Twenty examples of entropy stable schemes

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Outline

- Entropy variables
- Entropy conservative and entropy stable schemes
- The scalar problem
- Systems of conservation laws I
- Entropy conservative schemes revisited II
- Fully discrete schemes - the homotopy approach
- Higher order extensions

Linear stability – the energy method

$$\mathbf{u}^\top \mathbf{u}_x \longrightarrow \frac{1}{2}\Big(\|\mathbf{u}\|^2\Big)_x : \text{ Does } \quad \mathbf{u}^\top D_{\Delta x}\mathbf{u} \longrightarrow \frac{1}{2} D_{\Delta x}\Big(\ldots\Big) \text{ ?}$$

Nonlinear entropy stability, $U = U(\mathbf{u})$

$$if \quad U_\mathbf{u}^\top \mathbf{f}(\mathbf{u})_x = F_x : \text{ Does } \quad U_\mathbf{u}^\top \frac{\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}}{\Delta x_\nu} \longrightarrow \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x_\nu} \text{ ?}$$

- Energy flux $\frac{1}{2}\|\mathbf{u}\|^2 \iff$ entropy flux $F(\mathbf{u})$: $\quad U_\mathbf{u}^\top \mathbf{f}_\mathbf{u} = F_\mathbf{u}^\top$

Entropy stability

$$\frac{d}{dt}\mathbf{u}_\nu(t) + \frac{\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}}{\Delta x_\nu} = 0 : \qquad \frac{d}{dt}U(\mathbf{u}_\nu(t)) + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x_\nu} \leq 0.$$

- Three main tools of the trade:

⊙ Comparison arguments

Monotone schemes — comparison w/constant solutions
(Harten- Hyman-Lax, Crandall-Majda)

E Schemes — comparison w/Godunov scheme
(Osher, Tadmor, ...)

Systems — comparison w/ entropy conservative schemes...

⊙ A homotopy approach — (Lax, Tadmor, ...) — fully-discrete

$$\frac{U(\mathbf{u}_\nu(t+\Delta t)) - U(\mathbf{u}_\nu(t))}{\Delta t} + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x_\nu} \leq 0.$$

⊙ Kinetic formulation (Bouchut, Makridakis-Perthame...)

Entropy and symmetric forms

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u}(x,t) = (u_1(x,t), \ldots, u_N(x,t))^\top$$

- An entropy: a convex $U(\mathbf{u})$ which symmetrizes 'on the left'

$$U_{\mathbf{uu}}A = [U_{\mathbf{uu}}A]^\top, \quad A(\mathbf{u}) := \mathbf{f_u}(\mathbf{u}),\ \mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_N(\mathbf{u}))^\top$$

⊙ Friedrichs and Lax, ...

⊙ 'Symmetrize on the right' – $A(U_{\mathbf{uu}})^{-1} = \left[A(U_{\mathbf{uu}})^{-1}\right]^\top$ ...

Entropy variables: $\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0$

- Entropy variables: $\mathbf{v} \equiv \mathbf{v}(\mathbf{u}) := \nabla_{\mathbf{u}} U(\mathbf{u})$.

- Symmetric form: convexity of $U(\cdot)$, $\mathbf{u} = \mathbf{u}(\mathbf{v})$ is 1-1

$$\boxed{\frac{\partial}{\partial t}\mathbf{u}(\mathbf{v}) + \frac{\partial}{\partial x}\mathbf{g}(\mathbf{v}) = 0, \quad \mathbf{g}(\mathbf{v}) := \mathbf{f}(\mathbf{u}(\mathbf{v}))}$$

⊙ Symmetric in the sense that the Jacobians are

$$H(\mathbf{v}) := \mathbf{u}_{\mathbf{v}}(\mathbf{v}) = H^{\top}(\mathbf{v}) > 0 \quad \text{and} \quad B(\mathbf{v}) := \mathbf{g}_{\mathbf{v}}(\mathbf{v}) = B^{\top}(\mathbf{v}).$$

$U(\cdot)$ is an entropy iff $\exists$ an entropy flux $F = F(\mathbf{u})$, $U_{\mathbf{u}}^{\top}\mathbf{f}_{\mathbf{u}} = F_{\mathbf{u}}^{\top}$:

$$\begin{aligned} \mathbf{u}(\mathbf{v}) &= \nabla_{\mathbf{v}}\phi(\mathbf{v}), \quad \phi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{u}(\mathbf{v})\rangle - U(\mathbf{u}(\mathbf{v})) \\ \mathbf{g}(\mathbf{v}) &= \nabla_{\mathbf{v}}\psi(\mathbf{v}), \quad \psi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{g}(\mathbf{v})\rangle - F(\mathbf{u}(\mathbf{v})), \end{aligned}$$

- The Jacobians $H(\mathbf{v}), B(\mathbf{v})$ are symmetric Hessians of $\phi(\mathbf{v}), \psi(\mathbf{v})$.

- The symmetry of $B = AH$ – symmetrization 'on the right'

⊙ Godunov, Mock, ...

Physically relevant solutions

- $\mathbf{u} = \lim_{\epsilon \downarrow 0} \mathbf{u}^\epsilon$ where $\mathbf{u}^\epsilon_t + \mathbf{f}(\mathbf{u}^\epsilon)_x = \epsilon(P\mathbf{u}^\epsilon_x)_x$

- Admissible $P = P(\mathbf{u}, \mathbf{u}_x)$'s: H-symmetric viscosity matrices

$$PH = \left[PH\right]^\top \geq 0, \quad H = (U_{\mathbf{uu}})^{-1}$$

- Entropy inequality: $\frac{\partial}{\partial t}U(\mathbf{u}) + \frac{\partial}{\partial x}F(\mathbf{u}) \leq 0$

- Euler equations: $\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ m \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} m \\ qm + p \\ q(E+p) \end{bmatrix} = 0.$

- Entropy pairs: $U(\mathbf{u}) = \frac{\gamma+1}{1-\gamma} \cdot (\rho p)^{\frac{1}{\gamma+1}}, \quad F(\mathbf{u}) = \frac{\gamma+1}{1-\gamma} q \cdot (\rho p)^{\frac{1}{\gamma+1}}$

$$\mathbf{v} \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -(\rho p)^{-\frac{\gamma}{\gamma+1}} \cdot \begin{bmatrix} E \\ -m \\ \rho \end{bmatrix} : \quad \frac{\partial}{\partial t}[\nabla_{\mathbf{v}} \rho] + \frac{\partial}{\partial x}[\nabla_{\mathbf{v}} m] = 0$$

Entropy-stable and entropy-conservative schemes

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\left[\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\right], \qquad \Delta x_\nu := \frac{1}{2}(x_{\nu+1} - x_{\nu-1})$$

⊙ The consistent numerical flux $\mathbf{f}_{\nu+\frac{1}{2}}$:

$$\mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p}), \qquad \mathbf{f}(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) \equiv \mathbf{f}(\mathbf{u})$$

- Entropy stability: $\frac{d}{dt}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\left[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\right] \leq 0$

Q. The consistent numerical entropy flux $F_{\nu+\frac{1}{2}}$:

$$F_{\nu+\frac{1}{2}} = F(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p}), \qquad F(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) = F(\mathbf{u})$$

⊙ Entropy-conservative: if $\leq 0 \quad \Longrightarrow \quad = 0$

Entropy-stable and entropy-conservative schemes

• Use of the entropy variables • Comparison with entropy-conservative schemes

• changes of variables $\mathbf{u}_\nu = \mathbf{u}(\mathbf{v}_\nu)$:

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\Big], \qquad \mathbf{v}_\nu(t) = \mathbf{v}(\mathbf{u}_\nu(t)),$$

⊙ Consistent numerical flux $\mathbf{g}_{\nu+\frac{1}{2}} \equiv \mathbf{g}(\mathbf{v}_{\nu-p+1}, \ldots, \mathbf{v}_{\nu+p})$:

$$\mathbf{g}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{u}(\mathbf{v}_{\nu-p+1}), \ldots, \mathbf{u}(\mathbf{v}_{\nu+p})), \qquad \mathbf{g}(\mathbf{v}, \mathbf{v}, \ldots, \mathbf{v}) = \mathbf{g}(\mathbf{v}) \equiv \mathbf{f}(\mathbf{u}(\mathbf{v})).$$

THEOREM [Ta 1987]: Entropy flux potential $\psi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{g}(\mathbf{v})\rangle - F(\mathbf{u}(\mathbf{v}))$

• Entropy stability: $\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}_{\nu+\frac{1}{2}} \Big\rangle \leq \Delta\psi_{\nu+\frac{1}{2}}, \qquad \Delta\mathbf{v}_{\nu+\frac{1}{2}} \equiv \mathbf{v}_{\nu+1} - \mathbf{v}_\nu$

• Entropy conservative: $\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}} \Big\rangle = \Delta\psi_{\nu+\frac{1}{2}}$

⊙ By parts: $\mathbf{v}_\nu\Big[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\Big]$ is conservative iff $\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}_{\nu+\frac{1}{2}}\rangle$ is

The scalar problem: $\frac{d}{dt}u_\nu(t) = -\frac{1}{\Delta x_\nu}\left[g_{\nu+\frac{1}{2}} - g_{\nu-\frac{1}{2}}\right]$

- The viscosity form: $Q_{\nu+\frac{1}{2}} := \Big(f(u_\nu)+f(u_{\nu+1})-2g_{\nu+\frac{1}{2}}\Big)/\Delta v_{\nu+\frac{1}{2}}$

$$\frac{d}{dt}u_\nu(t) = -\left[\frac{f(u_{\nu+1}) - f(u_{\nu-1})}{2\Delta x_\nu}\right] + \frac{1}{2\Delta x_\nu}\left[Q_{\nu+\frac{1}{2}}\Delta v_{\nu+\frac{1}{2}} - Q_{\nu-\frac{1}{2}}\Delta v_{\nu-\frac{1}{2}}\right]$$

- The entropy conservative flux (!): $g^*_{\nu+\frac{1}{2}} := \frac{\Delta\psi_{\nu+\frac{1}{2}}}{\Delta v_{\nu+\frac{1}{2}}} \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} g\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi$

$$g^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} (\xi)' g\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi$$

$$= \frac{1}{2}\left[f(u_\nu) + f(u_{\nu+1})\right] - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi g_v\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi\,\Delta v_{\nu+\frac{1}{2}}$$

- Entropy conservative viscosity: $Q^*_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi g_v\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi$

$$\frac{d}{dt}u_\nu(t) = -\frac{1}{\Delta x_\nu}\left[g^*_{\nu+\frac{1}{2}} - g^*_{\nu-\frac{1}{2}}\right] =$$

$$= -\frac{1}{2\Delta x_\nu}\left[f(u_{\nu+1}) - f(u_{\nu-1})\right] + \frac{1}{2\Delta x_\nu}\left[Q^*_{\nu+\frac{1}{2}}\Delta v_{\nu+\frac{1}{2}} - Q^*_{\nu-\frac{1}{2}}\Delta v_{\nu-\frac{1}{2}}\right]$$

Scalar examples

1 Burgers' $u_t + (\frac{1}{2}u^2)_x = 0$ w/logarithmic entropy $U(u) = -\ln u$:

$\odot$ Entropy flux $F(u) = -u$, entropy variable $v(u) = -1/u$,
Entropy flux potential $\psi(v) := vf(u(v)) - F(u(v)) = -\frac{1}{2v}$,

• Entropy-conservative flux:

$$g^*_{\nu+\frac{1}{2}} = \frac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu} = \frac{1}{2}\frac{1}{v_\nu v_{\nu+1}} = \frac{1}{2}u_\nu u_{\nu+1}.$$

• Entropy-conservative centered scheme
(Lax, Goodman, Hou, Levermore...):

$$\frac{d}{dt}u_\nu(t) = u_\nu(t)\frac{u_{\nu+1}(t) - u_{\nu-1}(t)}{2\Delta x_\nu}$$

Scalar examples

2 Toda flow: $u_t + (e^u)_x = 0$ w/exponential entropy $U(u) = e^u$:

$\odot$ Entropy flux $F(u) = e^{2u}/2$, entropy variable $v(u) = e^u$,
Entropy flux potential $\psi(v) := vf(u(v)) - F(u(v)) = -\frac{1}{2}v^2$,

• Entropy-conservative flux:

$$g^*_{\nu+\frac{1}{2}} = \frac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu} = \frac{1}{2}\frac{1}{v_\nu v_{\nu+1}} = \frac{1}{2}\Big[e^{u_\nu} + e^{u_{\nu+1}}\Big].$$

• Entropy-conservative centered scheme
(Lax, Deift, McLaughlin...):

$$\frac{d}{dt}u_\nu(t) = \frac{e^{u_{\nu+1}(t)} - e^{u_{\nu-1}(t)}}{2\Delta x_\nu}$$

Scalar examples - first order entropy stable schemes

3 Engquist-Osher scheme (1980)

$$Q^*_{\nu+\frac{1}{2}} \leq \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left| g_v\left(v_{\nu+\frac{1}{2}}(\xi)\right)\right| d\xi = \int \left| f_u\left(u\left(v_{\nu+\frac{1}{2}}(\xi)\right)\right)\right| \left|\frac{du(v_{\nu+\frac{1}{2}}(\xi))}{\Delta v_{\nu+\frac{1}{2}}}\right|$$

$$= \frac{1}{\Delta v_{\nu+\frac{1}{2}}}\left[\int_{u_\nu}^{u_{\nu+1}} |f_u(u)| du\right] =: Q^{EO}_{\nu+\frac{1}{2}},$$

4 Godunov scheme (1959): $\left(Q_{\nu+\frac{1}{2}}\frac{\Delta v_{\nu+\frac{1}{2}}}{\Delta u_{\nu+\frac{1}{2}}}\right)\Delta u_{\nu+\frac{1}{2}} - \left(Q_{\nu-\frac{1}{2}}\frac{\Delta v_{\nu-\frac{1}{2}}}{\Delta u_{\nu-\frac{1}{2}}}\right)\Delta u_{\nu-\frac{1}{2}}$

⊙ Uniform entropy stability: maximize

$$\sup_v \left[\frac{f(u_\nu) + f(u_{\nu+1}) - 2g^*_{\nu+\frac{1}{2}}}{\Delta u_{\nu+\frac{1}{2}}}\right], \quad g^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f\left(u\left(v_{\nu+\frac{1}{2}}(\xi)\right)\right) d\xi,$$

5 E-schemes: entropy stability ∀ entropies (Osher, Tadmor): $Q \geq Q^G$

$$Q^G_{\nu+\frac{1}{2}} = \max_{\min(u_\nu,u_{\nu+1})\leq u\leq \max(u_\nu,u_{\nu+1})} \left[\frac{f(u_\nu) + f(u_{\nu+1}) - 2f(u)}{\Delta u_{\nu+\frac{1}{2}}}\right].$$

Scalar examples - 2nd order entropy stable schemes

- E-schemes are first order: $Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi g_v\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi$ but..

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\xi^2 - \frac{1}{4}\right)' g_v\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi =$$

$$= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) g_{vv}\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi \cdot \Delta v_{\nu+\frac{1}{2}}$$

- Second order: $Q^*_{\nu+\frac{1}{2}} \le \max_{\min(v_\nu,v_{\nu+1})\le v\le\max(v_\nu,v_{\nu+1})} \frac{|g_{vv}(v)|}{6} \cdot \left|\Delta v_{\nu+\frac{1}{2}}\right|$

6 Lax-Wendroff (1960): $f_{uu} > 0,\ U(u) = \frac{u^2}{2},\ g(v) = f(u)$

$$Q^*_{\nu+\frac{1}{2}} \le \frac{1}{4}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f_{uu}\Big(u_{\nu+\frac{1}{2}}(\xi)\Big)d\xi \cdot \Delta u_{\nu+\frac{1}{2}} = \frac{1}{4}\Big[a(u_{\nu+1}) - a(u_\nu)\Big]^+$$

Scalar examples - 2nd order centered schemes

- convex $g(v) = f(u) : \psi(u) = \int f(\cdot)$ and entropy stablity if

$$\Delta u_{\nu+\frac{1}{2}} \cdot f_{\nu+\frac{1}{2}} \leq \int_{u_\nu}^{u_{\nu+1}} f(u)du.$$

⊙ Midpoint for rarefactions $\Delta u_{\nu+\frac{1}{2}} > 0$, trapezoidal rule for shocks

7 Second order centered entropy stable scheme:

$$g_{\nu+\frac{1}{2}} = f_{\nu+\frac{1}{2}} = \left[\begin{array}{ll} f\Big(\dfrac{u_\nu + u_{\nu+1}}{2}\Big), & \Delta u_{\nu+\frac{1}{2}} > 0 \\ \\ \dfrac{f(u_\nu) + f(u_{\nu+1})}{2}, & \Delta u_{\nu-\frac{1}{2}} \leq 0. \end{array} \right.$$

- The $\frac{1}{3}$-rule: $\frac{\partial}{\partial t}u + \frac{1}{3}\frac{\partial}{\partial x}[u^2] + \frac{1}{3}u\frac{\partial}{\partial x}[u] = 0 \quad Q^*_{\nu+\frac{1}{2}} = \frac{1}{6}\Delta u_{\nu+\frac{1}{2}}$

8 Entropy stable "$\frac{1}{3}$-rule"

$$\begin{aligned} \frac{d}{dt}u_\nu(t) = &-\frac{1}{2\Delta x_\nu}\Big[\frac{1}{2}u^2_{\nu+1} - \frac{1}{2}u^2_{\nu-1}\Big] \\ &+ \frac{1}{2\Delta x_\nu}\Big[\frac{1}{6}\Big(\Delta u_{\nu+\frac{1}{2}}\Big)^+\Delta u_{\nu+\frac{1}{2}} - \frac{1}{6}\Big(\Delta u_{\nu-\frac{1}{2}}\Big)^+\Delta u_{\nu-\frac{1}{2}}\Big]. \end{aligned}$$

Systems of conservation laws

$$\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}} \rangle = \Delta \psi_{\nu+\frac{1}{2}}$$

- On the choice of path of integration:

- [Tadmor 1987]: $\mathbf{v}_{\nu+\frac{1}{2}}(\xi) = \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi \Delta \mathbf{v}_{\nu+\frac{1}{2}}$

$$\boxed{\mathbf{g}^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big) d\xi}$$

$\odot$ $$\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}} \rangle = \int_{\mathbf{v}_\nu}^{\mathbf{v}_{\nu+1}} \langle d\mathbf{v}, \mathbf{g}(\mathbf{v}) \rangle = \Delta \psi_{\nu+\frac{1}{2}}$$

- Viscosity form: $B(\mathbf{v}) = \mathbf{g}_{\mathbf{v}}(\mathbf{v})$

$$\begin{aligned}
\mathbf{g}^*_{\nu+\frac{1}{2}} &= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} (\xi)' \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big) d\xi \\
&= \xi \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big|_{\xi=-\frac{1}{2}}^{\frac{1}{2}} - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi \mathbf{g}_{\mathbf{v}}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big) (\mathbf{v}_{\nu+\frac{1}{2}}(\xi))' d\xi \\
&= \frac{1}{2}\Big[\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1})\Big] - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big) d\xi \Delta \mathbf{v}_{\nu+\frac{1}{2}}
\end{aligned}$$

Systems of conservation laws I

- Viscosity form:

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\Big[\frac{\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})}{2\Delta x_\nu}\Big] + \frac{1}{2\Delta x_\nu}\Big[Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} - Q^*_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big]$$

- Viscosity coefficient: $Q^*_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi$

- A comparison principle:

Entropy stability if (– for 3-pt schemes only if) $D_{\nu+\frac{1}{2}} := Q_{\nu+\frac{1}{2}} - Q^*_{\nu+\frac{1}{2}} \geq 0$

$$\frac{d}{dt}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big] =$$

$$= -\frac{1}{4\Delta x_\nu}\Big[\Big\langle\Delta\mathbf{v}_{\nu-\frac{1}{2}}, D_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big\rangle + \frac{1}{4}\Big\langle\Delta\mathbf{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big] \leq 0$$

Examples of entropy stable schemes for systems I

9 Rusanov 1961, Lax 1954 scalar viscosity coefficient

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\Big[\frac{\mathbf{f}(\mathbf{u}_{\nu+1}) - f(\mathbf{u}_{\nu-1})}{2\Delta x_\nu}\Big] + \frac{1}{2\Delta x_\nu}\Big[p_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} - p_{\nu-\frac{1}{2}}\Delta\mathbf{u}_{\nu-\frac{1}{2}}\Big]$$

but $\Delta\mathbf{u}_{\nu+\frac{1}{2}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d}{d\xi}\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} H\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}$

$$p_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} = Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} \quad \Longrightarrow \quad Q_{\nu+\frac{1}{2}} = p_{\nu+\frac{1}{2}}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} H(\xi)d\xi$$

• Entropy stability: $2\xi B(\xi) \le p_{\nu+\frac{1}{2}}H(\xi),\ B = AH$; by congruence

$$p_{\nu+\frac{1}{2}} \ge \max_{\lambda,|\xi|\le\frac{1}{2}}\Big|2\xi\lambda\Big[H^{-\frac{1}{2}}(\xi)A(\xi)H^{\frac{1}{2}}(\xi)\Big]\Big| = \max_{\lambda,|\xi|\le\frac{1}{2}}\Big|\lambda\Big[A\Big(\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big)\Big]\Big|$$

⊙ Lax-Friedrichs 1954: $p_{\nu+\frac{1}{2}} = \max_{\lambda,\mathbf{u}}\Big|\lambda\Big[A(\mathbf{u})\Big]\Big|$

Examples of entropy stable 2nd-order schemes

$$\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle$$
$$= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi\left\langle H^{\frac{1}{2}}(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}}, H^{-\frac{1}{2}}(\xi)A(\xi)H^{\frac{1}{2}}(\xi)\cdot H^{\frac{1}{2}}(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle d\xi.$$

- Eigen-decomposition: $\{a_k(\xi), \mathbf{r}_k(\xi)\}_{k=1}^N$ the eigenpairs of $A(\xi)$:

$$\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle = \sum_{k=1}^N \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi a_k(\xi)\cdot\left|\left\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle\right|^2 d\xi$$

⊙ Integrate by parts: $\left(a_k(\xi)\cdot\left|\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\rangle\right|^2\right)' \sim (\Delta\mathbf{v})^3\ldots$

10 Second order entropy stability:

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\left[\frac{\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})}{2\Delta x_\nu}\right] + \frac{1}{2\Delta x_\nu}\left[P_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} - P_{\nu-\frac{1}{2}}\Delta\mathbf{u}_{\nu-\frac{1}{2}}\right]$$

is entropy stable if $\min_\lambda \lambda\left[P_{\nu+\frac{1}{2}}\right] \geq \gamma_{\nu+\frac{1}{2}}\left|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\right|$

Note. $P_{\nu+\frac{1}{2}}$ need not be symmetric but H-symmetric

Examples of entropy stability — Roe's scheme

- Roe's decomposition: $\Delta \mathbf{f}_{\nu+\frac{1}{2}} \equiv \overline{A}_{\nu+\frac{1}{2}} \Delta \mathbf{u}_{\nu+\frac{1}{2}}$:

$$P_{\nu+\frac{1}{2}} = p\left(\overline{A}_{\nu+\frac{1}{2}}\right) = R_{\nu+\frac{1}{2}} \begin{bmatrix} p(\overline{a}_1) & & \\ & \ddots & \\ & & p(\overline{a}_N) \end{bmatrix} R^{-1}_{\nu+\frac{1}{2}}$$

- Roe's scheme: $p(\overline{a}_k) = |\overline{a}_k|$ unstable

11 First-order entropy fix (Osher 1985 Harten 1983)

$$p(\overline{a}_k) = \max\left\{ |\overline{a}_k|, \gamma_{\nu+\frac{1}{2}} \left| \Delta \mathbf{u}_{\nu+\frac{1}{2}} \right| \right\}$$

⊙ Sharp resolution of steady shocks is lost

12 Roe's scheme w/2nd-order entropy fix (Harten-Hyman 1983)

$$p(\overline{a}_k) = |\overline{a}_k| + \left[\frac{1}{6} \Delta a_k(\mathbf{u}_\nu) + \text{Const} \left| \Delta \mathbf{u}_{\nu+\frac{1}{2}} \right|^2 \right]^+$$

- Add $\mathcal{O}(|\Delta \mathbf{u}|)$-entropy dissipation when $\Delta a_k(\mathbf{u})$ is dominated by k-rarefaction

Examples of entropy stable schemes - systems

- computational scheme:

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\Big[\frac{\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})}{2\Delta x_\nu}\Big] + \frac{1}{2\Delta x_\nu}\Big[P_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} - P_{\nu-\frac{1}{2}}\Delta\mathbf{u}_{\nu-\frac{1}{2}}\Big],$$

13 Entropy correction (Khalfallah and Lerat 1988)

Set $e_{\nu+\frac{1}{2}} := \Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, E_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}}\Big\rangle, \quad E_{\nu+\frac{1}{2}} := P^*_{\nu+\frac{1}{2}} - P_{\nu+\frac{1}{2}}$

correct : $P_{\nu+\frac{1}{2}} \longrightarrow P_{\nu+\frac{1}{2}} + p^c_{\nu+\frac{1}{2}} I_{N\times N}, \quad p^c := \dfrac{(e_{\nu+\frac{1}{2}})^+}{\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{u}_{\nu+\frac{1}{2}}\Big\rangle}$

Entropy conservative schemes for systems II

- Choice of path: N linearly independent directions $\left\{\mathbf{r}^j_{\nu+\frac{1}{2}}\right\}_{j=1}^N$

- Intermediate states $\left\{\mathbf{v}^j_{\nu+\frac{1}{2}}\right\}_{j=1}^N$:

Starting with $\mathbf{v}^1_{\nu+\frac{1}{2}} = \mathbf{v}_\nu$, and followed by

$$\mathbf{v}^{j+1}_{\nu+\frac{1}{2}} = \mathbf{v}^j_{\nu+\frac{1}{2}} + \left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \mathbf{r}^j_{\nu+\frac{1}{2}}, \ \ j = 1, 2, \ldots, N \quad (\mathbf{v}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{v}_{\nu+1})$$

THEOREM [Ta, Acta Numerica 2003].
The conservative scheme $\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\left[\mathbf{g}^*_{\nu+\frac{1}{2}} - \mathbf{g}^*_{\nu-\frac{1}{2}}\right]$
is entropy conservative with a numerical flux

$$\mathbf{g}^*_{\nu+\frac{1}{2}} = \sum_{j=1}^N \frac{\psi\left(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\right) - \psi\left(\mathbf{v}^j_{\nu+\frac{1}{2}}\right)}{\left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle} \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}$$

⊙ Case I: the case of $\{\mathbf{r}^j_{\nu+\frac{1}{2}}\}$ collapse to $\Delta\mathbf{v}$ ...

Entropy stability for systems II

- The viscosity form: intermediate states $\left\{\mathbf{u}^j_{\nu+\frac{1}{2}} := \mathbf{u}\left(\mathbf{v}^j_{\nu+\frac{1}{2}}\right)\right\}_{j=1}^{N+1}$

$$\mathbf{r}^j_{\nu+\frac{1}{2}} \sim \mathbf{v}^{j+1}_{\nu+\frac{1}{2}} - \mathbf{v}^j_{\nu+\frac{1}{2}}, \ \left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \mathbf{r}^k_{\nu+\frac{1}{2}} \right\rangle = \delta_{jk}, \ Q^{j+\frac{1}{2},*}_{\nu+\frac{1}{2}} := \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\xi B\left(\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\right) d\xi$$

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\Bigg[& \sum_{j=1}^{N} \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu+\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\right), \mathbf{r}^j_{\nu+\frac{1}{2}} \right\rangle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}} \\
& - \sum_{j=1}^{N} \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu-\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{u}^{j+1}_{\nu-\frac{1}{2}}\right), \mathbf{r}^j_{\nu-\frac{1}{2}} \right\rangle \boldsymbol{\ell}^j_{\nu-\frac{1}{2}} \Bigg] \\
+ \frac{1}{2\Delta x_\nu}\Bigg[& \sum_{j=1}^{N} \left\langle \mathbf{r}^j_{\nu+\frac{1}{2}}, Q^{j+\frac{1}{2},*}_{\nu+\frac{1}{2}} \mathbf{r}^j_{\nu+\frac{1}{2}} \right\rangle \left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}} \\
& - \sum_{j=1}^{N} \left\langle \mathbf{r}^j_{\nu-\frac{1}{2}}, Q^{j+\frac{1}{2},*}_{\nu-\frac{1}{2}} \mathbf{r}^j_{\nu-\frac{1}{2}} \right\rangle \left\langle \boldsymbol{\ell}^j_{\nu-\frac{1}{2}}, \Delta \mathbf{v}_{\nu-\frac{1}{2}} \right\rangle \boldsymbol{\ell}^j_{\nu-\frac{1}{2}} \Bigg],
\end{aligned}$$

- Entropy stability (per wave):

$$\boxed{q^{j+\frac{1}{2}}_{\nu+\frac{1}{2}} \geq \left\langle \mathbf{r}^j_{\nu+\frac{1}{2}}, Q^{j+\frac{1}{2},*}_{\nu+\frac{1}{2}} \mathbf{r}^j_{\nu+\frac{1}{2}} \right\rangle}$$

Entropy stable schemes for systems II

- Choice of path: $\left\{\mathbf{u}^j_{\nu+\frac12}\right\}_{j=1}^N$ along Riemann path:

$$\left\langle \mathbf{r}^j_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^j_{\nu+\frac12}\right\rangle \approx \int_{\xi=-\frac12}^{\frac12}\left(\frac14-\xi^2\right)\left\langle \nabla_{\mathbf{u}} a_j\left(\mathbf{u}^{j+\frac12}_{\nu+\frac12}(\xi)\right), \mathbf{r}^j_{\nu+\frac12}\right\rangle d\xi$$

$$\int_{\xi=-\frac12}^{\frac12}\left(\frac14-\xi^2\right)\left\langle \nabla_{\mathbf{u}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac12}\right\rangle^+ d\xi \le \frac14 \frac{\left[a_j\left(\mathbf{u}^{j+1}_{\nu+\frac12}\right)-a_j\left(\mathbf{u}^j_{\nu+\frac12}\right)\right]^+}{\left\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta v_{\nu+\frac12}\right\rangle}$$

14 Entropy stable Lax-Wendroff scheme 1960

$$\begin{aligned}\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\Bigg[\sum_{j=1}^N &\left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu+\frac12}\right)+\mathbf{f}\left(\mathbf{u}^{j+1}_{\nu+\frac12}\right), \mathbf{r}^j_{\nu+\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu+\frac12} \\ &-\left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu-\frac12}\right)+\mathbf{f}\left(\mathbf{u}^{j+1}_{\nu-\frac12}\right), \mathbf{r}^j_{\nu-\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu-\frac12}\Bigg] \\ +\frac{1}{8\Delta x_\nu}\Bigg[&\sum_{j=1}^N \left[a_j\left(\mathbf{u}^{j+1}_{\nu+\frac12}\right)-a_j\left(\mathbf{u}^j_{\nu+\frac12}\right)\right]^+ \boldsymbol{\ell}^j_{\nu+\frac12} \\ &-\sum_{j=1}^N \left[a_j\left(\mathbf{u}^{j+1}_{\nu-\frac12}\right)-a_j\left(\mathbf{u}^j_{\nu-\frac12}\right)\right]^+ \boldsymbol{\ell}^j_{\nu-\frac12}\Bigg]\end{aligned}$$

Entropy stability for fully discrete schemes

15 The backward Euler scheme - unconditional stability

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x_\nu}\Big[\mathbf{g}_{\nu+\frac{1}{2}}(\mathbf{v}^{n+1}) - \mathbf{g}_{\nu-\frac{1}{2}}(\mathbf{v}^{n+1})\Big], \qquad \mathbf{v}^{n+1} = \mathbf{v}(\mathbf{u}(t^{n+1}))$$

16 Crank-Nicolson: $\overline{\mathbf{v}}^{n+\frac{1}{2}} := \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{v}\Big(\frac{1}{2}\big(\mathbf{u}^n + \mathbf{u}^{n+1}\big) + \xi \Delta \mathbf{u}^{n+\frac{1}{2}}\Big) d\xi \not\approx \mathbf{u}^{n+\frac{1}{2}}$ (!)

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x_\nu}\Big[\mathbf{g}_{\nu+\frac{1}{2}}\big(\overline{\mathbf{v}}^{n+\frac{1}{2}}\big) - \mathbf{g}_{\nu-\frac{1}{2}}\big(\overline{\mathbf{v}}^{n+\frac{1}{2}}\big)\Big]$$

is an entropy stable ($-$ conservative) scheme iff the semi-discrete is

Forward Euler $-$ conditional stability, provided

$$\gamma^3 \frac{\Delta t}{\Delta x_\nu}\Big(B_{\nu+\frac{1}{2}} + Q_{\nu+\frac{1}{2}}\Big)^2 \leq D_{\nu+\frac{1}{2}}, \qquad D := Q - Q^* \geq 0$$

17 Modify LxF: $\mathbf{u}_\nu^{n+1} = \frac{1}{4}\big(\mathbf{u}_{\nu+1}^n + 2\mathbf{u}_\nu^n + \mathbf{u}_{\nu-1}^n\big) + \frac{\Delta t}{2\Delta x}\big[\mathbf{f}\big(\mathbf{u}_{\nu+1}^n\big) - \mathbf{f}\big(\mathbf{u}_{\nu-1}^n\big)\big]$

$$D_{\nu+\frac{1}{2}}^{LxF} = \frac{\Delta x}{2\Delta t} I_{N\times N} \Longrightarrow \text{ CFL : } \frac{\Delta t}{\Delta x_\nu} \max_\lambda |\lambda(A + Q^*)| \leq \frac{\sqrt{2}-1}{2}$$

Fully discrete schemes – the homotopy approach

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - \frac{\Delta t}{2\Delta x}\Big[\mathbf{f}\big(\mathbf{u}_{\nu+1}^n\big) - \mathbf{f}\big(\mathbf{u}_{\nu-1}^n\big)\Big] + \frac{\Delta t}{2\Delta x}\Big[P_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} - P_{\nu-\frac{1}{2}}\Delta\mathbf{u}_{\nu-\frac{1}{2}}\Big]$$

- Decomposition: $\mathbf{u}_\nu^{n+1} = \Big(\mathbf{u}_{\nu+\frac{1}{2}}^{n+1} + \mathbf{u}_{\nu-\frac{1}{2}}^{n+1}\Big)/2$

$$\mathbf{u}_{\nu+\frac{1}{2}}^{n+1} := \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x_\nu}\Big[\mathbf{f}\big(\mathbf{u}_{\nu+1}^n\big) - \mathbf{f}\big(\mathbf{u}_\nu^n\big)\Big] + \frac{\Delta t}{\Delta x_\nu}P_{\nu+\frac{1}{2}}\big(\mathbf{u}_{\nu+1}^n - \mathbf{u}_\nu^n\big),$$
$$\mathbf{u}_{\nu-\frac{1}{2}}^{n+1} := \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x_\nu}\Big[\mathbf{f}\big(\mathbf{u}_\nu^n\big) - \mathbf{f}\big(\mathbf{u}_{\nu-1}^n\big)\Big] - \frac{\Delta t}{\Delta x_\nu}P_{\nu+\frac{1}{2}}\big(\mathbf{u}_\nu^n - \mathbf{u}_{\nu-1}^n\big),$$

- Quasi-cell entropy inequality: $\mathbf{u}_{\nu+\frac{1}{2}}^n(s) := \mathbf{u}_\nu^n + s(\mathbf{u}_{\nu+1}^n - \mathbf{u}_\nu^n)$

$$\mathcal{I}_+ := U\Big(\mathbf{u}_{\nu+\frac{1}{2}}^{n+1}\Big) - U\big(\mathbf{u}_\nu^n\big) + \frac{\Delta t}{\Delta x_\nu}\Big[F\big(\mathbf{u}_{\nu+1}^n\big) - F\big(\mathbf{u}_\nu^n\big)\Big]$$
$$- \frac{\Delta t}{\Delta x_\nu}\int_{s=0}^1 \Big\langle U'\Big(\mathbf{u}_{\nu+\frac{1}{2}}^n(s)\Big), P_{\nu+\frac{1}{2}}\Delta\mathbf{u}\Big\rangle ds \leq 0$$

$$\mathbf{u}_{\nu+\frac{1}{2}}^{n+1}(s) := \mathbf{u}_\nu^n - \tfrac{\Delta t}{\Delta x_\nu}\Big[\mathbf{f}\Big(\mathbf{u}_{\nu+\frac{1}{2}}^n(s)\Big) - \mathbf{f}(\mathbf{u}_\nu^n)\Big] + \tfrac{\Delta t}{\Delta x_\nu}P_{\nu+\frac{1}{2}}\Big(\mathbf{u}_{\nu+\frac{1}{2}}^n(s) - \mathbf{u}_\nu^n\Big) \quad \begin{vmatrix} \text{at } s=1:\ \mathbf{u}_{\nu+\frac{1}{2}}^{n+1} \\ \text{at } s=0:\ \mathbf{u}_\nu^n \end{vmatrix}$$

$\odot$ $U\Big(\mathbf{u}_{\nu+\frac{1}{2}}^{n+1}\Big) - U(\mathbf{u}_\nu^n) = \int_{s=0}^1 \frac{d}{ds}U\Big(\mathbf{u}_{\nu+\frac{1}{2}}^{n+1}(s)\Big)ds \ldots$

The homotopy approach - cont'd

$$\mathcal{I}_+ = \int_{s=0}^{1} \left\langle U'\left(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)\right) - U'\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\right), -\frac{\Delta t}{\Delta x_\nu}\left(P_{\nu+\frac{1}{2}} - A\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\right)\right)\mathbf{\Delta u}\right\rangle$$

- Set $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(r,s) = \mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s) - \frac{\Delta t}{\Delta x_\nu}\left(\mathbf{f}\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\right) - \mathbf{f}\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\right)\right)$

$$\mathcal{I}_+ = -\int_{r,s=0}^{1} s\left\langle \left(I + \frac{\Delta t}{\Delta x_\nu}A\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\right) - \frac{\Delta t}{\Delta x_\nu}P_{\nu+\frac{1}{2}}\right)\mathbf{\Delta u}, \right.$$
$$\left. U''\left(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(r,s)\right)\left(-\frac{\Delta t}{\Delta x_\nu}A\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\right) + \frac{\Delta t}{\Delta x_\nu}P_{\nu+\frac{1}{2}}\right)\mathbf{\Delta u}\right\rangle drds$$

COROLLARY (symmetric systems): Entropy stability under CFL

$$\frac{\Delta t}{\Delta x_\nu}\left|A\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\right)\right| \le \frac{\Delta t}{\Delta x_\nu}P_{\nu+\frac{1}{2}} \le I - \frac{\Delta t}{\Delta x_\nu}\left|A\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\right)\right|$$

18 Modified LxF: $P_{\nu+\frac{1}{2}} = \frac{\Delta x}{2\Delta t}I_{N\times N}$

19 Upwind scheme: $P_{\nu+\frac{1}{2}} = p\left(A\left(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\right)\right),\ p(\cdot) \ge |\cdot|$

entropy stable under CFL condition $\frac{\Delta t}{\Delta x_\nu}\sup_{s,\lambda}\left|\lambda\left(A\left(\mathbf{u}_{\nu+\frac{1}{2}}(s)\right)\right)\right| \le \frac{1}{2}$ (!)

- On the $\frac{1}{2}$ factor, no Riemann solver, the scalar work, 2nd-order NT scheme, ...

Beyond second-order accuracy

- The weak formulation $\int_\Omega \langle \mathbf{w}(x,t), \frac{\partial}{\partial t}\mathbf{u}(\mathbf{v})\rangle = \int_\Omega \langle \frac{\partial}{\partial x}\mathbf{w}(x,t), \mathbf{g}(\mathbf{v})\rangle dxdt$
- The finite-element discretization [Ta 1986]: $\hat{\mathbf{v}} = \sum_j \mathbf{v}_j(t)\widehat{H}_j(x)$

$$\text{Time}: \int_{x_{\nu-1}}^{x_{\nu+1}} \widehat{H}_\nu \frac{\partial}{\partial t}\mathbf{u}\left(\sum_j \mathbf{v}_j(t)\widehat{H}_j(x)\right) dxdt = \Delta x_\nu \frac{d}{dt}\mathbf{u}(\mathbf{v}_\nu(t)) + \mathcal{O}\left(\left|\mathbf{v}_{\nu+\frac{1}{2}}\right|\right)$$

$$\text{Space}: \int_{x_{\nu-1}}^{x_{\nu+1}} \frac{\partial}{\partial x}\widehat{H}_\nu(x)\mathbf{g}\left(\sum_j \mathbf{v}_j(t)\widehat{H}_j(x)\right) dxdt =$$

$$= -\left[\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\left(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\right)d\xi - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\left(\mathbf{v}_{\nu-\frac{1}{2}}(\xi)\right)d\xi\right]$$

... $\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{\mathbf{g}^*_{\nu+\frac{1}{2}} - \mathbf{g}^*_{\nu-\frac{1}{2}}}{\Delta x_\nu}$ entropy conservative: $\hat{\mathbf{w}}(x,t) = \hat{\mathbf{v}}(x,t)$

20 LeFloch Rohde 2000 Third-order entropy conservative

$$\mathbf{g}^*_{\nu+\frac{1}{2}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\left(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\right)d\xi - \frac{1}{12}\left[Q^{**}_{\nu+\frac{3}{2}}\Delta\mathbf{v}_{\nu+\frac{3}{2}} - Q^{**}_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\right]$$

with secondary numerical viscosity coefficient $Q^{**}_{\nu+\frac{1}{2}} = Q^{**}(\mathbf{v}_{\nu-1}, \mathbf{v}_\nu, \mathbf{v}_{\nu+1})$

THANK YOU