Numerical Simulation of Constraints Preserving Boundary Conditions for Constrainted Hyperbolic Equations

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Abstract

In general relativity, the Einstein equations can be written in many ways into a system of hyperbolic equations and a set of constraint equations. One of the most challenging questions in this area is how to specify the boundary conditions such that the constraints will be preserved. As a model problem, we consider a system of wave equations with divergence free constraints, and give both homogeneous and in-homogeneous constraints preserving boundary conditions. Numerical results with discontinuous Galerkin method show that divergence free conditions are indeed preserved under these boundary conditions. We also provide numerical results for the constraints preserving boundary conditions of the linearized Einstein-Christoff formulation of the Einstein equations.

Constrainted wave equations

Considering n independent wave equations with divergence free constraints:

$$\begin{cases} \ddot{w}_i = \Delta w_i, \quad i=1,\dots,n, \ (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \\ w_i(x,0) = w_i^0(x), \ i=1,\dots,n, \quad x\in\mathbb{R}^n, \\ div\ w(x,t) = 0, \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+. \end{cases} \tag{1}$$

Let $u_{ij} = \partial w_i/\partial x_j$ and $v_i = \dot{w}_i$, $i,j = 1,\dots,n$.

$$\dot{w}_i = v_i \quad \dot{u_{ij}} = \partial_j v_i \quad \dot{v_i} = \partial^j u_{ij}. \tag{2}$$

Constraints:

$$\partial^i u_{ij} = 0, \quad j=1,\dots,n \quad \partial^i v_i = 0. \tag{3}$$

For the constraints $\partial^i v_i = 0$, let $C = \partial^i v_i$,

$$\ddot{C} = \Delta C. \tag{4}$$

So if the constraints are satisfied at the initial time $t = 0$, then $C(x,0) = \partial^i v_i(x,0) = 0$ and $\dot{C}(x,0) = \partial^j\partial^i u_{ij}(x,0) = 0$, which implies $C(x,t) \equiv 0$. This shows that if the divergence free constraints (3) are satisfied at the initial time, they will be satisfied all the time.

Homogeneous constraints preserving BC in 2-D: Suppose Ω is a polygon domain in $\mathbb{R}^2$.

$$n^i n^j u_{ij} = s^i v_i = 0,$$

where, n^i and s^i are the unit exterior normal and tangential vectors on the boundary, respectively.

Proof. Let $C = \partial^i v_i$, then $\ddot{C} = \Delta C$. $C(0) = \dot{C}(0) = 0$ by assumption.

On the boundary

$$\partial^i v_i = \delta^{ij}\partial_j v_i = n^i n^j \partial_j v_i + s^i s^j \partial_j v_i = (n^i n^j u_{ij})^{\cdot} + s^j \partial_j (s^i v_i) = 0.$$

So $C \equiv 0$.

Inhomogeneous constraints preserving BC in 2-D:

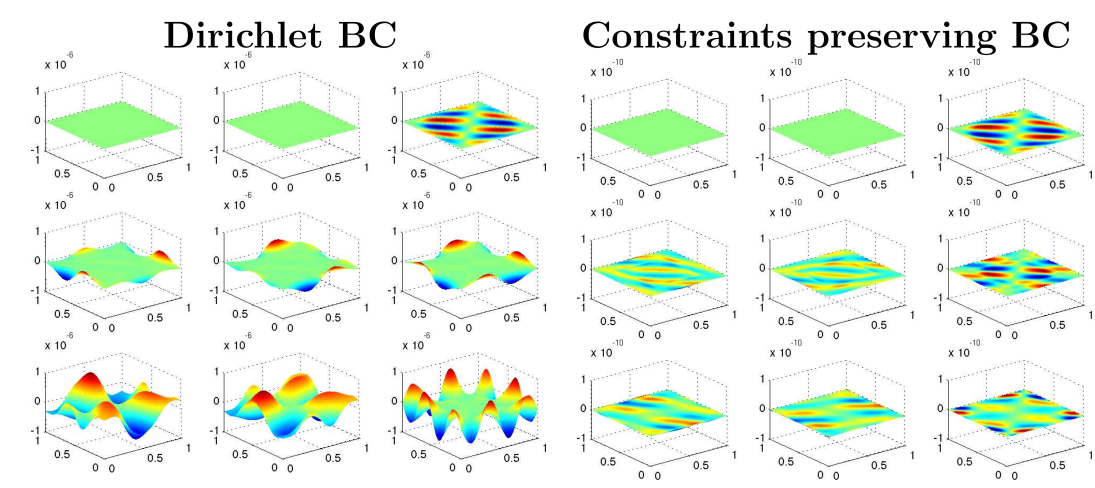
$$n^i n^j u_{ij} = n^i n^j \phi_{ij}, \quad s^i v_i = s^i \mu_i,$$

where, ϕ_{ij} and μ_i are given functions which satisfy the constraints

$$\partial^i \phi_{ij} = 0, \quad j = 1,\dots,n \quad \partial^i \mu_i = 0,$$

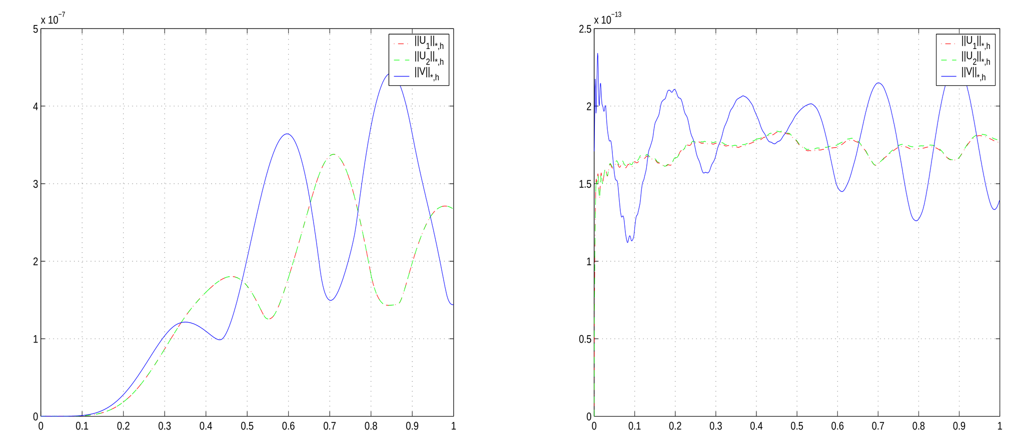
and $\dot{\phi_{ij}} = \partial_j \mu_i$.

Numerical results–homogeneous boundary conditions



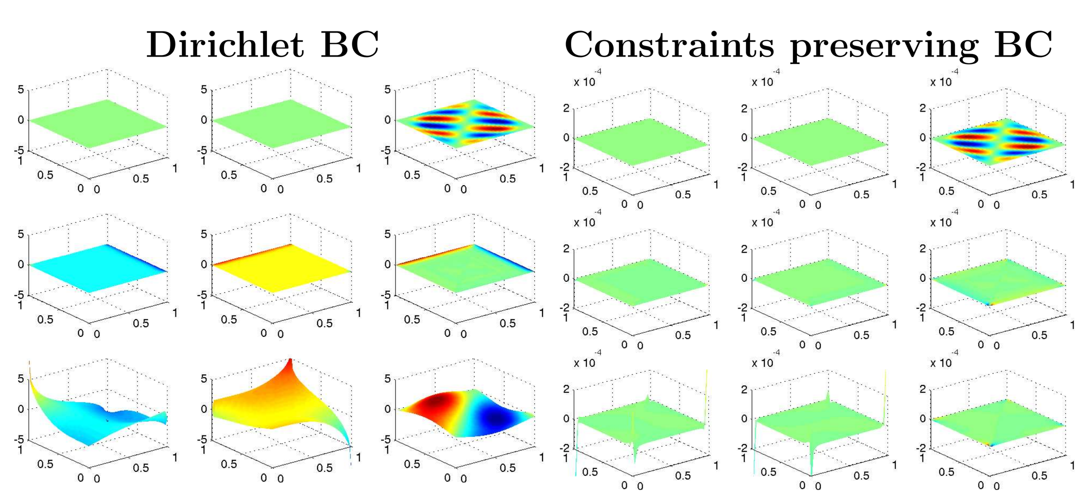
Divergences of $u_{\cdot 1}$, $u_{\cdot 2}$ and v. Top: $t = 0$; Middle: $t = 0.2$; Bottom: $t = 1.0$.

Divergences of $u_{\cdot 1}$ $u_{\cdot 2}$ and v. Top: $t = 0$; Middle: $t = 0.2$; Bottom: $t = 1.0$.



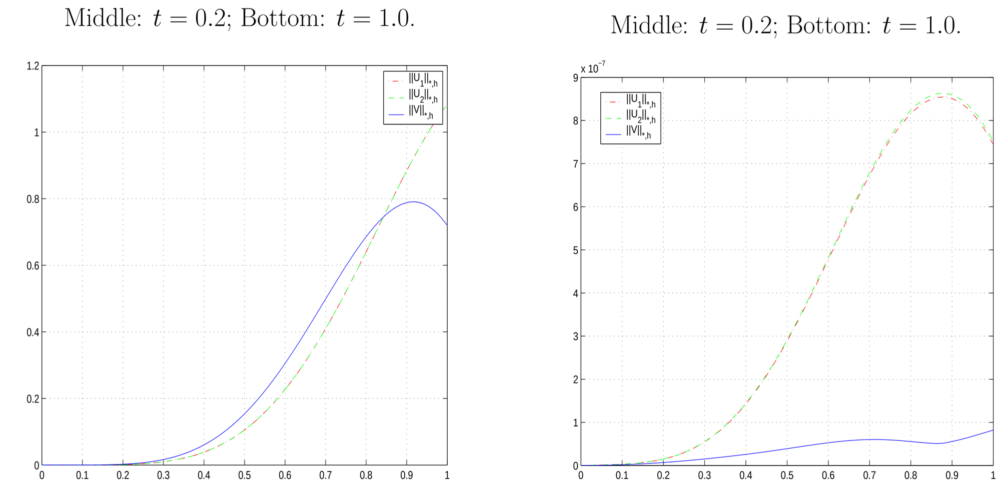
Changes of $\|\nabla\cdot u\|_{*,h}$ and $\|\nabla\cdot v\|_{*,h}$ over $t \in [0,1]$. Left: Dirichlet BC; Right: CP-BC.

Numerical results–inhomogeneous boundary conditions



Divergences of $u_{\cdot 1}$, $u_{\cdot 2}$ and v. Top: $t = 0$; Middle: $t = 0.2$; Bottom: $t = 1.0$.

Divergences of $u_{\cdot 1}$ $u_{\cdot 2}$ and v. Top: $t = 0$; Middle: $t = 0.2$; Bottom: $t = 1.0$.



Changes of $\|\nabla\cdot u\|_{*,h}$ and $\|\nabla\cdot v\|_{*,h}$ over $t \in [0,1]$. Left: Dirichlet BC; Right: CP-BC.

Linearized Einstein-Christoff formulation of the Einstein equations

The system:

$$\begin{aligned} \partial_t \gamma_{ij} &= -2\kappa_{ij} + \partial_i\beta_j + \partial_j\beta_i, \\ \partial_t \kappa_{ij} &= -\partial^k f_{kij} - \partial_i\partial_j\alpha, \\ \partial_t f_{kij} &= -\partial_k \kappa_{ij} + L_{kij}. \end{aligned} \tag{5}$$

where,γ_{ij} are the spatial metric, κ_{ij} are the extrinsic curvatures and

$$\begin{aligned} f_{kij} &= \frac{1}{2}\Big(\partial_k\gamma_{ij} - (\partial^l\gamma_{li} - \partial_i\gamma_l^l)\delta_{jk} - (\partial^l\gamma_{lj} - \partial_j\gamma_l^l)\delta_{ik}\Big) \\ L_{kij} &= \partial_k\partial_{(i}\beta_{j)} - \partial^l\partial_{[l}\beta_{i]}\delta_{jk} - \partial^l\partial_{[l}\beta_{j]}\delta_{ik}. \end{aligned}$$

The constraints:

$$C := \partial^j(\partial^l\gamma_{lj} - \partial_j\gamma_l^l) = 0, \quad C_j := \partial^l\kappa_{lj} - \partial_j\kappa_l^l = 0.$$

Suppose Ω is a polyhedral domain in $\mathbb{R}^3$. Let n^i be the exterior unit normal vector of $\partial\Omega$ and m^i and l^i be the unit tangential vectors which form an orthonormal basis with n^i.

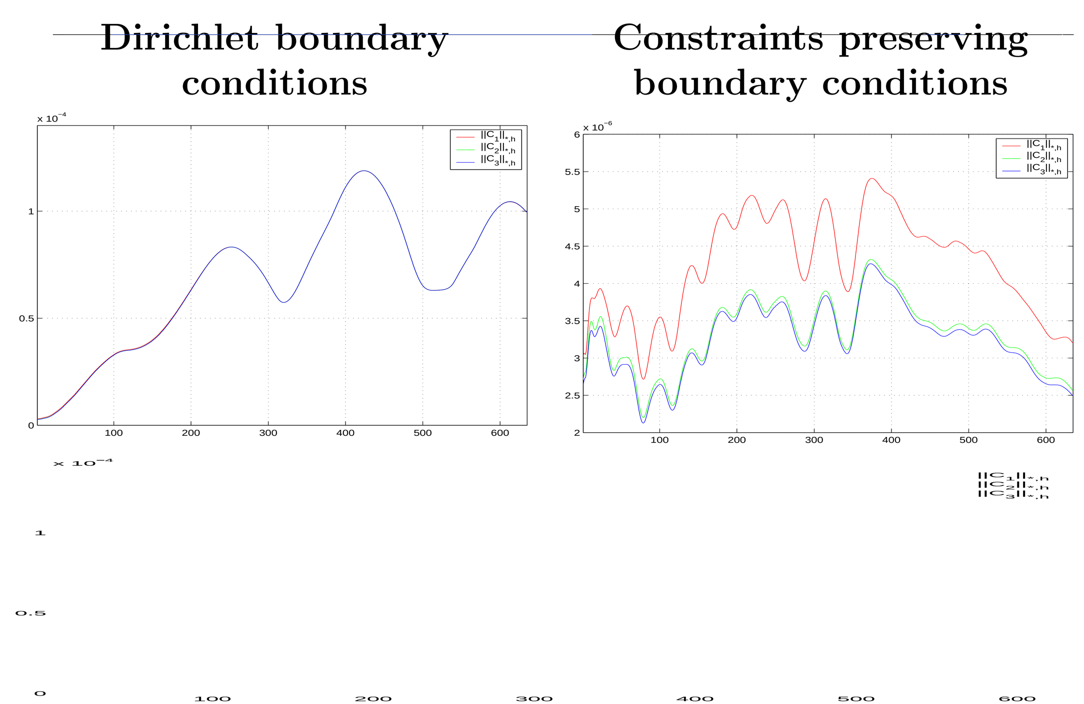
Constraints preserving Boundary conditions (Arnold-Tarfulea[1])

$$n^i m^i \kappa_{ij} = n^i l^j \kappa_{ij} = n^k n^i n^j f_{kij} = n^k m^i m^j f_{kij} = n^k l^i l^j f_{kij} = n^k m^i l^j f_{kij} = 0.$$

Measurement of the Momentum constraints:

$$\|C_i\|_{*,h} = \sum_{k\in K}\int_k |\partial^l\kappa_{li} - \partial_i\kappa_l^l|dx + \sum_{e\in E}\int_e |[\kappa_{li}n^l - \kappa_l^l n_i]|ds.$$

Numerical results



Changes of the momentum constraints over time $t \in [0,1]$

References

[1] D. N. Arnold, N. Tarfulea, Maximal nonnegative constraint-preserving boundary conditions for the linearized Einstein equations, preprint.

[2] B. Cockburn, C. W. Shu, Runge-Kutta discontinuous Galerkin methods for convection-dominated problems, J. Sci. Comp., 16 (2001)173–261.

[3] B. Cockburn, Fengyan Li and C. W. Shu, Locally divergence-free discontinuous Galerkin methods for the Maxwell equations, preprint.