From the de Rham sequence to mixed finite elements for elasticity

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For the mixed elasticity system, derived from the Hellinger–Reissner variational principle, the construction of stable mixed finite elements have been surprisingly difficult to construct.

Attempts have for example been made by Fraejis de Veubeke (1965), Watwood and Hartz (1968), Johnson and Mercier (1978), Arnold, Brezzi and Douglas (1984), Stenberg (1986), Arnold and Falk(1988), Morley (1989), Stein and Rolfes (1990), Becache, Joly and Tsogka (2002),.... However, none of these elements seems to be “the perfect choice.”

My own contribution: D. Arnold and R. Winther, Mixed finite elements for elasticity, Numer. Math. (2002)
D. Arnold and R. Winther, Nonconforming mixed elements for elasticity, M^3AS (2003)

The equations of linear elasticity can be written as a system of equations of the form

$$\begin{aligned} A\sigma &= \varepsilon u \\ \operatorname{div}\sigma &= f \quad \text{in } \Omega. \end{aligned}$$

Here

- $\sigma(x) \in \mathbb{S}$ is the stressfield and $u(x) \in \mathbb{R}^3$ denotes displacement field
- $f = f(x)$ is the given body force
- $\epsilon(u)$ is the symmetric gradient of u and div of a matrix fields is taken row wise
- $\mathbb{S}$ denotes the space of symmetric matrices
- If the body is clamped on the boundary $\partial\Omega$ of Ω, then the proper boundary condition for the system is $u = 0$ on $\partial\Omega$.
- $A = A(x) : \mathbb{S} \mapsto \mathbb{S}$ is the given, uniformly positive definite, compliance tensor which is material dependent

The pair (σ, u) can alternatively be characterized as the unique critical point of the Hellinger–Reissner functional

$$\mathcal{J}(\tau, v) = \int_\Omega \Big(\frac{1}{2} A\tau : \tau + \operatorname{div}\tau \cdot v - f \cdot v\Big)\, dx.$$

in $H(\operatorname{div}, \Omega, \mathbb{S}) \times L_2(\Omega, \mathbb{R}^3)$.

Equivalently, $(\sigma, u) \in H(\operatorname{div}, \Omega, \mathbb{S}) \times L_2(\Omega, \mathbb{R}^3)$ is the unique solution to the following weak formulation of the system

$$\begin{aligned}(A\sigma, \tau) + (\operatorname{div}\tau, u) &= 0, \qquad \tau \in H(\operatorname{div}, \Omega, \mathbb{S}),\\ (\operatorname{div}\sigma, v) &= (f, v) \;\; v \in L_2(\Omega, \mathbb{R}^3).\end{aligned}$$

Alternative approach; Displacement formulation

$$\begin{aligned} A\sigma &= \varepsilon u \\ \operatorname{div} \sigma &= f \quad \text{in } \Omega. \end{aligned}$$

Eliminate σ to obtain

$$\operatorname{div} A^{-1}\varepsilon u = f \quad \text{in } \Omega$$

for the displacement u.

A weak solution of this equation can be characterized as the global minimizer of the energy functional

$$\mathcal{E}(u) = \int_\Omega \Big(\frac{1}{2}A^{-1}\varepsilon u : \varepsilon u + f \cdot u\Big)\, dx$$

over the Sobolev space $H_0^1(\Omega, \mathbb{R}^3)$.

Weak mixed formulations

Poisson's equation ($\Delta p = f, p = 0$ on the boundary):

Find $(u, p) \in H(\mathrm{div}, \Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R})$ such that

$$\begin{aligned} (u, v) \quad &+ (p, \mathrm{div}\, v) = 0 \quad &&\forall v \in H(\mathrm{div}, \Omega, \mathbb{R}^3), \\ (\mathrm{div}\, u, q) \quad & \qquad = (f, q) &&\forall q \in L^2(\Omega, \mathbb{R}). \end{aligned}$$

Linear elasticity:

Find $(\sigma, u) \in H(\mathrm{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^3)$ such that

$$\begin{aligned} (A\sigma, \tau) \quad &+ (u, \mathrm{div}\, \tau) = 0 \quad &&\forall \tau \in H(\mathrm{div}, \Omega, \mathbb{S}), \\ (\mathrm{div}\, \sigma, v) \quad & \qquad = (f, v) &&\forall v \in L^2(\Omega, \mathbb{R}^3). \end{aligned}$$

The de Rham complex

We assume that the domain Ω is contractible.

The following sequence is exact

$$\mathbb{R} \xrightarrow{\subset} C^\infty \xrightarrow{\text{grad}} C^\infty(\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty \longrightarrow 0,$$

Also

$$\mathbb{R} \xrightarrow{\subset} H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

The Brezzi stability conditons

Consider a general saddle point problem of the form:

Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} a(u,v) \; +b(v,p) &= F(v) \;\; \forall v \in V, \\ b(u,q) \qquad\qquad &= G(q) \;\; \forall q \in Q. \end{aligned}$$

The spaces $V_h \times Q_h \subset V \times Q$ defines a stable discretization if

(A1) $\exists c_1 > 0$, independent of h, such that

$$a(v,v) \geq c_1 \|v\|_V^2 \quad \forall v \in V_{0,h}$$

where

$$V_{0,h} \equiv \{v \in V_h : b(v,q) = 0 \quad \forall q \in Q_h\}.$$

(A2) $\exists c_2 > 0$, independent of h, such that

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q} \geq c_2.$$

The Brezzi conditions; Poisson's equation

$$\begin{aligned}(u,v) \quad &+(p,\operatorname{div} v) = 0 \quad && \forall v \in H(\operatorname{div},\Omega,\mathbb{R}^3),\\ (\operatorname{div} u, q) \quad && = (f,q) \quad && \forall q \in L^2(\Omega,\mathbb{R}).\end{aligned}$$

Discretization

$$V_h \times Q_h \subset H(\operatorname{div},\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathbb{R})$$

$$V_{0,h} = \{v \in V_h : (\operatorname{div} v, q) = 0 \ \forall q \in Q_h\}$$

and

$$\|v\|_{\operatorname{div}}^2 = \|v\|_0^2 + \|\operatorname{div} v\|^2.$$

(A1) $\exists c_1 > 0$, **independent of** h, **such that**

$$\|v\|_0 \geq c_1 \|v\|_{\operatorname{div}} \quad \forall v \in V_{0,h}$$

(A2) $\exists c_2 > 0$, **independent of** h, **such that**

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\operatorname{div} v, q)}{\|v\|_{\operatorname{div}} \|q\|_0} \geq c_2.$$

Alternative stronger conditions:

(A1′) $\operatorname{div} V_h \subset Q_h$.

(A2′) There exists a linear operator $\Pi_h^d : H^1(\Omega, \mathbb{R}^3) \mapsto V_h$, bounded in $\mathcal{L}(H^1, L^2)$ uniformly with respect to h, and such that

$$\operatorname{div} \Pi_h^d v = \Pi_h^0 \operatorname{div} v.$$

Here $\Pi_h^0 : L^2(\Omega, \mathbb{R}^3) \mapsto Q_h$ is the L^2 projection.

The property $\operatorname{div} \Pi_h^d v = \Pi_h^0 \operatorname{div} v$ can be illustrated by the commutative diagram

$$\begin{array}{ccc} \mathcal{D}(\Pi_h^d) & \xrightarrow{\operatorname{div}} & L^2(\Omega, \mathbb{R}) \\ \big\downarrow \Pi_h^d & & \big\downarrow \Pi_h^0 \\ V_h & \xrightarrow{\operatorname{div}} & Q_h \end{array}$$

Observe that any element of the form $\Pi_h^d \operatorname{curl} z$ is divergence free, since

$$\operatorname{div} \Pi_h^d \operatorname{curl} z = \Pi_h^0 \operatorname{div} \operatorname{curl} z = 0.$$

Hence, there exists an $z_h \in H(\operatorname{curl})$ such that $\operatorname{curl} z_h = \Pi_h^d \operatorname{curl} z$.

Question: Do we have $z_h \in Z_h$ where where Z_h is a finite element subspace of $H(\operatorname{curl})$, and $z \mapsto z_h$ coincides with $z \mapsto \Pi_h^c z$ where Π_h^c maps smooth elements of $H(\operatorname{curl})$ onto Z_h?

Answer: Yes, in all known examples.

This imples:

$$\operatorname{curl} \varPi_h^c z = \varPi_h^d \operatorname{curl} z$$

As a consequence we obtain the commutative diagram:

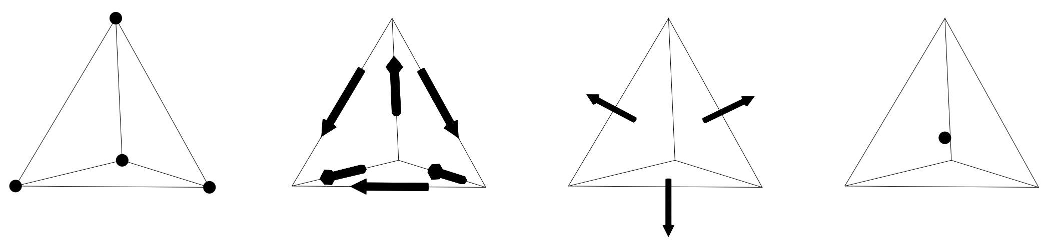
$$\begin{array}{ccccc} \mathcal{D}_h^c & \xrightarrow{\operatorname{curl}} & \mathcal{D}(\varPi_h^d) & \xrightarrow{\operatorname{div}} & L_2(\Omega) \\ \downarrow \varPi_h^c & & \downarrow \varPi_h^d & & \downarrow \varPi_h^0 \\ Z_h & \xrightarrow{\operatorname{curl}} & V_h & \xrightarrow{\operatorname{div}} & Q_h \end{array}$$

where $\mathcal{D}_h^c \subset H(\operatorname{curl})$

$$\mathcal{D}_h^c = \{z \in \mathcal{D}(\varPi_h^c) : \operatorname{curl} z \in \mathcal{D}(\varPi_h^d)\}.$$

Conclusion; The following diagram commutes

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\subset} & C^\infty & \xrightarrow{\text{grad}} & C^\infty\mathbb{R}^3) & \xrightarrow{\text{curl}} & C^\infty(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty & \longrightarrow & 0 \\
\downarrow id & & \downarrow \Pi_h^1 & & \downarrow \Pi_h^c & & \downarrow \Pi_h^d & & \downarrow \Pi_h^0 & & \\
\mathbb{R} & \xrightarrow{\subset} & S_h & \xrightarrow{\text{grad}} & Z_h & \xrightarrow{\text{curl}} & V_h & \xrightarrow{\text{div}} & Q_h & \longrightarrow & 0
\end{array}
$$



Back to linear elasticity:

Find $(\sigma, u) \in H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ such that

$$\begin{aligned} (A\sigma, \tau) \quad + (u, \operatorname{div} \tau) &= 0 && \forall \tau \in H(\operatorname{div}, \Omega, \mathbb{S}), \\ (\operatorname{div} \sigma, v) \quad &= (f, v) && \forall v \in L^2(\Omega, \mathbb{R}^2). \end{aligned}$$

The corresponding exact sequence in this case:

$$RM \xrightarrow{\subset} C^\infty(\mathbb{R}^3) \xrightarrow{\varepsilon} C^\infty(\mathbb{S}) \xrightarrow{J} C^\infty(\mathbb{S}) \xrightarrow{\operatorname{div}} C^\infty(\mathbb{R}^3) \longrightarrow 0,$$

Here $J\sigma = \operatorname{curl}(\operatorname{curl} \sigma)^T$.

The 2D case

The results of our (Arnold-W) paper from 2002 was to construct commuative diagrams of the form

$$\begin{array}{ccccccccc}
\mathcal{P}_1 & \xrightarrow{\subset} & C^\infty & \xrightarrow{J} & C^\infty(\mathbb{S}) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^2) & \longrightarrow & 0 \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \varPi_h^2} & & \downarrow{\scriptstyle \varPi_h^d} & & \downarrow{\scriptstyle \varPi_h^0} & & \\
\mathcal{P}_1 & \xrightarrow{\subset} & Q_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\text{div}} & V_h & \longrightarrow & 0
\end{array}$$

Here Σ_h is a piecewise cubic space and

$$Jq = \begin{pmatrix} \partial^2 q/\partial y^2 & -\partial^2 q/\partial x \partial y \\ -\partial^2 q/\partial x \partial y & \partial^2 q/\partial x^2 \end{pmatrix}.$$

Weakly imposed symmetry

(de Veubeke, Arnold-Brezzi-Douglas, Stenberg, Morley)

$$\begin{aligned}
(A\sigma,\tau) + (\operatorname{div}\tau, u) + (\tau, p) &= 0, && \tau \in H(\operatorname{div},\Omega,\mathbb{M}),\\
(\operatorname{div}\sigma, v) &= (f,v), && v \in L_2(\Omega,\mathbb{R}^3),\\
(\sigma, q) &= 0, && q \in L_2(\Omega,\mathbb{K}).
\end{aligned}$$

Here $\mathbb{M} = \mathbb{R}^{3\times 3}$ and $\mathbb{K}$ is the space of skew symmetric matrices.

Let $\mathbb{W} = \mathbb{R}^3 \times \mathbb{K}$. The relevant exact sequence in this case is

$$\mathbb{T} \xrightarrow{\subset} C^\infty(\mathbb{W}) \xrightarrow{(\operatorname{grad}, I)} C^\infty(\mathbb{M}) \xrightarrow{J} C^\infty(\mathbb{M}) \xrightarrow{(\operatorname{div},\operatorname{skw})^T} C^\infty(\mathbb{W}) \longrightarrow 0.$$

$\mathbb{T} = \{(v,q) : v \in RM, q = -\operatorname{skw}(\operatorname{grad} v)\}$

Here $J : C^\infty(\mathbb{M}) \mapsto C^\infty(\mathbb{M})$ denotes the extension of the previous operator

$$J\tau = \operatorname{curl} S^{-1} \operatorname{curl} \tau,$$

$$S\mu = \mu^T - \operatorname{tr}(\mu)\delta, \qquad S^{-1}\mu = \mu^T - \frac{1}{2}\operatorname{tr}(\mu)\delta.$$

The significance of the operator S can be seen from the fact that it maps $H(\operatorname{curl})$ to $H(\operatorname{div})$. In fact

$$\operatorname{div} S\tau = -\operatorname{skw} \operatorname{curl} \tau.$$

There is a close connection between the two elasticity sequences. Consider the diagram:

$$\begin{array}{ccccccccccc}
\mathbb{T} & \xrightarrow{\subset} & C^\infty(\mathbb{W}) & \xrightarrow{(\mathrm{grad}, I)} & C^\infty(\mathbb{M}) & \xrightarrow{J} & C^\infty(\mathbb{M}) & \xrightarrow{(\mathrm{div},\mathrm{skw})^T} & C^\infty(\mathbb{W}) & \longrightarrow & 0 \\
\Big\downarrow \pi_0 & & \Big\downarrow \pi_0 & & \Big\downarrow \pi_1 & & \Big\downarrow \pi_2 & & \Big\downarrow \pi_3 & & \\
RM & \xrightarrow{\subset} & C^\infty(\mathbb{R}^3) & \xrightarrow{\varepsilon} & C^\infty(\mathbb{S}) & \xrightarrow{J} & C^\infty(\mathbb{S}) & \xrightarrow{\mathrm{div}} & C^\infty(\mathbb{R}^3) & \longrightarrow & 0,
\end{array}$$

where the operators π_k are defined by

$$\pi_0(u,q) = u, \quad \pi_1(\sigma) = \pi_2(\sigma) = \mathrm{sym}(\sigma), \quad \pi_3(u,q) = u - \mathrm{div}\, q.$$

Note that if RM is identified with $\mathbb{T}$ and analogously $C^\infty(\mathbb{R}^3)$ with

$$\{(u,q) : u \in C^\infty(\mathbb{R}^3), q = -\mathrm{skw}(\mathrm{grad}\, u)\}$$

then the spaces in the bottom row are subspaces of the ones in the upper row, and the operators π_k are all projections. Futhermore, the diagram commutes. Therefore, exactness of the upper row implies exactness of the bottom row.

From de Rham to elasticity

k-forms are defined as $\Lambda^k = C^\infty(\Omega, \mathrm{Alt}_k)$, i.e.

$$\omega(x) = \sum_{i_1<i_2<\ldots<i_k} f_{i_1\ldots i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k} = \sum_I f_I(x) dx_I$$

with coefficients $f_I \in C^\infty(\Omega, \mathbb{R})$.

In particular, 0–forms can be identified with scalar functions, and 1–forms with vector fields under the identification $f_i dx_i \leftrightarrow f_i e_i$. Furthermore, when $\Omega \subset \mathbb{R}^3$, 2 forms also corresponds to vector fields under the identification $f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2 \leftrightarrow (f_1, f_2, f_3)$ and 3–forms can be be identified with the scalar function $f_{1,2,3}$.

The exterior derivate $d : \Lambda^k \mapsto \Lambda^{k+1}$ is defined by

$$d\omega(x) = \sum_{j,I} \frac{\partial f_I(x)}{\partial x_j} (dx_j \wedge dx_I).$$

Basic property:

$$d^2 = d \circ d = 0$$

The de Rham sequence now takes the form

$$\mathbb{R} \xrightarrow{\subset} \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \longrightarrow 0.$$

Let $\mathbb{V} = \mathbb{R}^3 \times \mathbb{R}^3_*$. The vector valued version of de Rham is also an exact sequence

$$\mathbb{V} \xrightarrow{\subset} \Lambda^0(\mathbb{V}) \xrightarrow{d} \Lambda^1(\mathbb{V}) \xrightarrow{d} \Lambda^2(\mathbb{V}) \xrightarrow{d} \Lambda^3(\mathbb{V}) \longrightarrow 0,$$

Here $\Lambda^k(\mathbb{V})$ consists of elements of the form

$$\omega(x) = \sum_{i_1<i_2<\ldots<i_k} f_{i_1\ldots i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k} = \sum_I f_I(x) dx_I$$

with coefficients $f_I \in C^\infty(\Omega, \mathbb{V})$.

The BGG resolution

Bernstein, Gelfand, Gelfand (1975), Eastwood (2000)

Define $K : \Lambda^k(\mathbb{R}^3_*) \mapsto \Lambda^k(\mathbb{R}^3)$ by

$$\mu(x) = \sum_I f_I(x) dx_I \mapsto K\mu(x) = \sum_I (f_I(x) \times x) dx_I.$$

Furthermore, $\Phi : \Lambda^k(\mathbb{V}) \mapsto \Lambda^k(\mathbb{V})$ by

$$\Phi(\omega, \mu) = (\omega + K\mu, \mu)$$

defines an isomorphism on $\Lambda^k(\mathbb{V})$, with inverse given by

$$\Phi^{-1}(\omega, \mu) = (\omega - K\mu, \mu).$$

Mappings of complexes

Define the operator $\mathcal{A} : \Lambda^k(\mathbb{V}) \mapsto \Lambda^{k+1}(\mathbb{V})$ by $\mathcal{A} = \Phi d \Phi^{-1}$. Also the sequence

$$\mathbb{V} \xrightarrow{\subset} \Lambda^0(\mathbb{V}) \xrightarrow{\mathcal{A}} \Lambda^1(\mathbb{V}) \xrightarrow{\mathcal{A}} \Lambda^2(\mathbb{V}) \xrightarrow{\mathcal{A}} \Lambda^3(\mathbb{V}) \longrightarrow 0,$$

is exact.

The operator $\mathcal{A}$ has a simple form

$$\begin{aligned} \mathcal{A}(\omega, \mu) = \Phi \circ d(\omega - K\mu, \mu) &= \Phi(d\omega - dK\mu, d\mu) \\ &= (d\omega - S\mu, d\mu), \end{aligned}$$

where $S = dK - Kd : \Lambda^k(\mathbb{R}^3_*) \mapsto \Lambda^{k+1}(\mathbb{R}^3)$ is an algebraic operator.

$$\omega(x) = \sum_I f_I(x) dx_I \mapsto S\omega(x) = \sum_{I,j} (f_I(x) \times e_j) dx_j \wedge dx_I.$$

Useful identity:

$$dS = d^2K - dKd = -(dK - Kd)d = -Sd.$$

Consider the following projection of the sequence above

$$\begin{array}{ccccccccccc}
\mathbb{V} & \xrightarrow{\subset} & \Lambda^0(\mathbb{V}) & \xrightarrow{\mathcal{A}} & \Lambda^1(\mathbb{V}) & \xrightarrow{\mathcal{A}} & \Lambda^2(\mathbb{V}) & \xrightarrow{\mathcal{A}} & \Lambda^3(\mathbb{V}) & \longrightarrow & 0 \\
\Big\downarrow id & & \Big\downarrow id & & \Big\downarrow \pi^1 & & \Big\downarrow \pi^2 & & \Big\downarrow id & & \\
\mathbb{V} & \xrightarrow{\subset} & \Lambda^0(\mathbb{V}) & \xrightarrow{\mathcal{A}} & \Gamma^1 & \xrightarrow{\mathcal{A}} & \Gamma^2 & \xrightarrow{\mathcal{A}} & \Lambda^3(\mathbb{V}) & \longrightarrow & 0
\end{array}$$

where $\Gamma^k \subset \Lambda^k(\mathbb{V})$ is given by

$$\Gamma^1 = \{(\omega, \mu) : d\omega = S\mu\}, \quad \Gamma^2 = \{0\} \times \Lambda^2(\mathbb{R}^3_*).$$

The projections π^1 and π^2 are given by

$$\pi^1(\omega, \mu) = (\omega, S^{-1}d\omega) \quad \pi^2(\omega, \mu) = (0, \mu + dS^{-1}\omega)$$

The diagram commutes, and therefore the bottom sequence is exact.

The bottom sequence above:

$$\mathbb{V} \xrightarrow{\subset} \Lambda^0(\mathbb{V}) \xrightarrow{\mathcal{A}} \Gamma^1 \xrightarrow{\mathcal{A}} \Gamma^2 \xrightarrow{\mathcal{A}} \Lambda^3(\mathbb{V}) \longrightarrow 0$$

where

$$\Gamma^1 = \{(\omega, \mu) : d\omega = S\mu\}, \quad \Gamma^2 = \{0\} \times \Lambda^2(\mathbb{R}^3_*).$$

By identifying elements $(\omega, \mu) \in \Gamma^1$ with $\omega \in \Lambda^1(\mathbb{R}^3)$, and elements $(0, \mu) \in \Gamma^2$ with $\mu \in \Lambda^2(\mathbb{R}^3_*)$, the bottom row is equivalent to

$$\mathbb{V} \xrightarrow{\subset} \Lambda^0(\mathbb{V}) \xrightarrow{(d,S)} \Lambda^1(\mathbb{R}^3) \xrightarrow{d \circ S^{-1} \circ d} \Lambda^2(\mathbb{R}^3_*) \xrightarrow{(d,S)^T} \Lambda^3(\mathbb{V}) \longrightarrow 0$$

since $\mathcal{A}(\omega, S^{-1}d\omega) = (0, dS^{-1}d\omega)$.

The last sequence can again be identified with the elasiticity sequence with weakly imposed symmetry. By identifying

$$\mathbb{V} \cong \mathbb{R}^3 \times \mathbb{K} = \mathbb{W}, \quad \Lambda^0(\mathbb{V}) \cong C^\infty(\mathbb{W}) \quad \Lambda^1(\mathbb{R}^3) \cong \Lambda^2(\mathbb{R}^3_*) \cong C^\infty(\mathbb{M})$$

the two sequences

$$\mathbb{V} \xrightarrow{\subset} \Lambda^0(\mathbb{V}) \xrightarrow{(d,S)} \Lambda^1(\mathbb{R}^3) \xrightarrow{d \circ S^{-1} \circ d} \Lambda^2(\mathbb{R}^3_*) \xrightarrow{(d,S)^T} \Lambda^3(\mathbb{V}) \longrightarrow 0$$

and

$$\mathbb{T} \xrightarrow{\subset} C^\infty(\mathbb{W}) \xrightarrow{(\mathrm{grad}, I)} C^\infty(\mathbb{M}) \xrightarrow{J} C^\infty(\mathbb{M}) \xrightarrow{(\mathrm{div}, \mathrm{skw})^T} C^\infty(\mathbb{W}) \longrightarrow 0$$

are equivalent.

The discrete case

Let Λ_h^k be discrete k–forms:

$$\omega(x) = \sum_{i_1<i_2<\ldots<i_k} f_{i_1\ldots i_k}(x)dx_{i_1}\wedge\ldots\wedge dx_{i_k} = \sum_I f_I(x)dx_I$$

where $\{f_I\}$ are piecewise polynomial functions with values in $\mathbb{R}$.

We assume that

$$\mathbb{R} \xrightarrow{\subset} \Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \Lambda_h^2 \xrightarrow{d} \Lambda_h^3 \longrightarrow 0$$

is exact.

Furthermore, we assume that there exist corresponding interpolation operators Π_h onto Λ_h^k such that the following diagram commutes

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\subset} & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \longrightarrow & 0 \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Pi_h} & & \downarrow{\scriptstyle \Pi_h} & & \downarrow{\scriptstyle \Pi_h} & & \downarrow{\scriptstyle \Pi_h} & & \\
\mathbb{R} & \xrightarrow{\subset} & \Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \Lambda_h^2 & \xrightarrow{d} & \Lambda_h^3 & \longrightarrow & 0
\end{array}$$

Let $\mathbb{V} = \mathbb{R}^3 \times \mathbb{R}^3_*$. By essentially taking six, possibly different, discretizations of the de Rham sequence we obtain a corresonding $\mathbb{V}$-valued sequence

$$\mathbb{V} \xrightarrow{\subset} \Lambda_h^0(\mathbb{V}) \xrightarrow{d} \Lambda_h^1(\mathbb{V}) \xrightarrow{d} \Lambda_h^2(\mathbb{V}) \xrightarrow{d} \Lambda_h^3(\mathbb{V}) \longrightarrow 0,$$

where

$$\Lambda_h^k(\mathbb{V}) = \Lambda_h^k(\mathbb{R}^3) \times \Lambda_h^k(\mathbb{R}^3_*).$$

We define $K_h : \Lambda_h^k(\mathbb{R}^3_*) \mapsto \Lambda_h^k(\mathbb{R}^3)$ by $K_h = \varPi_h K$.

Furthermore, $S_h : \Lambda_h^k(\mathbb{R}^3_*) \mapsto \Lambda_h^{k+1}(\mathbb{R}^3)$ is defined by $S_h = dK_h - K_h d$. We obtain

$$dS_h = -S_h d$$

and

$$S_h = d\varPi_h K - \varPi_h K d = \varPi_h(dK - Kd) = \varPi_h S.$$

Let $\mathcal{A}_h : \Lambda_h^k(\mathbb{V}) \mapsto \Lambda_h^{k+1}(\mathbb{V})$ be defined as $\mathcal{A}_h = \Phi_h d \Phi_h^{-1}$, which leads to

$$\mathcal{A}_h(\omega, \mu) = (d\omega - S_h \mu, d\mu).$$

We obtain:

$$\mathbb{V} \xrightarrow{\subset} \Lambda_h^0(\mathbb{V}) \xrightarrow{\mathcal{A}_h} \Lambda_h^1(\mathbb{V}) \xrightarrow{\mathcal{A}_h} \Lambda_h^2(\mathbb{V}) \xrightarrow{\mathcal{A}_h} \Lambda_h^3(\mathbb{V}) \longrightarrow 0.$$

In order to perform the proper projection of this sequence we define subspaces Γ_h^k of $\Lambda_h^k(\mathbb{V})$, $k = 1, 2$ by

$$\Gamma_h^1 = \{(\omega, \mu) : d\omega = S_h \mu\}, \quad \Gamma_h^2 = \{0\} \times \Lambda_h^2(\mathbb{R}_*^3).$$

Extra assumption on the discrete spaces:

(A) The operator $S_h : \Lambda_h^1(\mathbb{R}_*^3) \mapsto \Lambda_h^2(\mathbb{R}^3)$ is onto.

The operator $S_h : \Lambda_h^1(\mathbb{R}^3_*) \mapsto \Lambda_h^2(\mathbb{R}^3)$ has a right inverse $S_h^\dagger$.

Discrete projection operators π_h^1 and π_h^2, are now defined by

$$\pi_h^1(\omega, \mu) = (\omega, \mu - S_h^\dagger S_h \mu + S_h^\dagger d\omega) \quad \pi_h^2(\omega, \mu) = (0, \mu + dS_h^\dagger \omega)$$

The following diagram commutes and the rows are exact:

$$\begin{array}{ccccccccccc}
\mathbb{V} & \xrightarrow{\subset} & \Lambda_h^0(\mathbb{V}) & \xrightarrow{\mathcal{A}_h} & \Lambda_h^1(\mathbb{V}) & \xrightarrow{\mathcal{A}_h} & \Lambda_h^2(\mathbb{V}) & \xrightarrow{\mathcal{A}_h} & \Lambda_h^3(\mathbb{V}) & \longrightarrow & 0 \\
\big\downarrow id & & \big\downarrow id & & \big\downarrow \pi_h^1 & & \big\downarrow \pi_h^2 & & \big\downarrow id & & \\
\mathbb{V} & \xrightarrow{\subset} & \Lambda_h^0(\mathbb{V}) & \xrightarrow{\mathcal{A}_h} & \Gamma_h^1 & \xrightarrow{\mathcal{A}_h} & \Gamma_h^2 & \xrightarrow{\mathcal{A}_h} & \Lambda_h^3(\mathbb{V}) & \longrightarrow & 0
\end{array}$$

Note that the constraints in a finite element formulation, with weakly imposed symmetry, corresponds exactly to the operator $\mathcal{A}_h : \Gamma_h^2 \mapsto \Lambda_h^3(\mathbb{V})$.

Finite elements for mixed elasticity with weakly imposed symmetry

Find $(\sigma, u, p) \in \Sigma_h \times V_h \times Q_h \subset H(\operatorname{div}, \Omega, \mathbb{M}) \times L_2(\Omega, \mathbb{R}^3) \times L_2(\Omega, \mathbb{K})$ such that

$$\begin{aligned}
(A\sigma, \tau) + (\operatorname{div} \tau, u) + (\tau, p) &= 0, & \tau &\in \Sigma_h, \\
(\operatorname{div} \sigma, v) &= (f, v), & v &\in V_h, \\
(\sigma, q) &= 0, & q &\in Q_h.
\end{aligned}$$

Simplest stable element:

- Σ_h = piecewise linear matrix fields; $\Sigma_h \cong \mathcal{P}_1\Lambda_h^2(\mathbb{R}^3_*)$
- V_h = piecewise constant vector fields; $V_h \cong \mathcal{P}_0\Lambda_h^3(\mathbb{R}^3_*)$
- Q_h = piecewise constant elements of $\mathbb{K}$; $Q_h \cong \mathcal{P}_0\Lambda_h^3(\mathbb{R}^3)$

This discretization arises from the following 2×3 discrete de Rham sequences:

$$\mathbb{R}^3 \xrightarrow{\subset} \mathcal{P}_1\Lambda_h^0(\mathbb{R}^3) \xrightarrow{d} \mathcal{P}_0^+\Lambda_h^1(\mathbb{R}^3) \xrightarrow{d} \mathcal{P}_0^+\Lambda_h^2(\mathbb{R}^3) \xrightarrow{d} \mathcal{P}_0\Lambda_h^3(\mathbb{R}^3) \longrightarrow 0$$

and

$$\mathbb{R}^3_* \xrightarrow{\subset} \mathcal{P}_2\Lambda_h^0(\mathbb{R}^3_*) \xrightarrow{d} \mathcal{P}_1^+\Lambda_h^1(\mathbb{R}^3_*) \xrightarrow{d} \mathcal{P}_1\Lambda_h^2(\mathbb{R}^3_*) \xrightarrow{d} \mathcal{P}_0\Lambda_h^3(\mathbb{R}^3_*) \longrightarrow 0.$$

Assumption (A) is verified by the commuting diagram:

$$\begin{array}{ccc} \Lambda^1(\mathbb{R}^3_*) & \xleftarrow{S^{-1}} & \Lambda^2(\mathbb{R}^3) \\ \downarrow \varPi_h & & \downarrow \varPi_h \\ \mathcal{P}_1^+\Lambda_h^1(\mathbb{R}^3_*) & \xrightarrow{S_h} & \mathcal{P}_0\Lambda_h^2(\mathbb{R}^3) \end{array}$$

or

$$\varPi_h = S_h \varPi_h S^{-1} \qquad \text{on } \Lambda^2(\mathbb{R}^3).$$

Verification of the inf-sup condition

Given $(\omega, \mu) \in \mathcal{P}_0\Lambda_h^3(\mathbb{R}^3) \times \in \mathcal{P}_0\Lambda_h^3(\mathbb{R}_*^3)$. Find $\sigma \in \mathcal{P}_1\Lambda_h^2(\mathbb{R}_*^3)$ such that

$$\mathcal{A}_h(0, \sigma) \equiv (-S_h\sigma, d\sigma) = (\omega, \mu)$$

and

$$\|\sigma\|_0 + \|d\sigma\|_0 \leq c(\|\omega\|_0 + \|\mu\|_0).$$

Solution: $\sigma = \sigma_1 + \sigma_2$ where

$$d\sigma_2 = \mu \qquad \text{three copies of mixed Poisson argument}$$
$$\sigma_1 = d\mathit{\Pi}_h S^{-1}\tau \qquad \text{where } d\tau = \omega.$$

Here $\tau \in \mathcal{P}_0^+\Lambda_h^2(\mathbb{R}^3)$ and $\mathit{\Pi}_h \mapsto \mathcal{P}_1^+\Lambda_h^1(\mathbb{R}_*^3)$.