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Consistent Material Operators for Geometric Discretization Methods on Generalized Grids

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Abstract

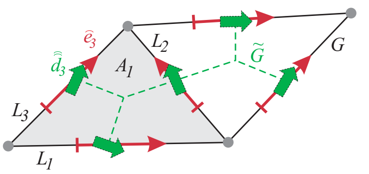
Geometric methods for the spatial discretization of Maxwell's Equations
- preserve important conservation properties
- Examples: FD, Finite Elements (FE), FIT, Cell method, ...

Finite Integration Technique (FIT)
- a natural and efficient notation of such approaches
- separate matrix operators for:
  - topological relations (*exact*)
  - material relations (*approximations*)

Realization of material operators: a key point of each method
- Consistency property: Accuracy, rate of convergence
- Symmetry: Stability (Time domain), discrete energy
- strongly depends on the type of computational grid

In this paper:
- Accuracy and efficiency of some recently developed material operators for tetrahedral grids
- Consistency requirement leads to important consequences for the overall simulation scheme.

Finite Integration Technique (FIT)



State variables of FIT:

integrated fields on edges $L_i, \widetilde{L}_i$, and facets $A_i, \widetilde{A}_i$ of grid G, dual grid $\widetilde{G}$

electric grid voltage $\widehat{e}_i = \int_{L_i} \vec{E}\cdot d\vec{s}$ electric grid flux $\widehat{\widehat{d}}_i = \int_{\widetilde{A}_i} \vec{D}\cdot d\vec{A}$

magnetic grid voltage $\widehat{h}_i = \int_{\widetilde{L}_i} \vec{H}\cdot d\vec{s}$ magnetic grid flux $\widehat{\widehat{b}}_i = \int_{A_i} \vec{B}\cdot d\vec{A}$

$$\widehat{e}_1 + \widehat{e}_2 - \widehat{e}_3 = -\frac{d}{dt}\widehat{\widehat{b}}_1 \quad \Rightarrow \quad \forall \text{ facets}$$

$$\mathbf{C}\widehat{\mathbf{e}} = -\frac{d}{dt}\widehat{\widehat{\mathbf{b}}}$$ **Faraday's Law (exact)**

matrix equations

$$\widetilde{\mathbf{C}}\widehat{\mathbf{h}} = \frac{d}{dt}\widehat{\widehat{\mathbf{d}}} + \widehat{\widehat{\mathbf{j}}}$$ **Ampere's Law (exact)**

Relations between grids: Discrete material operators:

$\widehat{\widehat{\mathbf{d}}} = \mathbf{M}_\varepsilon \widehat{\mathbf{e}}$ or $\widehat{\mathbf{e}} = \mathbf{M}_{\varepsilon^{-1}} \widehat{\widehat{\mathbf{d}}}$

$\widehat{\widehat{\mathbf{b}}} = \mathbf{M}_\mu \widehat{\mathbf{h}}$ or $\widehat{\mathbf{h}} = \mathbf{M}_{\mu^{-1}} \widehat{\widehat{\mathbf{b}}}$

square matrices (linear case) approximations of the method

High frequency time-domain formulation:

$$\mathbf{M}_{\varepsilon^{-1}}\widetilde{\mathbf{C}}\mathbf{M}_{\mu^{-1}}\mathbf{C}\,\widehat{\mathbf{e}} = -\frac{d^2}{dt^2}\widehat{\mathbf{e}} \quad (+excitation)$$

Stability :	$\mathbf{M}_{\varepsilon^{-1}}$ $\mathbf{M}_{\mu^{-1}}$	*s.p.d.*	
Accuracy :	$\mathbf{M}_{\varepsilon^{-1}}$ $\mathbf{M}_{\mu^{-1}}$	*consistent*	
Efficiency :	$\mathbf{M}_{\varepsilon^{-1}}$ $\mathbf{M}_{\mu^{-1}}$	*sparse*	$\mathbf{M}_{\varepsilon^{-1}} = (\mathbf{M}_\varepsilon)^{-1}$?

References:

[Ma03] M.Marrone: *A New Consistent Way to Build Symmetric Constitutive Matrices on General 2D Grids.* Compumag 2003 / IEEE Trans.Mag.40,2, 2004, pp. 1420-1423.

[Ci04] M.Cinalli, F. Edelvik, R.Schuhmann, T.Weiland: *Consistent Material Operators for Tetrahedral Grids Based on Geometrical Principles.* Int. Journal of Numerical Modelling, 2004, in print.

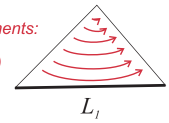
for more references see [Ci04].

Other geometric approaches:

FDTD: on Cartesian grids *computationally equivalent*

FE (edge elements) $\vec{E}(\vec{r}) = \sum E_i \vec{w}_i^{(1)}(\vec{r})$ $\vec{B}(\vec{r}) = \sum B_i \vec{w}_i^{(2)}(\vec{r})$

edge elements: $\vec{w}_i^{(1)}(\vec{r})$



- tangential continuity
- associated to primary edges:

$$\int_{L_i} \vec{w}_j^{(1)}(\vec{r})\cdot d\vec{s} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

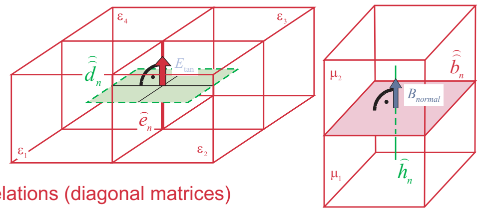
$$\Rightarrow \quad \widehat{e}_i = \int_{L_i} \vec{E}(\vec{r})\cdot d\vec{s} = E_i \qquad \widehat{\widehat{b}}_i = \int_{A_i} \vec{B}(\vec{r})\cdot d\vec{A} = B_i$$

Others (Cell Method, ...)

Material Matrix: Implementations

Cartesian:

(and other dual-orthogonal grids)



FIT / FDTD

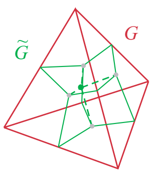
$\Rightarrow$ 1:1 relations (diagonal matrices)

Simplicial with barycentric dual:

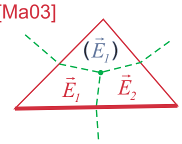
FE approach
- variational (not geometrical) ansatz:

$$M_{\varepsilon,ij} = \int_\Omega \varepsilon\, \vec{w}_i^{(1)}(\vec{r})\cdot\vec{w}_j^{(1)}(\vec{r})\, dV$$

- correlated to energy
- always s.p.d., but no sparse $\mathbf{M}_{\varepsilon^{-1}} = (\mathbf{M}_\varepsilon)^{-1}$



Cell Method: *assume constant fields in "Microcells"* [Ma03]
- geometric derivation
- always consistent (= exact for constant fields)
- non-symmetric (instable in time domain)



Symmetrized Cell Method: [Ci04]
- additional averaging symmetrizes matrix (stability in time domain)
- consistent, but reduced accuracy [Ci04]
- no sparse solution for $\mathbf{M}_{\varepsilon^{-1}}$

Consistency criterion used:

Operators must be *exact* for constant fields

- ok for FE and Cell method approach
- consequence for simplicial grids: (proof: [Ci04])

$\forall$ *consistent* $\mathbf{M}_\varepsilon^{(1)}, \mathbf{M}_\varepsilon^{(2)}$: $\mathbf{M}_\varepsilon^{(1)}\mathbf{G} = \mathbf{M}_\varepsilon^{(2)}\mathbf{G}$ (ε grad)

$\forall$ *consistent* $\mathbf{M}_{\mu^{-1}}^{(1)}, \mathbf{M}_{\mu^{-1}}^{(2)}$: $\mathbf{M}_{\mu^{-1}}^{(1)}\mathbf{C} = \mathbf{M}_{\mu^{-1}}^{(2)}\mathbf{C}$ (μ^{-1} curl)

$\Rightarrow$ All consistent time domain schemes differ only in $\mathbf{M}_{\varepsilon^{-1}}$