]^2 dt \\ \leq u_h^2(t_n) + \int_{t_n}^{t_{n+1}} f^2 dt. \end{aligned}$$

Again sum to get:

$$u_h^2(t_n) + \int_{t_0}^{t_n} [\Pi_{n-1} u_h]^2 dt \leq g^2 + \int_{t_0}^{t_n} f^2 dt.$$

What is this scheme if  $n=1$  and trapezoidal rule used to evaluate right hand side integral?

$$(u_{n+1} - u_n)/k + (u_{n+1} + u_n)/2 = (f_{n+1} + f_n)/2.$$

Next consider discontinuous approximation, where continuity condition implemented weakly by:

$$\begin{aligned} \int_{t_n}^{t_{n+1}} (u_h' v_h + u_h v_h) dt + u_h^+(t_n) v_h^+(t_n) \\ = \int_{t_n}^{t_{n+1}} f v_h dt + u_h^-(t_n) v_h^+(t_n). \end{aligned}$$

Look for  $u_h \in P_n$  and take  $v_h \in P_n$ . Now test function  $v_h = u_h$  legal. Note:

$$\begin{aligned} \int_{t_n}^{t_{n+1}} u_h' u_h dt + [u_h^+(t_n) - u_h^-(t_n)] u_h^+(t_n) \\ = \frac{1}{2} [u_h^2(t_{n+1}^-) - u_h^2(t_n^+)] + [u_h^+(t_n) - u_h^-(t_n)] u_h^+(t_n) \\ = \frac{1}{2} u_h^2(t_{n+1}^-) - \frac{1}{2} u_h^2(t_n^-) + \frac{1}{2} [u_h^+(t_n) - u_h^-(t_n)]^2 \end{aligned}$$

When we sum, we get cancellation as before plus control of another nonnegative term (the jumps in  $u_h$  across the mesh points).

Scheme makes sense even for  $n = 0$ . Get

$$(u_{n+1/2} - u_{n-1/2})/k + u_{n+1/2} = f_{n+1/2}$$

if we integrate by midpoint rule. Backward Euler.

Return to wave equation to use these ideas for time discretization.

Space-time Variational formulation:

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} u_{tt} v \, dx \, dt + \int_{t_n}^{t_{n+1}} \int_{\Omega} \nabla u \cdot \nabla v \, dx \, dt = 0$$

Initially, know  $u$  and  $u_t$ . To design method, might try to get stability by using standard trick for stability of wave equation, i.e., take  $v = u_t$ . Gives conservation of energy,

$$\int_{\Omega} (u_t^2 + |\nabla u|^2)(t_{n+1}) \, dx = \int_{\Omega} (u_t^2 + |\nabla u|^2)(t_n) \, dx.$$

Try for method where  $v_h = (u_h)_t$  is legal test function.

If on first subinterval, we implement the 2 IC strongly, might seek  $u_h$  such that  $u_h$  is tensor product of  $C^0$  piecewise linear functions in space with polynomials of degree  $\leq 2$  in time.

In time, have only one condition to determine, so natural choice of  $v_h$  is tensor product of  $C^0$  piecewise linear functions in space with polynomials of degree 0 in time. Problem:  $v_h = [u_h]_t$  does not belong to this space.

Instead, implement  $u(x, 0)$  IC strongly and  $u_t(x, 0)$  weakly, i.e.,

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} u_{tt} v \, dx \, dt + \int_{t_n}^{t_{n+1}} \int_{\Omega} \nabla u \cdot \nabla v \, dx \, dt + \int_{\Omega} [u_t(t_n^+) - u_t(t_n^-)] v(t_n^+) \, dx = 0.$$

On first subinterval, implement  $u(x, 0)$  strongly. Seek  $u_h$  such that  $u_h$  is tensor product of continuous piecewise linear functions in space with polynomials of degree  $\leq 1$  in time.

In time, we have only one condition to determine, so natural choice of  $v_h$  would be tensor product of continuous piecewise linear functions in space with polynomials of degree 0 in time. Now  $v_h = (u_h)_t$  belongs to this space. Then (dropping subscript  $h$ ),

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} u_{tt} u_t dx dt + \int_{\Omega} [u_t(t_n^+) - u_t(t_n^-)] u_t(t_n^+) dx = 0.$$

Get result similar to before with sum of squares of jumps of  $u_t$  across  $t_n$ .

Can also develop methods with exact energy conservation. Let  $w = u_t$  and consider system:  $u_t = w$ ,  $w_t - \Delta u = 0$ . Space time variational formulation:

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} u_t z dx dt = \int_{t_n}^{t_{n+1}} \int_{\Omega} w z dx dt,$$

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} w_t v dx dt + \int_{t_n}^{t_{n+1}} \int_{\Omega} \nabla u \cdot \nabla v dx dt = 0.$$

Choose  $v = u_t$  and  $z = w_t$  and subtract first equation from second. Then

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} (w_t w + \nabla u \cdot \nabla u_t) dx dt = 0$$

and so

$$\begin{aligned} \int_{\Omega} (w^2 + |\nabla u|^2)(t_{n+1}) dx \\ = \int_{\Omega} (w^2 + |\nabla u|^2)(t_n) dx. \end{aligned}$$

Take approximations  $u_h$  and  $w_h$  piecewise polynomials in time of degree  $\leq n$  (continuous) tensor product a spacial finite element space and  $v_h$  and  $z_h$  piecewise polynomials in time of degree  $\leq n-1$ , (discontinuous) tensor product the same spacial finite element space.

Then test functions  $v = u_t$  and  $z = w_t$  also legal on finite element space, so we get same estimate.

## First order hyperbolic systems

Let  $\Omega$  – bounded domain in  $\mathbb{R}^N$  with boundary  $\Gamma$ .

Let  $\Omega_T = \Omega \times (0, T)$ ,  $\Gamma_T = \Gamma \times (0, T)$ .

$$\begin{aligned} Lu &\equiv \partial u / \partial t + \sum_{i=1}^N A_i \partial u / \partial x_i + Bu = F \\ &\text{in } \Omega \times (0, T), \\ (M - D)u &= 0 \quad \text{on } \Gamma \times (0, T), \\ u(x, 0) &= g \quad \text{in } \Omega \end{aligned}$$

$A_i, B, M$  given  $m \times m$  matrices depending on  $x$ ,  $u$  an  $m$ -vector.

$D = \sum_{i=1}^N A_i n_i$ , where  $n = (n_i)$  is unit outward normal to  $\Gamma$ .

### IBVP for 2-D Damped Wave equation: $m = 3$ :

Let  $\Omega \subset \mathbb{R}^2$ ,  $\beta \geq 0$ .

$$\begin{aligned} w_{tt} + \beta w_t - w_{xx} - w_{yy} &= f, \quad \text{in } \Omega, \\ w, w_t \text{ given at } t = 0, \quad w &= 0 \text{ on } \Gamma. \end{aligned}$$

Let  $\mathbf{u} = (u_0, u_1, u_2) = (w_t, w_x, w_y)$ . Then

$$\begin{aligned} \mathbf{u}_t + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{u}_x + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mathbf{u}_y \\ + \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{u} &= \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Here

$$D = \begin{pmatrix} 0 & -n_x & -n_y \\ -n_x & 0 & 0 \\ -n_y & 0 & 0 \end{pmatrix}, M = \begin{pmatrix} 2 & -n_x & -n_y \\ n_x & 0 & 0 \\ n_y & 0 & 0 \end{pmatrix}$$

$$D - M = 2 \begin{pmatrix} -1 & 0 & 0 \\ -n_x & 0 & 0 \\ -n_y & 0 & 0 \end{pmatrix}.$$

Consider systems positive in sense of Friedrichs.

Let  $\sigma \geq 0$  constant and assume:

$$A_i(\mathbf{x}) \text{ symmetric, } i = 1, \dots, N,$$

$$M + M^T \geq 0 \text{ on } \Gamma,$$

$$B + B^T - \sum_{i=1}^N \frac{\partial A_i}{\partial x_i} \geq \sigma I, \text{ in } \Omega,$$

where

$$E \geq F \text{ means } \mathbf{x}^T (E - F) \mathbf{x} \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^m,$$

With these properties, get stability estimate for continuous problem.

By Green's formula:

$$\begin{aligned} \int_{\Omega_T} Lu \cdot v &= \int_{\Omega} \mathbf{u}(T) \mathbf{v}(T) - \int_{\Omega} \mathbf{u}(0) \mathbf{v}(0) \\ &\quad + \int_{\Gamma_T} D\mathbf{u} \cdot \mathbf{v} + \int_{\Omega_T} \mathbf{u} \cdot L^* \mathbf{v}, \end{aligned}$$

where

$$L^* \mathbf{v} = -\frac{\partial \mathbf{v}}{\partial t} - \sum_{i=1}^N A_i \frac{\partial \mathbf{v}}{\partial x_i} - \sum_{i=1}^N \frac{\partial A_i}{\partial x_i} \mathbf{v} + B^* \mathbf{v}.$$

$$\begin{aligned} \int_{\Omega_T} L\mathbf{u} \cdot \mathbf{u} &= \frac{1}{2} \int_{\Omega_T} (L + L^*)\mathbf{u} \cdot \mathbf{u} \\ &\quad + \frac{1}{2} \int_{\Gamma_T} D\mathbf{u} \cdot \mathbf{u} + \frac{1}{2} \int_{\Omega} \mathbf{u}(T)v(T) \\ &\quad - \frac{1}{2} \int_{\Omega} \mathbf{u}(0)v(0) \end{aligned}$$

$$\text{But } L + L^* = B + B^* - \sum_{i=1}^N \partial A_i / \partial x_i \geq \sigma I,$$

and  $(D - M)\mathbf{u} = 0$  on  $\Gamma$ , so

$$D\mathbf{u} \cdot \mathbf{u} = M\mathbf{u} \cdot \mathbf{u} = \frac{1}{2}(M + M^*)\mathbf{u} \cdot \mathbf{u}.$$

Hence, stability follows from:

$$\begin{aligned} \int_{\Omega_T} \mathbf{F} \cdot \mathbf{u} + \frac{1}{2} \int_{\Omega} \mathbf{g}^2 &= \int_{\Omega_T} L\mathbf{u} \cdot \mathbf{u} \\ &\geq \frac{1}{2} \int_{\Omega} \mathbf{u}(T)^2 + \frac{\sigma}{2} \int_{\Omega_T} \mathbf{u} \cdot \mathbf{u} + \frac{1}{4} \int_{\Gamma_T} (M + M^*)\mathbf{u} \cdot \mathbf{u} \\ &\geq \frac{\sigma}{2} \|\mathbf{u}\|_{L^2(\Omega_T)}^2 + \frac{1}{2} \int_{\Omega} \mathbf{u}(T)^2. \end{aligned}$$

## Numerical Methods

Let  $\tau_h^n$  be family of “triangulations” of strip  $S_n = \Omega \times (t_n, t_{n+1})$  indexed by  $h$ , maximum diameter of elements in strip.

$P_k(K)$  set of polynomials on  $K$  of total degree  $\leq k$ .

$$V_h^n = \{v \in H^1(S_n) : v|_K \in P_k(K) \quad \forall K \in \tau_h^n\},$$

$$W_h = \{v \in L^2(S_n) : v|_K \in P_k(K) \quad \forall K \in \tau_h^n\},$$

$$V_h = [V_h]^m, \quad W_h = [W_h]^m.$$

Set

$$(u, v)_n = \int_{S_n} u \cdot v \, dx \, dt,$$
$$\langle u, v \rangle_n = \int_{t_n}^{t_{n+1}} \int_{\Gamma} u \cdot v \, ds \, dt.$$

**Standard Galerkin method:** with weakly imposed B.C. and continuity conditions

Find  $\mathbf{u}_h^n \in \mathbf{V}_h$  such that  $\forall \mathbf{v} \in \mathbf{V}_h$

$$\begin{aligned} (L\mathbf{u}_h^n, \mathbf{v})_n + \frac{1}{2} \langle (M - D)\mathbf{u}_h^n, \mathbf{v} \rangle_n \\ + \int_{\Omega} \mathbf{u}_+^n(t_n) \cdot \mathbf{v}_+(t_n) dx \\ = (\mathbf{F}, \mathbf{v}) + \int_{\Omega} \mathbf{u}_-^n(t_n) \cdot \mathbf{v}_+(t_n) dx. \end{aligned}$$

**Stability Estimate:**

Choose test function  $\mathbf{v} = \mathbf{u}_h^n$ . Estimate follows in similar way to continuous problem with two simple tricks.

**Error Estimate:**

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_T)} \leq Ch^k \|\mathbf{u}\|_{k+1}.$$

Note: less than optimal order by one power of  $h$ .

## Streamline Diffusion Method:

Find  $\mathbf{u}_h^n \in V_h$  such that  $\forall \mathbf{v} \in V_h$

$$\begin{aligned} (L\mathbf{u}_h^n, \mathbf{v} + hL_0\mathbf{v})_n + \frac{1}{2} \langle (M - D)\mathbf{u}_h^n, \mathbf{v} \rangle_n \\ + \int_{\Omega} \mathbf{u}_+^n(t_n) \cdot \mathbf{v}_+(t_n) dx \\ = (\mathbf{F}, \mathbf{v} + hL_0\mathbf{v}) + \int_{\Omega} \mathbf{u}_-^n(t_n) \cdot \mathbf{v}_+(t_n) dx, \end{aligned}$$

where  $L_0 = \partial/\partial t + \sum_{i=1}^N A_i \partial/\partial x_i$ .

## Improved Stability Estimate:

$$\begin{aligned} \sqrt{h} \|L_0 \mathbf{u}_h\|_{L^2(\Omega_T)} + \sigma \|\mathbf{u}_h\|_{L^2(\Omega_T)} + \|\mathbf{u}_h(T)\|_{L^2(\Omega)} \\ \leq C \|\mathbf{F}\|_{L^2(\Omega_T)} + \|\mathbf{g}\|_{L^2(\Omega)}. \end{aligned}$$

## Improved Error Estimate:

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_T)} \leq Ch^{k+1/2} \|\mathbf{u}\|_{k+1}.$$

Another possibility:  $\delta = 0$  or  $1$ .

**Discontinuous Galerkin** on each strip.

Let  $W_h^n \subset L^2(S_n)$ ,  $\mathbf{W}_h^n = [W_h^n]^m$ .

Use previous formulation, but replace

$$(Lu_h^n, v + \delta L_0 v)_{S_n} \quad \text{by} \quad \sum_{K \in \tau_h^n} (Lu_h^n, v + \delta L_0 v)_K$$

$$(F, v + \delta h L_0 v)_{S_n} \quad \text{by} \quad \sum_{K \in \tau_h^n} (F, v + \delta h L_0 v)_K$$

and add boundary integral terms

$$\sum_{K \in \tau_h^n} \frac{1}{2} \langle (M_K - D_K)[u_h^n], v \rangle_K .$$

Note: Within each strip methods are implicit: must solve for all unknowns at the same time.

## An Explicit Method

Divide space-time domain into mesh of elements  $K$ , union of simplices in  $\mathbb{R}^{N+1}$ .

Let  $\Gamma^*(K) = \Gamma(K) \cap (\Gamma \times (0, T))$ . On  $\Gamma(K) - \Gamma^*(K)$ , require  $n_t \neq 0$  and define  $D_t \equiv \text{sign}\{n_t\} D$ .

Define bilinear form

$$a_K(\mathbf{u}, \mathbf{v}) \equiv (L\mathbf{u}, \mathbf{v})_K + \int_{\Gamma_{in}(K)} [D_t \mathbf{u}] \cdot \mathbf{v} + \frac{1}{2} \int_{\Gamma^*(K)} (M - D)\mathbf{u} \cdot \mathbf{v},$$

where,  $\Gamma_{in}(K)$  [ $\Gamma_{out}(K)$ ] denotes portion of  $\Gamma(K) - \Gamma^*(K)$  where  $n_t < 0$  [ $n_t > 0$ ].

On each element  $K$ , approximate  $\mathbf{u}$  by  $\mathbf{u}_h \in \mathbf{P}_n(K)$ , polynomials of total degree  $\leq n$  over  $K$ .

Starting from appropriate interpolant  $\mathbf{u}_h^-(x, 0)$  of initial condition  $\mathbf{u}_0$ , develop solution explicitly by:

$$a_K(\mathbf{u}_h, \mathbf{v}) = (\mathbf{F}, \mathbf{v})_K \quad \text{for all } \mathbf{v} \in \mathbf{P}_n(K).$$

Key idea: Special mesh construction allowing explicit method without violating domain of dependence considerations.

Mesh for 1-D wave equation as first order system.  
 Need  $|n_t| > |n_x|$  for stability.

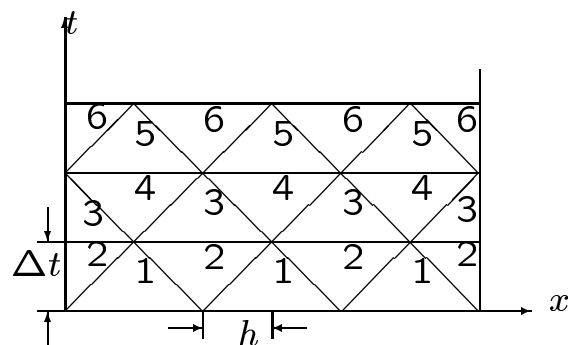


Fig. 1a

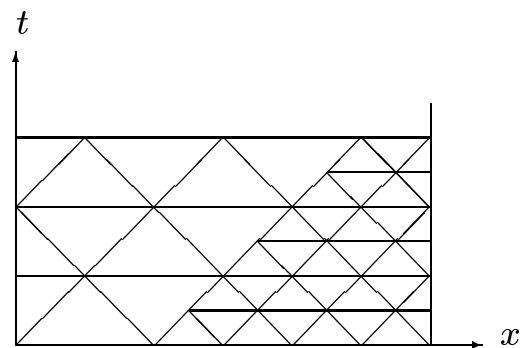


Fig. 1b

An alternative mesh: no discrete time levels, but half as many elements. Only one generic interior element: a rhombus standing on end, suitably compressed in the  $t$ -direction to satisfy stability condition.

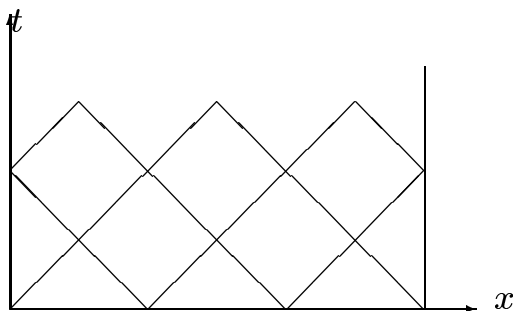


Fig. 2a

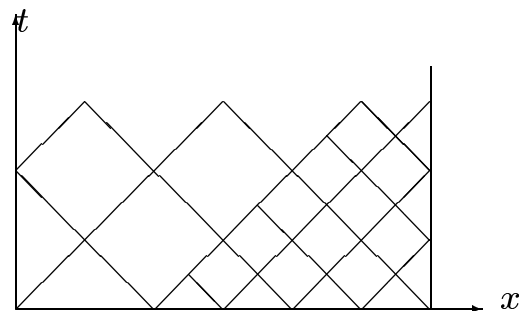


Fig. 2b

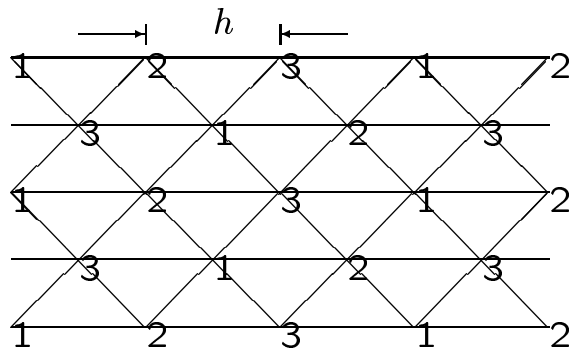
Example:  $\Omega \subset \mathbb{R}^2$ . Starting from triangulation of  $\Omega$ , obtain space-time elements  $\{K\}$  whose surfaces (except those contained by  $\Gamma \times [0, T]$ ) have outer normal  $\mathbf{n} = (n_x, n_t)$  satisfying

$$|n_x|/|n_t| \leq \theta$$

for arbitrary  $\theta > 0$ .

Choose  $\theta$  small enough to satisfy explicitness condition.

Consider mesh of equilateral triangles of  $\Omega$ .

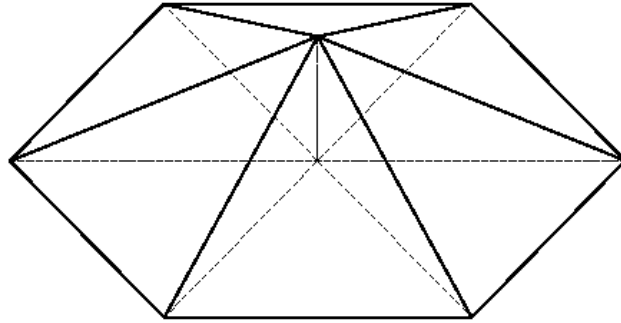


To construct solution in time slab  $\Omega \times [0, \Delta t]$ :

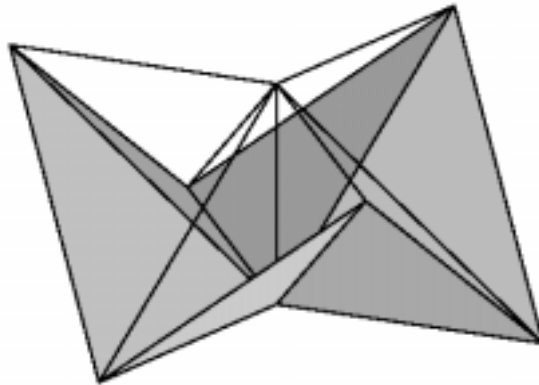
First step: Develop solution simultaneously in all elements  $K$  centered above vertices of type 1. Macroelement of type 1 consists of 6 tetrahedra.

Second step: Develop solution simultaneously in all elements  $K$  centered above vertices of type 2. Macroelement of type 2 consists of 6 tetrahedra.

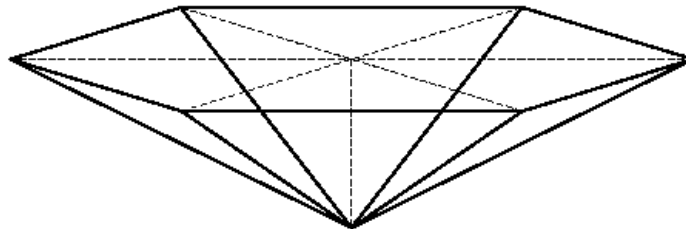
Third step: Develop solution simultaneously in all elements  $K$  centered above vertices of type 3. Brings solution to next time level.



Macroelement of type 1

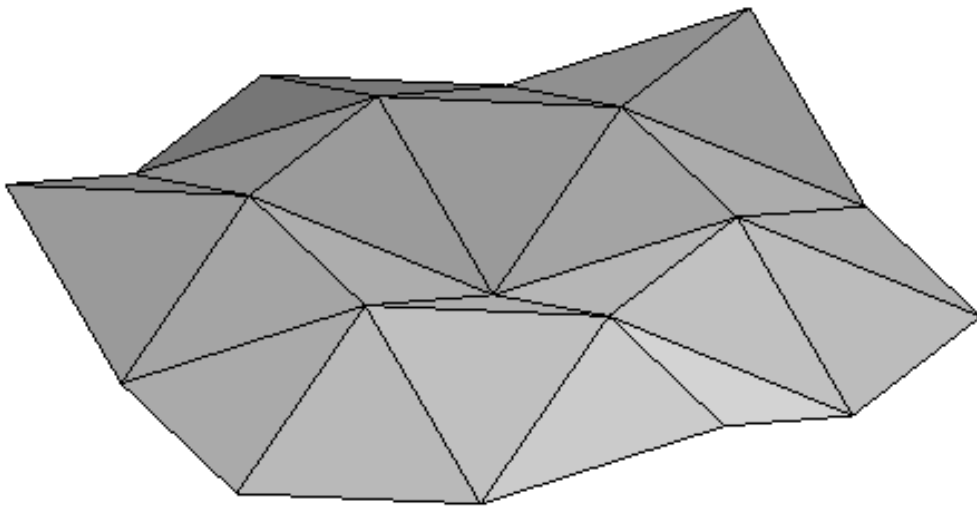


Macroelement of type 2



Macroelement of type 3

In Figure below, a developing mesh is shown, just after formation of elements of type 1 and 2. Note that an element of type 3 is now needed to bring the mesh up to the next time level.



Developing Mesh