Superconvergence in some locally conservative discretization methods

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Outline

- Overview of superconvergence in mixed finite element methods
- Superconvergence in control-volume mixed finite element methods
- Superconvergence in mimetic finite difference methods

Model problem: single phase flow in porous media

$$\mathbf{u} = -K\nabla p \qquad \text{in } \Omega \subset \mathbf{R}^d \ (d = 2,3) \qquad \text{(Darcy's law)}$$

$$\nabla \cdot \mathbf{u} = f \qquad \text{in } \Omega \qquad \text{(conservation of mass)}$$

$$\mathbf{u} \cdot \nu = 0 \qquad \text{on } \partial\Omega \qquad \text{(no flow BC)}$$

$\mathbf{u}$ - velocity, p - pressure, K - permeability tensor, q - source (wells)

Variational mixed formulation

$$H(\text{div};\Omega) = \{\mathbf{v} : \mathbf{v} \in (L^2(\Omega))^d, \ \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

$$\mathbf{V} = \{\mathbf{v} \in H(\text{div};\Omega) : \mathbf{v} \cdot \nu = 0 \text{ on } \partial\Omega\}; \quad \|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2}$$

$$W = L_0^2(\Omega) = \{w \in L^2(\Omega) : \int_\Omega w\,dx = 0\}.$$

Find $\mathbf{u} \in \mathbf{V}$, $p \in W$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}), \qquad \mathbf{v} \in \mathbf{V},$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w), \qquad w \in W.$$

The mixed finite element method

$\mathcal{T}_h$ - finite element partition

$\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ - mixed finite element spaces

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}) &= (p_h, \nabla \cdot \mathbf{v}), && \mathbf{v} \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, w) &= (f, w), && w \in W_h. \end{aligned}$$

Properties:

- Simultaneous (accurate) approximation of pressure and velocity
- Local mass conservation: for each element E,

$$w = \begin{cases} 1 \text{ on } E, \\ 0 \text{ otherwise} \end{cases} \Longrightarrow \int_E \nabla \cdot \mathbf{u}_h = \int_E f.$$

- Continuity of normal flux across element faces: for each $e = \partial E_1 \cap \partial E_2$,

$$\mathbf{u}_h\big|_{E_1} \cdot \nu_e = \mathbf{u}_h\big|_{E_2} \cdot \nu_e.$$

Mixed finite element spaces

Raviart-Thomas(1977); Nedelec(1980) Brezzi-Douglas-Marini(1985)

Brezzi-Douglas-Duran-Fortin(1987) Brezzi-Douglas-Fortin-Marini(1987)

Key properties:

- $\nabla \cdot \mathbf{V}_h = W_h$
- $\exists\ \Pi : (H^1(\Omega))^d \to \mathbf{V}_h$ such that for $\mathbf{q} \in (H^1(\Omega))^d$

$$(\nabla \cdot (\Pi \mathbf{q} - \mathbf{q}), w) = 0, \quad w \in W_h.$$

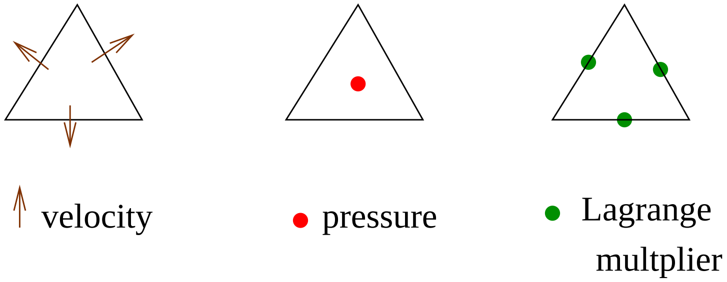
- inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}, w)}{\|\mathbf{v}\|_{\mathbf{V}}} \geq \beta \|w\|_W, \quad w \in W_h$$

Raviart-Thomas mixed finite element spaces

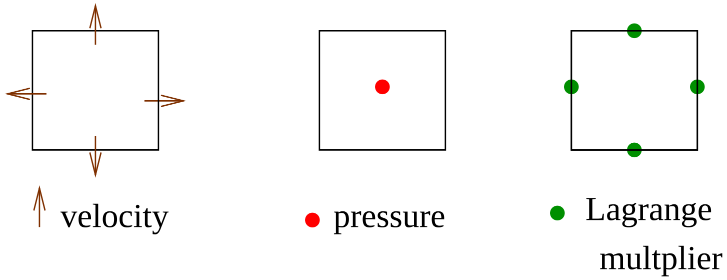
- Triangular element:

$$\mathbf{V}_h^k(E) = (P_k(E))^2 + \mathbf{x}P_k(E)$$
$$W_h^k(E) = P_k(E)$$



- Rectangular element:

$$\mathbf{V}_h^k(E) = P_{k+1,k}(E) \times P_{k,k+1}(E)$$
$$W_h^k(E) = P_{k,k}(E)$$



The operator Π onto $\mathbf{V}_h$ is defined locally:

$$\int_e (\Pi\mathbf{q} - \mathbf{q}) \cdot \nu = 0, \qquad \text{for each side } e \subset \partial E.$$

Approximation property:

$$\|\mathbf{q} - \Pi\mathbf{q}\| \leq Ch^m |\mathbf{q}|_m, \qquad 1 \leq m \leq k+1.$$

Convergence theory for mixed methods

Optimal convergence:

$$\|p - p_h\|_W + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} = O(h^{k+1})$$

Brezzi (1974), Fortin (1977)
Falk, Osborn (1980), Johnson, Thomee (1981)
Douglas, Roberts (1985)
Gastaldi, Nochetto (1987)

Superconvergence:

$$|||p - p_h|||_W + |||\mathbf{u} - \mathbf{u}_h|||_{\mathbf{V}} = O(h^{k+2})$$

Nakata, Weiser, Wheeler (1985), Weiser, Wheeler (1988)
Douglas, Roberts (1985), Douglas, Milner (1985), Arnold, Brezzi (1985)
Douglas, Wang (1989), Duran (1990), Ewing, Lazarov, Wang (1991)
Arbogast, Wheeler, Yotov(1997), Arbogast, Dawson, Keenan, Wheeler, Yotov(1998)
Brandts (1994), Dupont, Keenan (1998)
Ewing, Liu, Wang (1999)

Key steps in velocity superconvergence analysis

Error equations:

$$
\begin{aligned}
(K^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{v}) &= (p-p_h,\nabla\cdot\mathbf{v}), && \mathbf{v}\in\mathbf{V}_h,\\
(\nabla\cdot(\mathbf{u}-\mathbf{u}_h),w) &= 0, && w\in W_h.
\end{aligned}
$$

Taking $\mathbf{v}=\Pi\mathbf{u}-\mathbf{u}_h$ and using that $\nabla\cdot(\Pi\mathbf{u}-\mathbf{u}_h)=0$,

$$(K^{-1}(\Pi\mathbf{u}-\mathbf{u}_h),\Pi\mathbf{u}-\mathbf{u}_h) = (K^{-1}(\Pi\mathbf{u}-\mathbf{u}),\Pi\mathbf{u}-\mathbf{u}_h).$$

Lemma (Duran 1990): On rectangular grids, if $\mathbf{q}\in(P_{k+1}(E))^2$, then

$$\int_E(\mathbf{q}-\Pi\mathbf{q})\cdot\mathbf{v}=0,\quad \forall\,\mathbf{v}\in\mathbf{V}_h.$$

The Bramble-Hilbert lemma implies

Theorem (Duran 1990, Ewing, Lazarov, Wang 1991): For diagonal K,

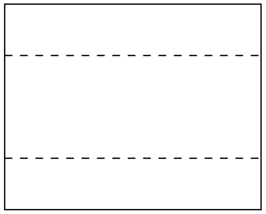
$$\|\Pi\mathbf{u}-\mathbf{u}_h\|\le Ch^{k+2}.$$

Superconvergence along Gaussian lines

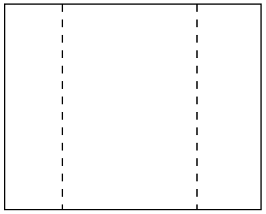
Following Nakata,Weiser,Wheeler(1985), Duran(1990), Ewing,Lazarov,Wang(1991).

For any element $E = [a_1, b_1] \times [a_2, b_2]$ define

$$|||q_1|||_{1,E}^2 = \sum_{j=1}^{k+1} A_j (b_2 - a_2) \int_{a_1}^{b_1} |q_1(x_1, g_j^2)|^2 \, dx_1, \quad k = 1:$$



$$|||q_2|||_{2,E}^2 = \sum_{j=1}^{k+1} A_j (b_1 - a_1) \int_{a_2}^{b_2} |q_2(g_j^1, x_2)|^2 \, dx_2, \quad k = 1:$$



where $g_1^i, \ldots, g_{k+1}^i$ are the Gaussian points on $[a_i, b_i]$ and $A_1, \ldots, A_{k+1}$ are the coefficients of Gaussian quadrature on $[-1, 1]$. Let

$$|||\mathbf{q}|||^2 = \sum_{i=1}^{2} \sum_{E \in \mathcal{T}_h} |||q_i|||_{i,E}^2.$$

Note: for $\mathbf{q} \in \mathbf{V}_h$, $|||\mathbf{q}||| = \|\mathbf{q}\|$.

Superconvergence along Gaussian lines

Theorem: For rectangular grids and diagonal K,

$$|||\mathbf{u} - \mathbf{u}_h||| \le Ch^{k+2}$$

Proof:

$$|||\mathbf{u} - \mathbf{u}_h||| \le |||\mathbf{u} - \Pi\mathbf{u}||| + |||\Pi\mathbf{u} - \mathbf{u}_h|||$$

$$|||\mathbf{q} - \Pi\mathbf{q}||| = 0 \text{ for } \mathbf{q} \in (P_{k+1}(E))^2 \Rightarrow |||\mathbf{u} - \Pi\mathbf{u}||| \le Ch^{k+2}$$

$$|||\Pi\mathbf{u} - \mathbf{u}_h||| = \|\Pi\mathbf{u} - \mathbf{u}_h\| \le Ch^{k+2} \quad \Box$$

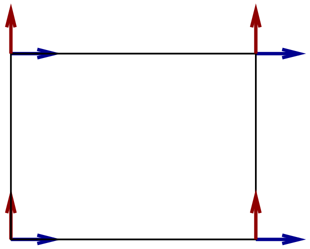
Postprocessing

Following Duran (1990).

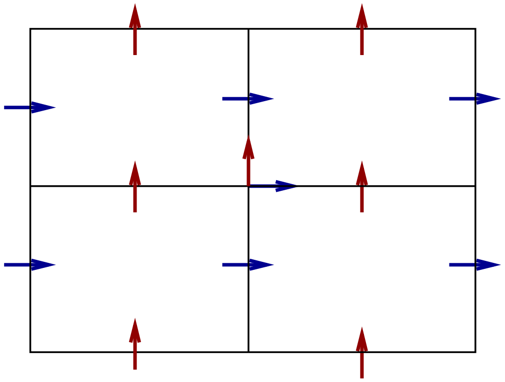
Idea: Postprocess the solution to obtain a higher-order approximation.

Construction for $RT_0 = P_{1,0}(E) \times P_{0,1}(E)$:

$R(u_h) \in P_{1,1}(E) \times P_{1,1}(E)$



Interpolation to define $R(u_h)$ at a vertex:



Theorem:

$$\|\mathbf{u} - R(\mathbf{u}_h)\| \leq Ch^{k+2}$$

Key steps in pressure superconvergence analysis

Following Douglas, Roberts (1985), Nakata,Weiser,Wheeler(1985).

Duality argument: Let Q_h be the orthogonal L^2-projection onto W_h. Solve

$$\begin{aligned} -\nabla \cdot K \nabla \varphi &= Q_h p - p_h, \quad \text{in } \Omega, \\ -K \nabla \varphi \cdot \nu &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Note: well posed since $\int_\Omega (Q_h p - p_h) = 0$. Elliptic regularity:

$$\|\varphi\|_2 \leq C \|Q_h p - p_h\|$$

Let $\phi = -\mathcal{K} \nabla \varphi$.

$$\begin{aligned} \|Q_h p - p_h\|^2 &= (Q_h p - p_h, \nabla \cdot \phi) = (Q_h p - p_h, \nabla \cdot \Pi \phi) \\ &= (K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi \phi) \leq Ch \|\mathbf{u} - \mathbf{u}_h\| \|\varphi\|_2. \end{aligned}$$

Theorem: For any regular finite element partition,

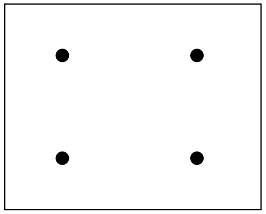
$$\|Q_h p - p_h\| \leq Ch^{k+2}.$$

Pressure superconvergence at Gaussian points

Following Nakata,Weiser,Wheeler(1985), Ewing,Lazarov,Wang(1991).

For any element $E = [a_1, b_1] \times [a_2, b_2]$ define

$$|||w|||^2 = \sum_E \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} A_i A_j (b_1 - a_1)(b_2 - a_2) w(g_i^1, g_j^2)^2, \quad k = 1 :$$



Note: for $w \in W_h$, $|||w||| = \|w\|$.

Theorem:

$$|||p - p_h||| \leq Ch^{k+2}$$

Proof:

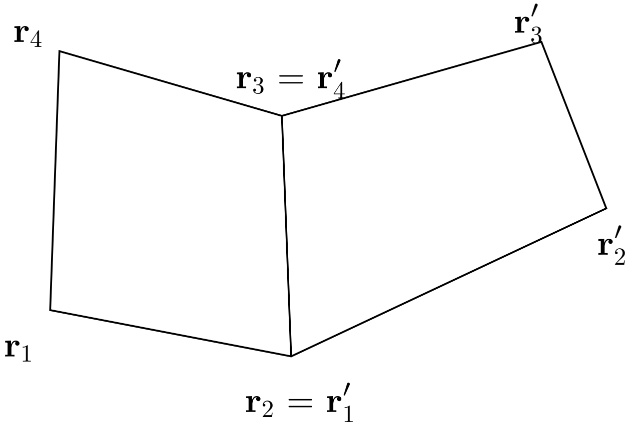
$$|||p - p_h||| \leq |||p - Q_h p||| + |||Q_h p - p_h|||.$$

$$|||w - Q_h w||| = 0 \text{ for } w \in P_{k+1}(E) \Rightarrow |||p - Q_h p||| \leq Ch^{k+2}.$$

$$|||Q_h p - p_h||| = \|Q_h p - p_h\| \leq Ch^{k+2}. \quad \square$$

Superconvergence on quadrilaterals

Following Ewing, Liu, Wang (1999). Assume h^2-uniform quadrilateral grid.



Each element is an h^2-parallelogram:

$$\|(\mathbf{r}_2 - \mathbf{r}_1) - (\mathbf{r}_3 - \mathbf{r}_4)\| \leq Ch^2.$$

Any two adjacent quadrilaterals form an h^2-parallelogram:

$$\|(\mathbf{r}_2 - \mathbf{r}_1) - (\mathbf{r}_2' - \mathbf{r}_1')\| \leq Ch^2,$$

Theorem: For the Neumann problem,

$$\|\Pi\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|Q_h p - p_h\|_W \leq Ch^{k+2}.$$

Algebraic system for the mixed method

$$\begin{pmatrix} M & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

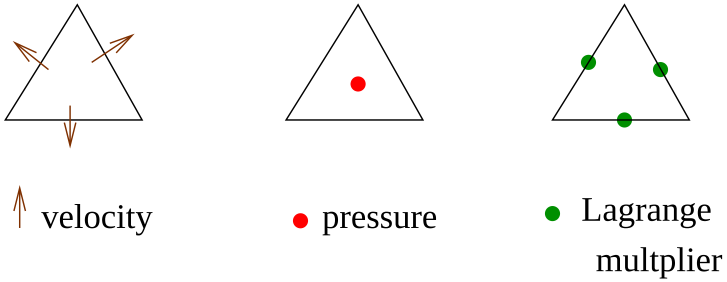
A saddle-point problem! The Shur complement system

$$-B^T M^{-1} B \ P = Q$$

is symmetric and positive semi-definite, but dense (inverting M required on each conjugate gradient iteration).

Hybrid mixed method

Arnold, Brezzi (1985) introduce pressure Lagrange multipliers λ on all element faces. V_h is discontinuous. The mass matrix M is block-diagonal.



$$\begin{pmatrix} M & B & L \\ B^T & 0 & 0 \\ L^T & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ F \\ 0 \end{pmatrix} \quad \Rightarrow S\lambda = g.$$

The Shur complement S is sparse, symmetric, and positive semi-definite.

Postprocessing for pressure: Using p_h and λ_h, construct $R(p_h) \in P_{k+1}(E)$ (non-conforming) such that

$$\|p - R(p_h)\| \leq Ch^{k+2}.$$

Drawback: face centered stencil - there are about 2-3 times more faces than cells.

Cell-centered finite differences

Russell, Wheeler (1983) - diagonal tensor K - employ quadrature rules to diagonalize the mass matrix M. The Shur complement for the pressure $B^TM^{-1}B$ is sparse. The stencil is cell-centered.

Arbogast, Wheeler, Y. (1997) - the expanded mixed method - generalize for full tensors.

This approximation preserves the superconvergence of the method.

Introduce the pressure gradient

$$\tilde{\mathbf{u}} = -\nabla p, \quad \mathbf{u} = K\tilde{\mathbf{u}}; \quad \tilde{\mathbf{u}} \in \tilde{\mathbf{V}} = L^2(\Omega).$$

Find $\mathbf{u}_h \in \mathbf{V}_h$, $\tilde{\mathbf{u}}_h \in \tilde{\mathbf{V}}_h$, and $p_h \in W_h$ such that

$$\begin{aligned} (\tilde{\mathbf{u}}_h, \mathbf{v}) &= (p_h, \nabla\cdot\mathbf{v}), & \mathbf{v} &\in \mathbf{V}_h, \\ (\mathbf{u}_h, \tilde{\mathbf{v}}) &= (K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}), & \tilde{\mathbf{v}} &\in \tilde{\mathbf{V}}_h, \\ (\nabla\cdot\mathbf{u}_h, w) &= (f, w), & w &\in W_h. \end{aligned}$$

Cell-centered finite difference approximation

Let $\mathbf{V}_h \times W_h$ be the RT$_0$ spaces on rectangles. Take $\tilde{\mathbf{V}}_h = \mathbf{V}_h$.

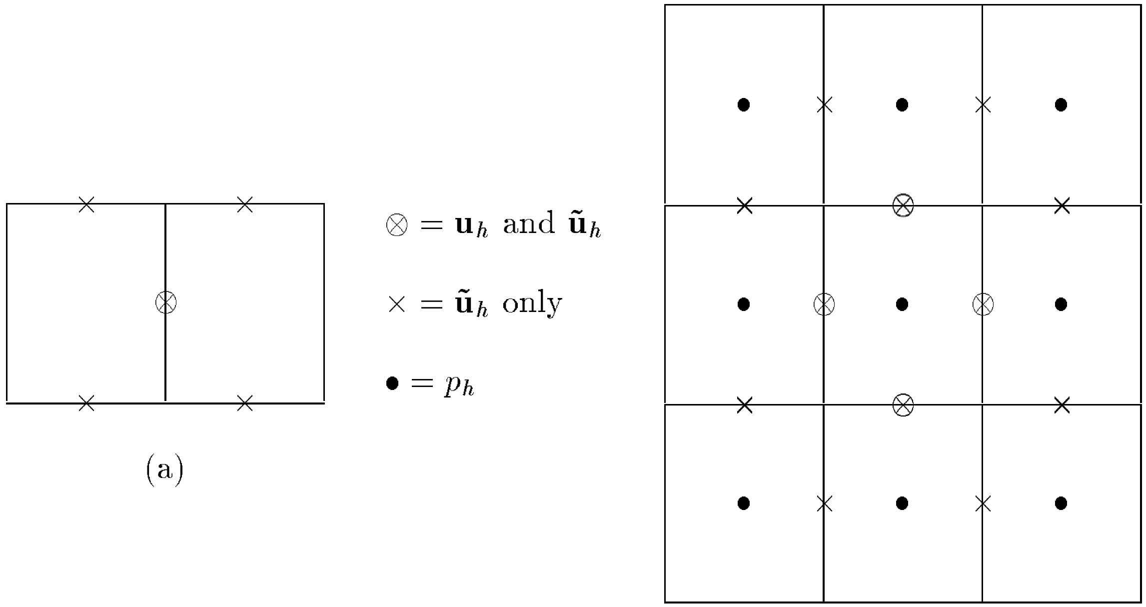
$$(\,\cdot\,,\,\cdot\,)_{\mathrm{M}} \text{ - midpoint rule,} \quad (\,\cdot\,,\,\cdot\,)_{\mathrm{T}} \text{ - trapezoidal rule}$$

$$(\mathbf{v}, \mathbf{q})_{\mathrm{TM}} = (v_1, q_1)_{\mathrm{T\times M\times M}} + (v_2, q_2)_{\mathrm{M\times T\times M}} + (v_3, q_3)_{\mathrm{M\times M\times T}}$$

Approximate

$$(\tilde{\mathbf{u}}_h, \mathbf{v})_{\mathrm{TM}}, \quad (\mathbf{u}_h, \tilde{\mathbf{v}})_{\mathrm{TM}}, \quad \text{and} \quad (\mathcal{K}\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\mathrm{T}}.$$

Directly eliminate $\hat{\tilde{u}}_h$ and $\hat{\mathbf{u}}_h$. and obtain finite difference scheme for p_h at the cell centers. Stencil: 9 points if $d = 2$ and 19 points if $d = 3$.



(a)

Convergence results

Theorem (Arbogast, Wheeler, Y.):

$$\|p - p_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\| + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \le Ch,$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM}} \le Ch^r,$$

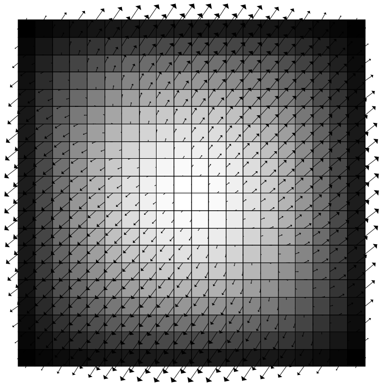
$$\|p - p_h\|_{\mathrm{M}} + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{\mathrm{M}} \le Ch^2,$$

where $r = 2$ if K is diagonal, $r = 3/2$ if the grids are C^2-map generated, and $r = 1$ otherwise.

If $\Omega_0 \subset\subset \Omega$ and the grids are C^2, then

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM},\Omega_0} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM},\Omega_0} \le C_\varepsilon h^{2-\varepsilon}.$$

Numerical results



$$\Omega = (0,1)^2, \quad p = (x - x^2)(y - y^2),$$

$$K = \begin{pmatrix} 11 & 9 \\ 9 & 13 \end{pmatrix}.$$

$\|p - p_h\|_{\mathrm{M}} \le C_p\, h^{\alpha_p}$ and $\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}} \le C_u\, h^{\alpha_u}$.

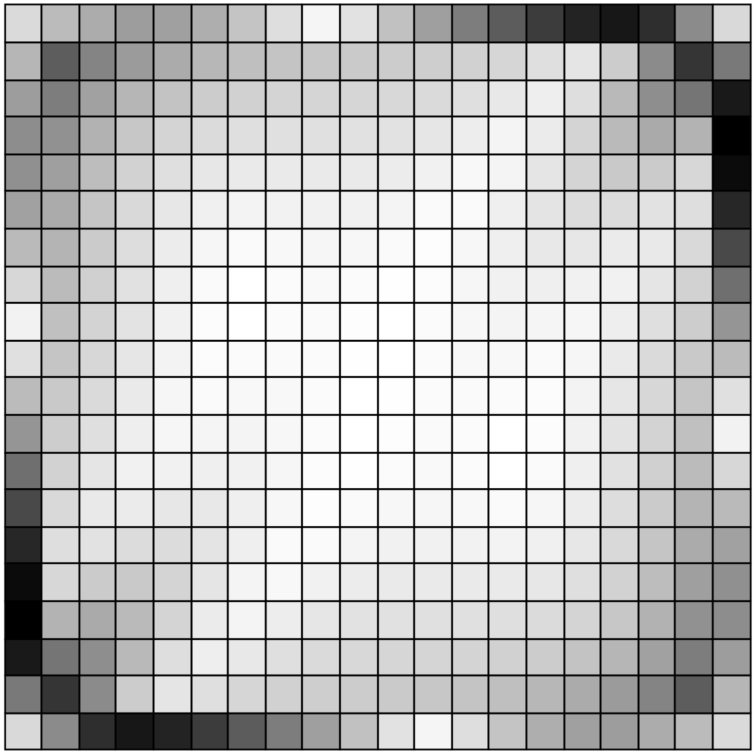
BC	C_p	α_p	C_u	α_u
Dirichlet	2.346	2.000	1.005	1.496
Neumann	3.340	1.998	0.839	1.472

Interior convergence rates:

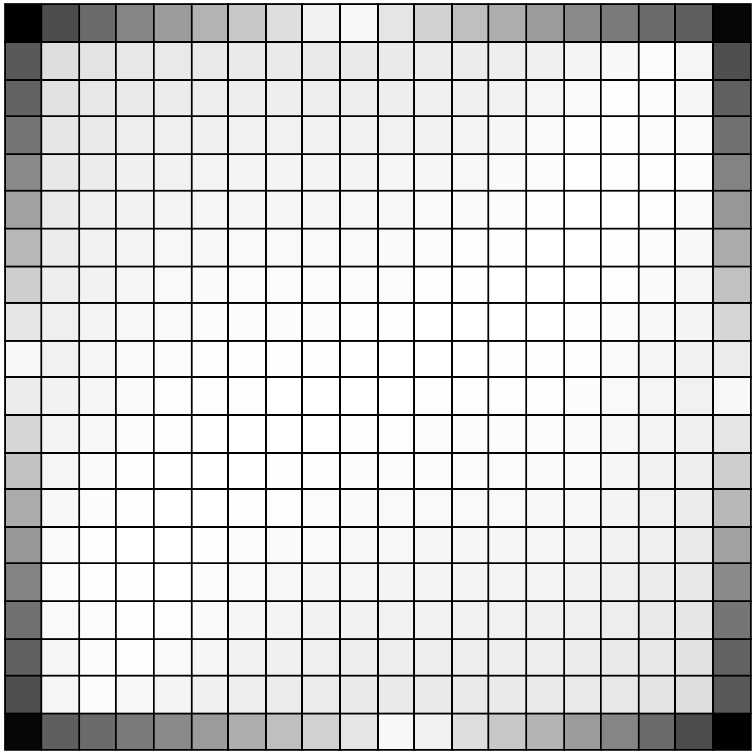
$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM},\Omega'_{\mathrm{h}}} \le C_u\, h^{\alpha_u}.$$

BC	C_u	α_u
Dirichlet	0.959	1.950
Neumann	1.897	1.836

Velocity error

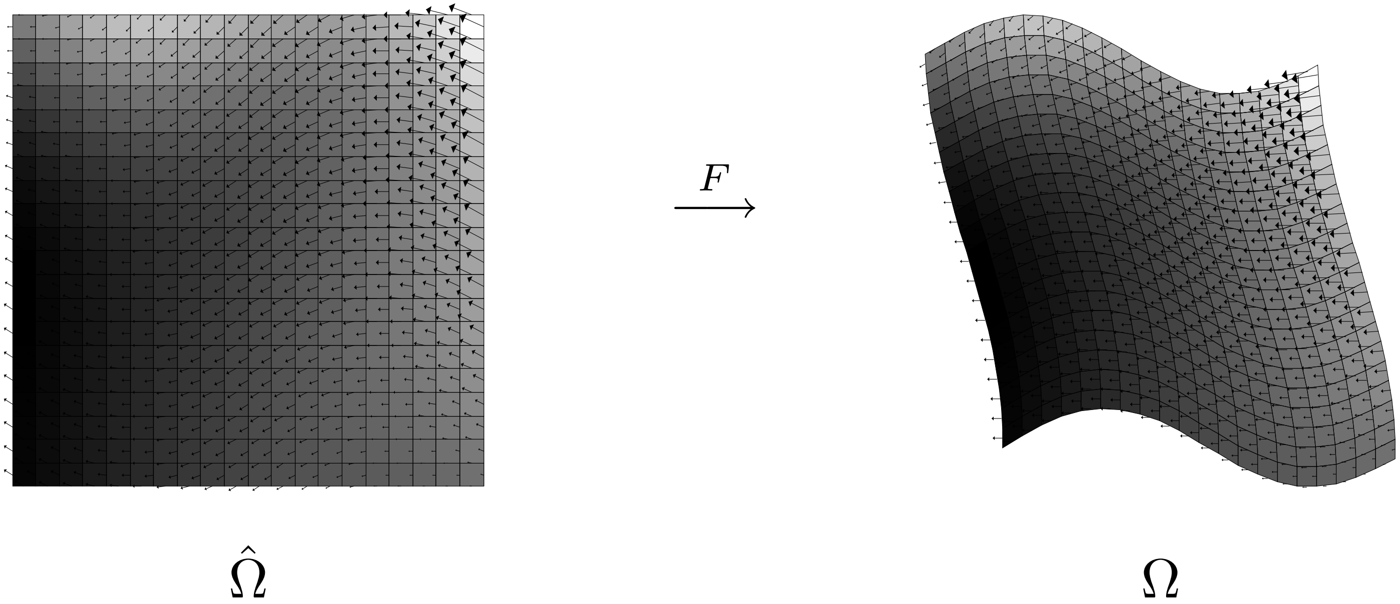


Neumann problem



Dirichlet problem

Smooth curvilinear grids



$$\Omega = F(\hat{\Omega}), \quad \mathcal{T}_h = F(\hat{\mathcal{T}}_h).$$

- DF is the Jacobian matrix of F : $(DF)_{ij} = \partial F_i / \partial \hat{x}_j$
- J is the Jacobian of F : $J = |\mathrm{det}(DF)|$

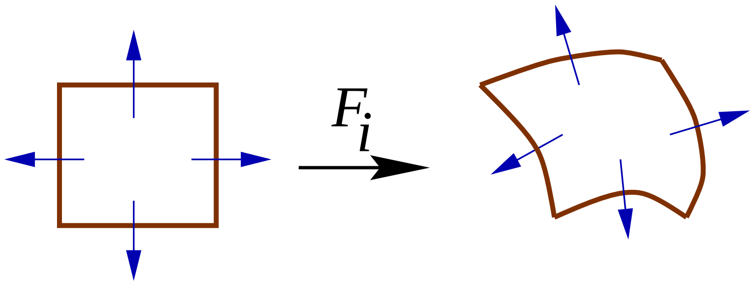
Mixed finite element spaces on curvilinear grids

Let

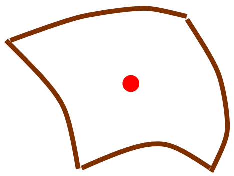
$$\hat{\mathbf{V}}_h \times \hat{W}_h \subset H(\mathrm{div};\hat{\Omega}) \times L^2(\hat{\Omega})$$

be any mixed spaces on $\hat{\mathcal{T}}_h$. Then $\mathbf{V}_h$ and W_h on $\mathcal{T}_h$ are defined as:

$$\mathbf{v}(x) = \frac{1}{J(\hat{x})} DF(\hat{x})\hat{\mathbf{v}}(\hat{x}) \quad (\mathbf{V}_h)$$



$$w(x) = \hat{w}(\hat{x}) \quad (W_h)$$



The velocity space is defined by the Piola transformation, which preserves the normal components across the element boundaries:

$$\int_e \mathbf{v} \cdot \nu = \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{\nu}.$$

The expanded mixed method on curvilinear grids

Let $G = J(DF^{-1})^T DF^{-1}$. Then

$$G\tilde{\mathbf{u}} = -\nabla p \qquad \xrightarrow{F^{-1}} \qquad \hat{\tilde{u}} = \hat{\nabla}\hat{p},$$

$$G\mathbf{u} = GKG\tilde{\mathbf{u}} \qquad \xrightarrow{F^{-1}} \qquad \hat{\mathbf{u}} = \mathcal{K}\hat{\tilde{u}}, \quad \mathcal{K} = JDF^{-1}K(DF^{-1})^T.$$

Find $\hat{\mathbf{u}}_h \in \hat{\mathbf{V}}_h$, $\hat{\tilde{u}}_h \in \hat{\tilde{\mathbf{V}}}_h$, and $\hat{p}_h \in \hat{W}_h$ such that

$$(\hat{\tilde{u}}_h, \hat{\mathbf{v}})_{\hat{\Omega}} = (\hat{p}_h, \hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h,$$

$$(\hat{\mathbf{u}}_h, \hat{\tilde{v}})_{\hat{\Omega}} = (\mathcal{K}\hat{\tilde{u}}_h, \hat{\tilde{v}})_{\hat{\Omega}}, \qquad \hat{\tilde{v}} \in \hat{\tilde{\mathbf{V}}}_h,$$

$$(\hat{\nabla}\cdot\hat{\mathbf{u}}_h, w)_{\hat{\Omega}} = (\hat{f}J, w)_{\hat{\Omega}}, \qquad w \in \hat{W}_h.$$

Logically rectangular grids: reduces to cell-centered finite differences on the reference grid.

The analysis on rectangular grids can be applied.

Convergence results for logically rectangular grids

Theorem (Arbogast, Dawson, Keenan, Wheeler, Y.):

If $F \in W^3_\infty$, then

$$\|p - p_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\| + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \le Ch,$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM}} \le Ch^r,$$

$$\|p - p_h\|_{\mathrm{M}} + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{\mathrm{M}} \le Ch^2,$$

where $r = 2$ if $\mathcal{K}$ is diagonal, and $r = 3/2$ otherwise.

If $\Omega_0 \subset\subset \Omega$, then

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM},\Omega_0} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM},\Omega_0} \le C_\varepsilon h^{2-\varepsilon}.$$

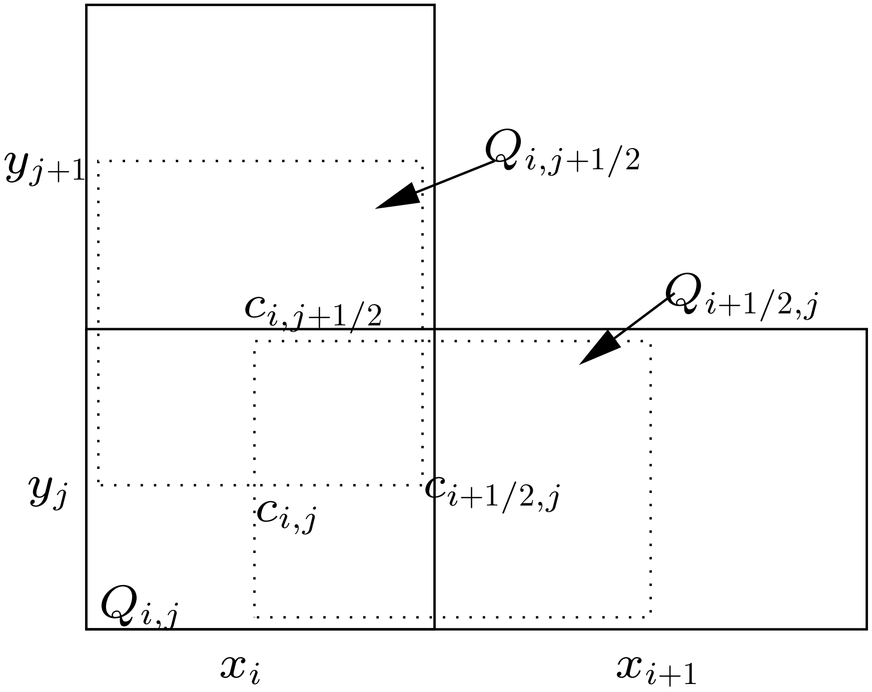
Drawback: Convergence deteriorates for non-smooth grids and coefficients.

Superconvergence can be recovered by introducing Lagrange multipliers on interfaces of discontinuities.

Control volume mixed finite element methods

Russell, Cai, Chou, Jones, Kim, Kwak, McCormick, Naff, Vassilevski, Wilson

Designed to be accurate on quadrilateral grids and discontinuous tensor coefficients.
$\mathcal{T}_h = \{Q_{i,j}\}$ - rectangular partition of Ω; $\mathbf{V}_h \times W_h$ - RT$_0$ spaces on $\mathcal{T}_h$



Integrate

$(K^{-1}\mathbf{u})^1 = -(\nabla p)^1$ over $Q_{i+1/2,j}$

$(K^{-1}\mathbf{u})^2 = -(\nabla p)^2$ over $Q_{i,j+1/2}$

$\nabla \cdot \mathbf{u} = f$ over $Q_{i,j}$

Optimal convergence (Chou, Kwak, Kim (2001)):

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_W \leq Ch$$

Superconvergence in CVMFEM on rectangular grids

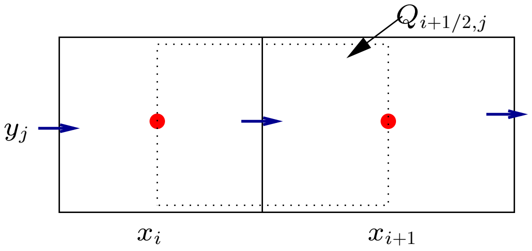
Joint work with T. Russell and M. F. Wheeler.

Velocity test space:

$$\begin{aligned}\mathbf{Y}_h = \{(v_h^1, v_h^2) :\, & v_h^1 \text{ is piecewise constant on } Q_{i+1/2,j},\ v_h^1 = 0 \text{ on border v1-volumes} \\ & v_h^2 \text{ is piecewise constant on } Q_{i,j+1/2},\ v_h^2 = 0 \text{ on border v2-volumes}\}\end{aligned}$$

Variational formulation:

For $\mathbf{v} \in \mathbf{Y}_h$,



$$\begin{aligned}(K^{-1}\mathbf{u}, \mathbf{v}) &= (-\nabla p, \mathbf{v}) \\ &= \sum_{i,j} \left(-\nabla p, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}\right)_{Q_{i+1/2,j}} + \sum_{i,j} \left(-\nabla p, \begin{pmatrix} 0 \\ v^2 \end{pmatrix}\right)_{Q_{i,j+1/2}} \\ &= -\sum_{i,j} \left\langle p, \begin{pmatrix} v^1 \\ 0 \end{pmatrix} \cdot \nu \right\rangle_{\partial Q_{i+1/2,j}} - \sum_{i,j} \left\langle p, \begin{pmatrix} 0 \\ v^2 \end{pmatrix} \cdot \nu \right\rangle_{\partial Q_{i,j+1/2}} \\ &=: b(\mathbf{v}, p)\end{aligned}$$

Variational formulation

For $w \in W_h$,

$$c(\mathbf{u}, w) := (\nabla \cdot \mathbf{u}, w) = (f, w)$$

$(\mathbf{u}, p) \in \mathbf{V} \times W$:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= 0, \quad \mathbf{v} \in \mathbf{Y}_h, \\ c(\mathbf{u}, w) &= (f, w), \quad w \in W_h. \end{aligned}$$

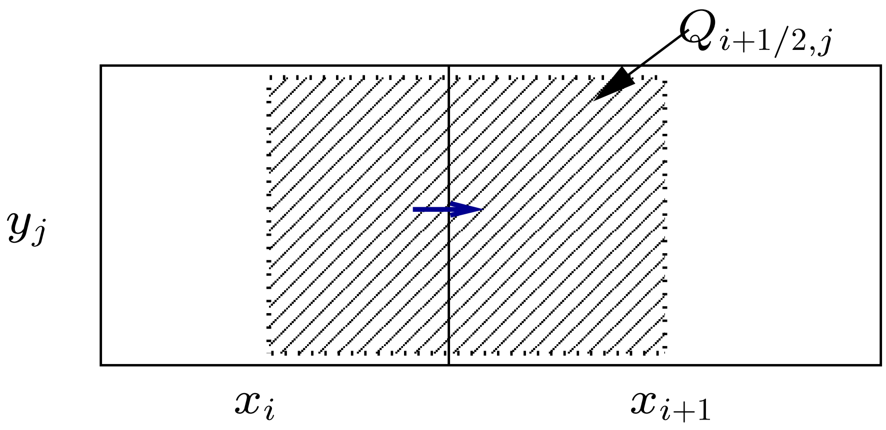
Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= 0, \quad \mathbf{v} \in \mathbf{Y}_h, \\ c(\mathbf{u}_h, w) &= (f, w), \quad w \in W_h. \end{aligned}$$

This is a Petrov-Galerkin formulation.

Transfer operator

$\gamma_h : \mathbf{V}_h \to \mathbf{Y}_h$:



$$\gamma_h \mathbf{v} = \left(\sum_{i,j} v^1(c_{i+1/2,j})\chi_{i+1/2,j}, \sum_{i,j} v^2(c_{i,j+1/2})\chi_{i,j+1/2} \right)$$

Lemma 1 (Chow, Kwak 2000): For $\mathbf{v} \in \mathbf{V}_h, w \in W_h$,

$$b(\gamma_h \mathbf{v}, w) = -c(\mathbf{v}, w)$$

$$a(\mathbf{v}, \gamma_h \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2$$

$$\|\gamma_h \mathbf{v}\| \leq C\|\mathbf{v}\|$$

The approach of Chow, Kwak (2000)

Rewrite the CVMFEM as a saddle-point problem.

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\gamma_h \mathbf{v}, p_h) &= 0, \quad \mathbf{v} \in \mathbf{V}_h, \\ b(\gamma_h \mathbf{u}_h, w) &= -(f, w), \quad w \in W_h. \end{aligned}$$

Mixed finite element method: find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} a(\tilde{\mathbf{u}}_h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \tilde{p}_h) &= 0, \quad \mathbf{v} \in \mathbf{V}_h, \\ -(\nabla \cdot \tilde{\mathbf{u}}_h, w) &= -(f, w), \quad w \in W_h. \end{aligned}$$

Bound $\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|$. Superconvergence is lost.

Superconvergence analysis

Use the Petrov-Galerkin formulation.

Error equations:

$$
\begin{aligned}
&a(\mathbf{u}-\mathbf{u}_h,\mathbf{v})+b(\mathbf{v},p-p_h)=0, \quad \mathbf{v}\in\mathbf{Y}_h,\\
&c(\mathbf{u}-\mathbf{u}_h,w)=0, \quad w\in W_h.
\end{aligned}
$$

Take $\mathbf{v}=\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h)$, $w=Q_hp-p_h$.

$$
\begin{aligned}
&a(\Pi\mathbf{u}-\mathbf{u}_h,\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h))=-a(\mathbf{u}-\Pi\mathbf{u},\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h))-b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-p_h),\\
&c(\Pi\mathbf{u}-\mathbf{u}_h,Q_hp-p_h)=0.
\end{aligned}
$$

$$
\begin{aligned}
b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-p_h)&=b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-Q_hp)+b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),Q_hp-p_h)\\
&=b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-Q_hp)-c(\Pi\mathbf{u}-\mathbf{u}_h,Q_hp-p_h)\\
&=b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-Q_hp)
\end{aligned}
$$

Velocity superconvergence

$$a(\Pi\mathbf{u}-\mathbf{u}_h,\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h)) = -a(\mathbf{u}-\Pi\mathbf{u},\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h)) - b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-Q_hp)$$

Lemma 2

$$|a(\mathbf{u}-\Pi\mathbf{u},\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h))| \leq Ch^2\|\mathbf{u}\|_2\|\Pi\mathbf{u}-\mathbf{u}_h\|$$

Lemma 3

$$|b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-Q_hp)| \leq Ch^2\|p\|_3\|\Pi\mathbf{u}-\mathbf{u}_h\|$$

Theorem 1

$$\|\Pi\mathbf{u}-\mathbf{u}_h\| \leq Ch^2(\|\mathbf{u}\|_2+\|p\|_3)$$

Corollary 1

$$|||\mathbf{u}-\mathbf{u}_h||| \leq Ch^2(\|\mathbf{u}\|_2+\|p\|_3),$$

where $|||\cdot|||$ is the norm along Gaussian lines.

Proof:

$$|||\mathbf{u}-\mathbf{u}_h||| \leq |||\mathbf{u}-\Pi\mathbf{u}||| + |||\Pi\mathbf{u}-\mathbf{u}_h|||. \quad \square$$

Sketch of proof of Lemma 2

Lemma 2

$$|a(\mathbf{u} - \Pi\mathbf{u}, \gamma_h(\Pi\mathbf{u} - \mathbf{u}_h))| \leq Ch^2\|\mathbf{u}\|_2\|\Pi\mathbf{u} - \mathbf{u}_h\|$$

Proof:

1. Show that if $\mathbf{u} \in (P_1(Q_{i,j}))^2$, $Q_{i,j} \in \mathcal{T}_h$, then

$$\int_{Q_{i,j}} (\mathbf{u} - \Pi\mathbf{u})\gamma_h \mathbf{v}\, dx\, dy = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

2. By the Bramble-Hilbert Lemma,

$$|(\mathbf{u} - \Pi\mathbf{u}, \gamma_h(\Pi\mathbf{u} - \mathbf{u}_h)| \leq Ch^2|\mathbf{u}|_2\|\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h)\|$$

Sketch of proof of Lemma 3

Lemma 3

$$|b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - Q_h p)| \leq Ch^2 \|p\|_3 \|\Pi\mathbf{u} - \mathbf{u}_h\|$$

Proof:

1. Show that if $p \in P_1(Q_{i,j})$, $Q_{i,j} \in \mathcal{T}_h$, then

$$b(\mathbf{v}, p - Q_h p) = 0, \quad \forall \mathbf{v} \in \mathbf{Y}_h.$$

2. By a Bramble-Hilbert lemma - type argument,

$$|b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - Q_h p)| \leq Ch^2 \|p\|_3 \|\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h)\|.$$

Pressure superconvergence

Auxiliary problem:

$$-\nabla \cdot K\nabla\varphi = Q_h p - p_h, \quad \text{in } \Omega,$$
$$-K\nabla\varphi \cdot \nu = 0, \quad \text{on } \partial\Omega.$$

Note: well posed since $\int_\Omega (Q_h p - p_h) ds = 0$.

Elliptic regularity:

$$\|\varphi\|_2 \le C\|Q_h p - p_h\|$$

Let $\phi = -\mathcal{K}\nabla\varphi$.

$$\begin{aligned}\|Q_h p - p_h\|^2 &= (Q_h p - p_h, \nabla \cdot \phi) = (Q_h p - p_h, \nabla \cdot \Pi\phi) = c(\Pi\phi, Q_h p - p_h)\\ &= b(\gamma_h \Pi\phi, Q_h p - p_h) = b(\gamma_h\Pi\phi, Q_h p - p) + b(\gamma_h\Pi\phi, p - p_h)\\ &= b(\gamma_h\Pi\phi, Q_h p - p) - a(\mathbf{u} - \mathbf{u}_h, \gamma_h\Pi\phi).\end{aligned}$$

Pressure superconvergence, cont.

$$\|Q_h p - p_h\|^2 = b(\gamma_h \Pi\phi, Q_h p - p) - a(\mathbf{u} - \mathbf{u}_h, \gamma_h \Pi\phi).$$

By Lemma 3,

$$|b(\gamma_h \Pi\phi, Q_h p - p)| \le Ch^2 \|p\|_3 \|\gamma_h \Pi\phi\|$$

By Lemma 2,

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \gamma_h \Pi\phi) &= a(\mathbf{u} - \Pi\mathbf{u}, \gamma_h \Pi\phi) + a(\Pi\mathbf{u} - \mathbf{u}_h, \gamma_h \Pi\phi) \\ &\le C(h^2 \|u\|_2 \|\gamma_h \Pi\phi\| + \|\Pi\mathbf{u} - \mathbf{u}_h\| \|\gamma_h \Pi\phi\|). \end{aligned}$$

Theorem 2

$$\|Q_h p - p_h\| \le Ch^2 (\|\mathbf{u}\|_2 + \|p\|_3)$$
$$|||p - p_h||| \le Ch^2 (\|\mathbf{u}\|_2 + \|p\|_3).$$

where

$$|||w|||^2 = \sum_{i,j} w(c_{i,j})^2 |Q_{i,j}|$$

Proof:

$$|||p - p_h||| \le |||p - Q_h p||| + |||Q_h p - p_h|||. \quad \square$$

Summary of superconvergence in CVMFEM

$$|||\mathbf{q}|||_{\mathbf{V}}^2 = |||\mathbf{q}|||^2 + |||\nabla \cdot \mathbf{q}|||^2$$

$$|||w|||_W = |||w|||$$

$$|||\mathbf{u} - \mathbf{u}_h|||_{\mathbf{V}} + |||p - p_h|||_W \leq Ch^2(\|\mathbf{u}\|_2 + \|\nabla \cdot \mathbf{u}\|_2 + \|p\|_3)$$

Mimetic Finite Difference Methods

Shashkov, Berndt, Hall, Hyman, Lipnikov, Morel, Moulton, Nicolaides, Roberts, Steinberg ...

Define discrete operators that mimic the properties of the differential operators.

Support operator methodology:

- Define discrete spaces for the physical variables and equip them with inner products
- Define a discrete approximation of the $prime$ operator
- Derive a discrete approximation of the $derived$ operator based on a calculus identity

Advantages to the finite element approach:

- Easier to define for general polygons.
- The inner product can be weighted with the (permeability) coefficient $\Rightarrow$ the methods work well for rough coefficients.

Superconvergence in Mimetic Finite Difference Methods

Joint work with M. Berndt, K. Lipnikov, M. Shashkov, and M. Wheeler.

$$\mathbf{u} = -K\nabla p, \quad \nabla \cdot \mathbf{u} = f \quad \text{in } \Omega$$

$$\mathbf{u} \cdot \nu = 0 \quad \text{on } \partial\Omega$$

Weighted gradient operator:

$$Gp \equiv K\nabla p$$

$$\mathbf{u} = -Gp, \quad \nabla \cdot \mathbf{u} = f.$$

Green's identity: for $\mathbf{v} \in \mathbf{V}$, $w \in W$,

$$(\mathbf{v}, \nabla w) = -(\nabla \cdot \mathbf{v}, w)$$

$$(\mathbf{v}, \nabla w) = (K^{-1}\mathbf{v}, K\nabla w) = (K^{-1}\mathbf{v}, Gw) =: (\mathbf{v}, Gw)_{\mathbf{V}}$$

Then

$$(\mathbf{v}, Gw)_{\mathbf{V}} = -(\nabla \cdot \mathbf{v}, w)_W, \text{ i.e., } G = -\text{div}^*.$$

Discrete Spaces and Scalar Products

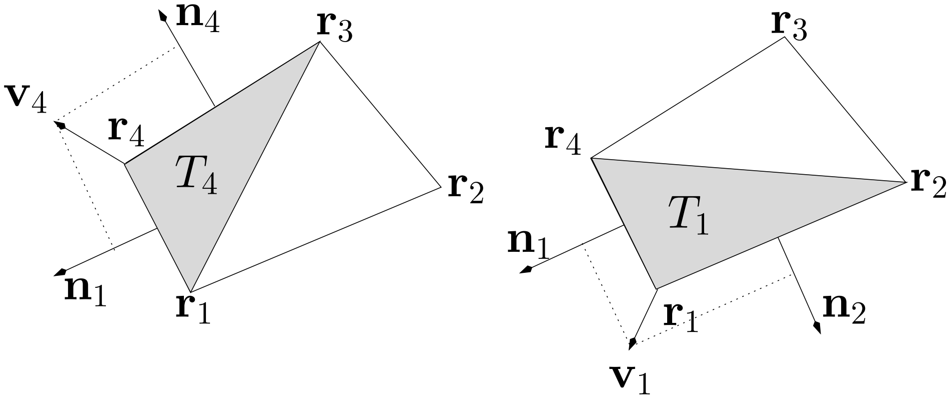
$\mathcal{T}_h$: h^2-uniform quadrilateral partition of Ω

Q_d: vector space of discrete cell-centered pressures; dim $Q_d = N_p = \#$ mesh cells

$$[p^d, q^d]_Q = \sum_{i=1}^{N_p} |E_i| p_i^d q_i^d, \qquad \forall\, p^d, q^d \in \mathcal{Q}$$

X_d: vector space of discrete edge-based velocities (normal components at the midpoints of mesh edges); dim $X_d = N_e = \#$ mesh edges

$$[\mathbf{u}_i^d, \mathbf{v}_i^d]_{X,i} = \frac{1}{2} \sum_{j=1}^{4} |T_j| K^{-1}(\mathbf{r}_j)\mathbf{u}^d(\mathbf{r}_j) \cdot \mathbf{v}^d(\mathbf{r}_j) \qquad [\mathbf{u}^d, \mathbf{v}^d]_X = \sum_{i=1}^{N_p} [\mathbf{u}_i^d, \mathbf{v}_i^d]_{X,i}$$



Discrete operators

$$\text{DIV} : X_d \to Q_d \qquad \text{GRAD} : Q_d \to X_d$$

$$\text{DIV} \;\approx \text{div}, \quad \text{GRAD} \;\approx G$$

Recall:

$$(\mathbf{v}, Gw)_{\mathbf{V}} = -(\nabla \cdot \mathbf{v}, w)_W$$

Prime operator DIV : using divergence theorem for element E,

$$\text{DIV } \mathbf{u}^d|_E = \frac{1}{|E|}(u_1^d|e_1| + u_2^d|e_2| + u_3^d|e_3| + u_4^d|e_4|)$$

Derived operator GRAD : defined implicitly by

$$[\mathbf{v}^d, \text{GRAD } w^d]_X \equiv -[\text{DIV } \mathbf{v}^d, w^d]_Q, \quad \forall\, \mathbf{v}^d \in X_d,\ \forall w^d \in Q_d \tag{1}$$

Mimetic finite difference discretization

$$\mathbf{u} = -Gp, \quad \nabla \cdot \mathbf{u} = f.$$

Find $\mathbf{u}^d \in X_d$, $p^d \in Q_d$:

$$\mathbf{u}^d = -\text{GRAD}\ p^d, \quad \text{DIV}\ \mathbf{u}^d = f^d$$

or equivalently

$$[\mathbf{u}^d, \mathbf{v}^d]_X = -[\text{GRAD}\ p^d, \mathbf{v}^d]_X, \quad \forall\ \mathbf{v}^d \in X_d,$$

$$[\text{DIV}\ \mathbf{u}^d, w^d]_Q = [f^d, w^d]_Q, \quad \forall\ w^d \in Q_d.$$

Optimal convergence (Berndt, Lipnikov, Moulton, Shashkov (2001)):

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_W \leq Ch$$

Relation between MFD and MFE methods

Following Berndt, Lipnikov, Moulton, Shashkov (2001).

Mimetic:

$$[\mathbf{u}^d, \mathbf{v}^d]_X = -[\text{GRAD } p^d, \mathbf{v}^d]_X, \quad \forall\ \mathbf{v}^d \in X_d,$$

$$[\text{DIV } \mathbf{u}^d, w^d]_Q = [f^d, w^d]_Q, \quad \forall\ w^d \in Q_d.$$

Recall

$$[\mathbf{v}^d, \text{GRAD } w^d]_X = -[\text{DIV } \mathbf{v}^d, w^d]_Q.$$

Then

$$[\mathbf{u}^d, \mathbf{v}^d]_X = [p^d, \text{DIV } \mathbf{v}^d]_X, \quad \forall\ \mathbf{v}^d \in X_d,$$

$$[\text{DIV } \mathbf{u}^d, w^d]_Q = [f^d, w^d]_Q, \quad \forall\ w^d \in Q_d.$$

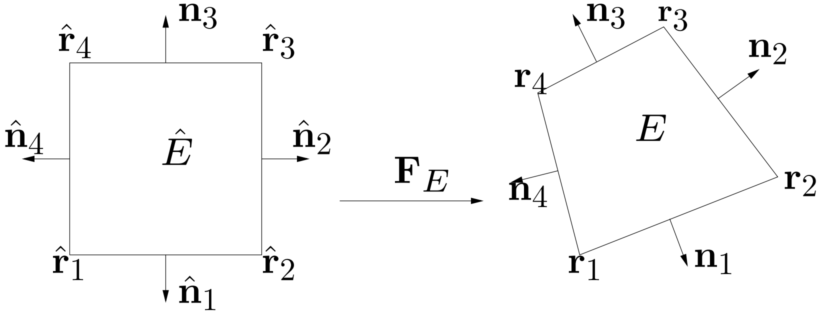
Mixed:

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h) = (p_h, \nabla \cdot \mathbf{v}_h), \qquad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(\nabla \cdot \mathbf{u}_h, w_h) = (f, w_h), \qquad w_h \in W_h.$$

Raviart-Thomas spaces on quadrilaterals

Mapping from a reference element: $F_E : \hat{E} \to E$, $DF_{E,kl} = \partial F_{E,k} / \partial \hat{x}_l$, $J_E = |\mathrm{det}(DF_E)|$



Reference velocity space:

$$\hat{\mathbf{v}}(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1 \hat{x}_1 \\ \alpha_2 + \beta_2 \hat{x}_2 \end{pmatrix}$$

Piola transformation:

$$\mathbf{v} = \frac{1}{J_E} DF_E \, \hat{\mathbf{v}} \circ F_E^{-1}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{J_E} \hat{\nabla} \cdot \hat{\mathbf{v}}, \quad \mathbf{v} \cdot \nu_e = \frac{1}{|e|} \hat{\mathbf{v}} \cdot \hat{\nu}_{\hat{e}}.$$

Mixed projection operator

Reference element: $\hat{\Pi} : \hat{\mathbf{V}}(\hat{E}) \to \hat{\mathbf{V}}_h(\hat{E})$,

$$\int_{\hat{e}} (\hat{\Pi}\hat{\mathbf{v}} - \hat{\mathbf{v}}) \cdot \hat{\nu}_{\hat{e}} = 0, \quad \forall \text{ edges } \hat{e}.$$

On any $E \in \mathcal{T}_h$, define $\Pi : \mathbf{V} \to \mathbf{V}_h$:

$$\Pi\mathbf{v} = \frac{1}{J_E} DF_E \hat{\Pi}\hat{\mathbf{v}} \circ F_E^{-1}, \text{i.e., } \widehat{\Pi\mathbf{v}} = \hat{\Pi}\hat{\mathbf{v}}.$$

Note:

$$(\nabla \cdot \Pi\mathbf{v}, w_h) = (\hat{\nabla} \cdot \hat{\Pi}\hat{\mathbf{v}}, \hat{w}_h) = (\hat{\nabla} \cdot \hat{\mathbf{v}}, \hat{w}_h) = (\nabla \cdot \mathbf{v}, w_h),$$

$$(\nabla \cdot (\Pi\mathbf{v} - \mathbf{v}), w_h) = 0, \quad \forall w_h \in W_h.$$

Isomorphisms

$\mathcal{I}_X : X_d \to \mathbf{V}_h, \quad \mathcal{I}_Q : Q_d \to W_h$

$\mathbf{v}_h \in \mathbf{V}_h$ is uniquely determined by its normal components on the edges

$w_h \in W_h$ is uniquely determined by its value at the center of gravity

Properties:

- $[q^d, w^d]_Q = (q_h, w_h), \quad q_h = \mathcal{I}_Q(p^d),\; w_h = \mathcal{I}_Q(w^d)$
- $[\text{DIV}\, \mathbf{v}^d, w^d]_Q = (\nabla \cdot \mathbf{v}_h, w_h), \quad \mathbf{v}_h = \mathcal{I}_X(\mathbf{v}^d)$

Let $\mathbf{u}_h = \mathcal{I}_X(\mathbf{u}^d)$, $p_h = \mathcal{I}_Q(p^d)$

Quadrature rule:

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_h \equiv [\mathbf{u}^d, \mathbf{v}^d]_X.$$

Then the mimetic FD method is

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h)_h &= (p_h, \nabla \cdot \mathbf{v}_h), & \mathbf{v}_h &\in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, w_h) &= (f, w_h), & w_h &\in W_h. \end{aligned}$$

Error analysis

Weak form: $\mathbf{u} \in \mathbf{V}$, $p \in W$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}), \qquad \mathbf{v} \in \mathbf{V},$$

$$(\nabla \cdot \mathbf{u}, w) = (q, w), \qquad w \in W.$$

Mimetic discretization: $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$ such that

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_h = (p_h, \nabla \cdot \mathbf{v}_h), \qquad \mathbf{v}_h \in \mathbf{V}_h,$$

$$(\nabla \cdot \mathbf{u}_h, w_h) = (q, w_h), \qquad w_h \in W_h.$$

Error equations: For all $\mathbf{v}_h \in \mathbf{V}_h$, $w_h \in W_h$,

$$(K^{-1}(\Pi\mathbf{u} - \mathbf{u}_h), \mathbf{v}_h)_h = (\mathcal{Q}_h p - p_h, \nabla \cdot \mathbf{v}_h) + (K^{-1}(\Pi\mathbf{u} - \mathbf{u}), \mathbf{v}_h) - E_h(K^{-1}\Pi\mathbf{u}, \mathbf{v}_h),$$

$$(\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h), w_h) = 0,$$

where

$$E_h(\mathbf{q}_h, \mathbf{v}_h) = (\mathbf{q}_h, \mathbf{v}_h) - (\mathbf{q}_h, \mathbf{v}_h)_h.$$

Velocity superconvergence

Error equations:

$$(K^{-1}(\Pi\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h)_h = (\mathcal{Q}_h p - p_h, \nabla\cdot\mathbf{v}_h) + (K^{-1}(\Pi\mathbf{u}-\mathbf{u}),\mathbf{v}_h) - E_h(K^{-1}\Pi\mathbf{u},\mathbf{v}_h),$$

$$(\nabla\cdot(\Pi\mathbf{u}-\mathbf{u}_h), w_h) = 0,$$

Take $\mathbf{v}_h = \Pi\mathbf{u}-\mathbf{u}_h$.

$$(K^{-1}(\Pi\mathbf{u}-\mathbf{u}_h),\Pi\mathbf{u}-\mathbf{u}_h)_h = (K^{-1}(\Pi\mathbf{u}-\mathbf{u}),\Pi\mathbf{u}-\mathbf{u}_h) - E_h(K^{-1}\Pi\mathbf{u},\Pi\mathbf{u}-\mathbf{u}_h).$$

Lemma 1 (Ewing, Liu, Wang):

$$(K^{-1}(\Pi\mathbf{u}-\mathbf{u}),\mathbf{v}_h) \le Ch^2\|\mathbf{u}\|_2\|\mathbf{v}_h\|.$$

Lemma 2: If $\nabla\cdot\mathbf{v}_h = 0$, then

$$E_h(K^{-1}\Pi\mathbf{u},\mathbf{v}_h) \le Ch^2\|\mathbf{u}\|_2\|\mathbf{v}_h\|.$$

Theorem 1

$$\|\Pi\mathbf{u}-\mathbf{u}_h\| \le Ch^2\|\mathbf{u}\|_2.$$

Sketch of proof of Lemma 2

On $E \in \mathcal{T}_h$,

$$E_h(K^{-1}\Pi\mathbf{u},\mathbf{v}_h)|_E = (K^{-1}\Pi\mathbf{u},\mathbf{v}_h)_E - (K^{-1}\Pi\mathbf{u},\mathbf{v}_h)_{h,E}.$$

$$(K^{-1}\Pi\mathbf{u},\mathbf{v}_h)_E = (K^{-1}\frac{1}{J}DF\hat{\Pi}\hat{\mathbf{u}},\frac{1}{J}DF\hat{\mathbf{v}}_h J)_{\hat{E}}$$

$$= (\frac{1}{J}DF^T K^{-1}DF\hat{\Pi}\hat{\mathbf{u}},\hat{\mathbf{v}}_h)_{\hat{E}} = (B\hat{\Pi}\hat{\mathbf{u}},\hat{\mathbf{v}}_h)_{\hat{E}},$$

where

$$B = \frac{1}{J}DF^T K^{-1}DF.$$

$$(K^{-1}\Pi\mathbf{u},\mathbf{v}_h)_{h,E} = \frac{1}{2}\sum_{j=1}^{4}|T_j|K^{-1}(\mathbf{r}_j)\Pi\mathbf{u}(\mathbf{r}_j)\cdot\mathbf{v}_h(\mathbf{r}_j)$$

$$= \frac{1}{4}\sum_{j=1}^{4}B(\hat{\mathbf{r}}_j)\hat{\Pi}\hat{\mathbf{u}}(\hat{\mathbf{r}}_j)\cdot\hat{\mathbf{v}}_h(\hat{\mathbf{r}}_j) = (B\hat{\Pi}\hat{\mathbf{u}},\hat{\mathbf{v}}_h)_{T,\hat{E}}.$$

Sketch of proof of Lemma 2, cont.

Error in trapezoidal rule: (Peano Kernel Theorem)

$$\begin{aligned}\int_0^1\int_0^1 g(\hat{x},\hat{y})d\hat{x}\,d\hat{y} - I_T(f) &= \int_0^1 \frac{\hat{x}(\hat{x}-1)}{2}\frac{\partial^2}{\partial\hat{x}^2}g(\hat{x},0)d\hat{x} + \int_0^1 \frac{\hat{y}(\hat{y}-1)}{2}\frac{\partial^2}{\partial\hat{y}^2}g(0,\hat{y})d\hat{y}\\ &= I_1 + I_2\end{aligned}$$

$$B\hat{\Pi}\hat{\mathbf{u}}\cdot\hat{\mathbf{v}} = B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1\hat{\mathbf{v}}_1 + B_{12}(\hat{\Pi}\hat{\mathbf{u}})_2\hat{\mathbf{v}}_1 + B_{21}(\hat{\Pi}\hat{\mathbf{u}})_1\hat{\mathbf{v}}_2 + B_{22}(\hat{\Pi}\hat{\mathbf{u}})_2\hat{\mathbf{v}}_2$$

Consider $g(\hat{x},\hat{y}) = B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1\hat{\mathbf{v}}_1$.

$$\begin{aligned}I_1 &= \int_0^1 \varphi(\hat{x})\left(\frac{\partial^2}{\partial\hat{x}^2}B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1(\hat{x},0)\hat{\mathbf{v}}_1(\hat{x},0) + \frac{\partial}{\partial\hat{x}}B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1(\hat{x},0)\frac{\partial}{\partial\hat{x}}\hat{\mathbf{v}}_1(\hat{x},0)\right)d\hat{x}\\ &= I_{11} + I_{12}.\end{aligned}$$

$$I_{11} \le C|B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1|_{2,\hat{E}}\|\hat{\mathbf{v}}_1\|_{0,\hat{E}}$$

Using that $\frac{\partial}{\partial\hat{x}}\hat{\mathbf{v}}_1 = -\frac{\partial}{\partial\hat{y}}\hat{\mathbf{v}}_2$ and IBP in $\hat{y}$,

$I_{12} =$ edge integrals on top and bottom $+ R$, $\quad R \le C|B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1|_{2,\hat{E}}\|\hat{\mathbf{v}}_2\|_{0,\hat{E}}$.

Sketch of proof of Lemma 2, cont.

Proposition

$$|B\hat{\mathbf{u}}|_{2,\hat{E}} \le Ch^2 \|K\|_{2,\infty,E} \|\mathbf{u}\|_{2,E}.$$

Proof uses bounds

$$|B|_{j,\infty,\hat{E}} \le Ch^j \|K\|_{j,\infty,E}, \quad j = 0, 1, 2.$$

$$|\hat{\mathbf{u}}|_{j,\hat{E}} \le Ch^j \|\mathbf{u}\|_{j,E}, \quad j = 0, 1, 2. \quad \square$$

When summing over $E \in \mathcal{T}_h$, differences of edge integrals can be estimated as $O(h^2)$. Term I_2 is treated similarly. Then

$$E_h(B_{11}(\hat{\Pi}\hat{\mathbf{u}})_1, \hat{\mathbf{v}}_1) \le Ch^2 \|K\|_{2,\infty} \|\mathbf{u}\|_2 \|\mathbf{v}\|.$$

Terms $B_{12}(\hat{\Pi}\hat{\mathbf{u}})_2\hat{\mathbf{v}}_1$, $B_{21}(\hat{\Pi}\hat{\mathbf{u}})_1\hat{\mathbf{v}}_2$, and $B_{22}(\hat{\Pi}\hat{\mathbf{u}})_2\hat{\mathbf{v}}_2$ are treated similarly. $\square$

Superconvergence at edge midpoints

Define

$$|||\mathbf{v}|||^2 = \sum_{E\in\mathcal{T}_h} \sum_{k=1}^{4} |e_k|^2 (\mathbf{v}\cdot\nu_k)^2(m_k),$$

where m_k is the midpoint of e_k.

Proposition:

$$c_1\|\mathbf{v}\| \le |||\mathbf{v}||| \le c_2\|\mathbf{v}\| \quad \forall\, \mathbf{v}\in\mathbf{V}_h. \quad \square$$

$$|||\mathbf{u}-\mathbf{u}^h||| \le |||\mathbf{u}-\Pi\mathbf{u}||| + |||\Pi\mathbf{u}-\mathbf{u}^h||| \le Ch^2 + c_2\|\Pi\mathbf{u}-\mathbf{u}^h\| \le Ch^2.$$

Pressure superconvergence

Weak form: $\mathbf{u} \in \mathbf{V}$, $p \in W$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v}) &= (p, \nabla \cdot \mathbf{v}), && \mathbf{v} \in \mathbf{V}, \\ (\nabla \cdot \mathbf{u}, w) &= (f, w), && w \in W. \end{aligned}$$

Mimetic discretization: $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h)_h &= (p_h, \nabla \cdot \mathbf{v}_h), && \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, w_h) &= (f, w_h), && w_h \in W_h. \end{aligned}$$

Error equations:

$$\begin{aligned} &(K^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v}_h) = (\mathcal{Q}_h p - p_h, \nabla \cdot \mathbf{v}_h) - E_h(K^{-1}\mathbf{u}_h, \mathbf{v}_h), && \mathbf{v}_h \in \mathbf{V}_h, \\ &(\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h), w_h) = 0, && w_h \in W_h, \end{aligned}$$

Pressure superconvergence, cont

Auxiliary problem:

$$-\nabla\cdot K\nabla\varphi = -(\mathcal{Q}_h p - p_h) \quad \text{in } \Omega$$
$$\varphi = 0 \quad \text{on } \partial\Omega$$

Elliptic regularity: $\|\varphi\|_2 \le C\|\mathcal{Q}_h p - p_h\|$

Taking $\mathbf{v}_h = \Pi K\nabla\varphi$ in

$$(K^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h) = (\mathcal{Q}_h p - p_h, \nabla\cdot\mathbf{v}_h) - E_h(K^{-1}\mathbf{u}_h,\mathbf{v}_h),$$

$$\begin{aligned}\|\mathcal{Q}_h p - p_h\|^2 &= (\mathcal{Q}_h p - p_h, \nabla\cdot K\nabla\varphi) = (\mathcal{Q}_h p - p_h, \nabla\cdot\Pi K\nabla\varphi)\\ &= (K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi K\nabla\varphi) + E_h(K^{-1}\mathbf{u}_h,\Pi K\nabla\varphi)\\ &\le Ch^2(\|\mathbf{u}\|_1 + \|\nabla\cdot\mathbf{u}\|_1)\|\varphi\|_2.\end{aligned}$$

Theorem 2

$$\|\mathcal{Q}_h p - p_h\| \le Ch^2(\|\mathbf{u}\|_1 + \|\nabla\cdot\mathbf{u}\|_1).$$

Summary of superconvergence in Mimetic FD Methods

For the Neumann problem on h^2-uniform quadrilateral grids,

$$|||\mathbf{u} - \mathbf{u}_h|||_V + |||p - p_h|||_W \leq Ch^2(\|\mathbf{u}\|_2 + \|\nabla \cdot \mathbf{u}\|_2)$$

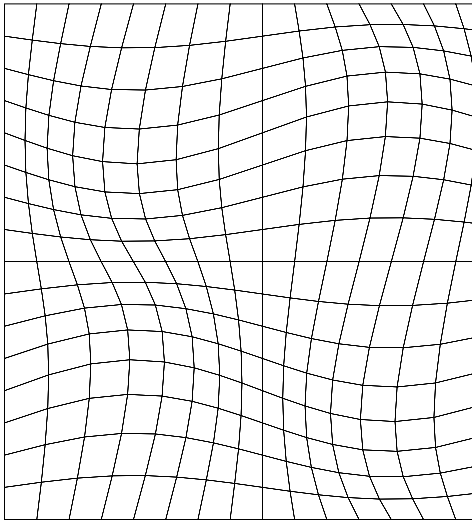
Loss of $O(h^{1/2})$ occurs for the Dirichlet problem.

Numerical experiments

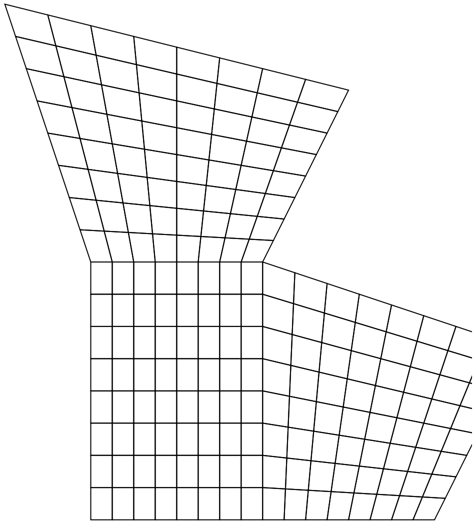
$$p(x,\, y) = x^3\, y^2 + x\, \cos(xy)\, \sin(x), \qquad K(x,\, y) = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}.$$

Example 1:

$$x(\xi,\eta) = \xi + 0.06\, \sin(2\pi\eta)\, \sin(2\pi\xi), \qquad y(\xi,\eta) = \eta + 0.06\, \sin(2\pi\eta)\, \sin(2\pi\xi),$$



Example 1



Example 2

Convergence studies

$1/h$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|_\infty$	$\|\|\|\mathbf{u}-\mathbf{u}^h\|\|\|$	$\|\|\|p-p^h\|\|\|_\infty$	$\|\|\|p-p^h\|\|\|$
8	1.36e-1	3.73e-2	6.67e-3	3.06e-3
16	4.83e-2	1.19e-2	2.31e-3	8.49e-4
32	1.33e-2	3.24e-3	6.47e-4	2.19e-4
64	3.45e-3	8.28e-4	1.69e-4	5.51e-5
128	8.70e-4	2.08e-4	4.29e-5	1.38e-5
256	2.20e-4	5.21e-5	1.08e-5	3.45e-6
Rate	1.97	1.99	1.97	2.00

$1/h$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|_\infty$	$\|\|\|\mathbf{u}-\mathbf{u}^h\|\|\|$	$\|\|\|p-p^h\|\|\|_\infty$	$\|\|\|p-p^h\|\|\|$
8	1.24e-1	3.49e-2	6.71e-3	4.24e-3
16	3.87e-2	8.94e-3	1.58e-3	1.01e-3
32	1.27e-2	2.25e-3	3.98e-4	2.47e-4
64	4.17e-3	5.64e-4	1.17e-4	6.11e-5
128	1.33e-3	1.41e-4	3.21e-5	1.52e-5
256	4.18e-4	3.53e-5	8.49e-6	3.79e-6
Rate	1.64	2.00	1.85	2.01

Summary and extensions

- Superconvergence $O(h^2)$ in discrete L^2-norms obtained for velocity and pressure in
  - CVMFEM on rectangular grids in $\mathbf{R}^2$ and $\mathbf{R}^3$,
  - MFDM on h^2-uniform quadrilateral grids in $\mathbf{R}^2$.

- Optimal convergence obtained for the above methods on non-matching grids via mortar finite elements.

- Current and future work involves extensions to
  - CVMFEM on quadrilateral and hexahedral grids,
  - MFDM on hexahedral grids,
  - Superconvergence on non-matching grids,
  - General polygons.