

Discontinuous Galerkin Methods for Maxwell's Equations in Frequency-Domain

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1 Maxwell's Equations

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, find the electric field \mathbf{u} such that

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \omega^2 (\varepsilon - i\omega^{-1}\sigma) \mathbf{u} &= \mathbf{j} & \text{in } \Omega, & (1) \\ \mathbf{n} \times \mathbf{u} &= \mathbf{0} & \text{on } \Gamma, & (2) \end{aligned}$$

where $\omega > 0$, μ , ε and σ are the temporal frequency, magnetic permeability, electric permittivity, and electric conductivity, respectively.

Assumption: μ and ε constant (for simplicity).

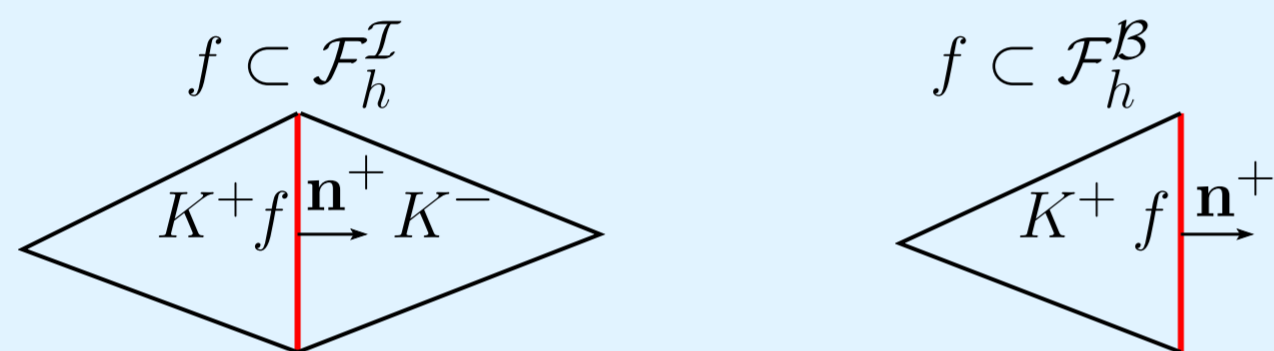
2 DG Discretizations

Main Advantages of DG Methods:

- DG methods are locally conservative;
- Greater robustness means that a wide range of problems may be treated within the same unified framework;
- Finite element meshes with hanging nodes and local spaces of different orders can easily be handled.

2.1 Notation

- $\mathcal{T}_h = \{K\}$ tetrahedral mesh of granularity h ;
- $\mathcal{F}_h^I/\mathcal{F}_h^B$ is the set of all interior/boundary faces;
- Trace operators:



$$\begin{aligned} [\mathbf{v}]_T &= \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-, & [\mathbf{v}]_T &= \mathbf{n} \times \mathbf{v}, \\ [\mathbf{v}]_N &= \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-, & [\mathbf{v}]_N &= \mathbf{v} \cdot \mathbf{n}, \\ [q]_N &= q^+ \mathbf{n}^+ + q^- \mathbf{n}^-, & [q]_N &= q \mathbf{n}, \\ \{\{\mathbf{v}\}\} &= (\mathbf{v}^+ + \mathbf{v}^-)/2, & \{\{\mathbf{v}\}\} &= \mathbf{v}, \\ \{\{q\}\} &= (q^+ + q^-)/2, & \{\{q\}\} &= q; \end{aligned}$$

- Finite Element Space:

$$S_h^\ell = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}^\ell(K) \quad \forall K \in \mathcal{T}_h\}, \quad \ell \geq 1.$$

- DG norms:

$$\begin{aligned} \|\mathbf{v}\|_{V(h)}^2 &= \|\varepsilon^{\frac{1}{2}} \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac{1}{2}} \nabla_h \times \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac{1}{2}} h^{-\frac{1}{2}} [\mathbf{v}]_T\|_{0,\mathcal{F}_h}^2, \\ \|q\|_{Q(h)}^2 &= \|\varepsilon^{\frac{1}{2}} \nabla_h q\|_{0,\Omega}^2 + \|\varepsilon^{\frac{1}{2}} h^{-\frac{1}{2}} [q]_N\|_{0,\mathcal{F}_h}^2, \\ \|\mathbf{v}\|_{DG}^2 &= \|\mu^{-\frac{1}{2}} \nabla_h \times \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac{1}{2}} h^{-\frac{1}{2}} [\mathbf{v}]_T\|_{0,\mathcal{F}_h}^2 \\ &\quad + \|\mu^{-\frac{1}{2}} \nabla_h \cdot \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac{1}{2}} h^{-\frac{1}{2}} [\varepsilon \mathbf{v}]_N\|_{0,\mathcal{F}_h}^2. \end{aligned}$$

2.2 Discrete Curl-Curl Operator

The interior penalty discretization is defined by:

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) &= (\mu^{-1} \nabla_h \times \mathbf{u}, \nabla_h \times \mathbf{v})_\Omega + \int_{\mathcal{F}_h} \mathbf{a} \mu^{-1} [\mathbf{u}]_T \cdot [\mathbf{v}]_T ds \\ &\quad - \int_{\mathcal{F}_h} ([\mathbf{u}]_T \cdot \{\{\mu^{-1} \nabla_h \times \mathbf{v}\}\} + [\mathbf{v}]_T \cdot \{\{\mu^{-1} \nabla_h \times \mathbf{u}\}\}) ds, \end{aligned}$$

where the interior penalty stabilization function \mathbf{a} is given by

$$\mathbf{a} := \alpha h^{-1}, \quad \alpha > \alpha_{\min} \geq 0.$$

3 Low-Frequency Problem

Neglecting $\omega^2 \varepsilon$ in (1) and assuming $\sigma \equiv 0$ gives:

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) &= \mathbf{j} & \text{in } \Omega, \\ \nabla \cdot (\varepsilon \mathbf{u}) &= 0 & \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0} & \text{on } \Gamma. \end{aligned}$$

Key Problem: The design of numerical schemes which incorporate the divergence free constraint.

3.1 Regularized DG Formulation

Find $\mathbf{u}_h \in (S_h^\ell)^3$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) + r_h(\mathbf{u}_h, \mathbf{v}) = (\mathbf{j}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in (S_h^\ell)^3,$$

where the form $r_h(\cdot, \cdot)$ is defined by

$$\begin{aligned} r_h(\mathbf{u}, \mathbf{v}) &= (\mu^{-1} \nabla_h \cdot \mathbf{u}, \nabla_h \cdot \mathbf{v})_\Omega - \int_{\mathcal{F}_h^I} [\varepsilon \mathbf{u}]_N \{\{\mu^{-1} \nabla_h \cdot (\varepsilon \mathbf{v})\}\} ds \\ &\quad - \int_{\mathcal{F}_h^I} [\varepsilon \mathbf{v}]_N \{\{\mu^{-1} \nabla_h \cdot (\varepsilon \mathbf{u})\}\} ds + \int_{\mathcal{F}_h^I} \mathbf{d} \mu^{-1} [\varepsilon \mathbf{u}]_N \cdot [\varepsilon \mathbf{v}]_N ds, \end{aligned}$$

and $\mathbf{d} := \delta h^{-1}$, $\delta > \delta_{\min} \geq 0$.

Theorem 1 ([4]) Assuming $\mathbf{u} \in H^{s+1}(\mathcal{T}_h)^3$, $s > 1/2$, we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{DG} \leq C h^{\min\{s,\ell\}} \|\mathbf{u}\|_{s+1,\mathcal{T}_h}.$$

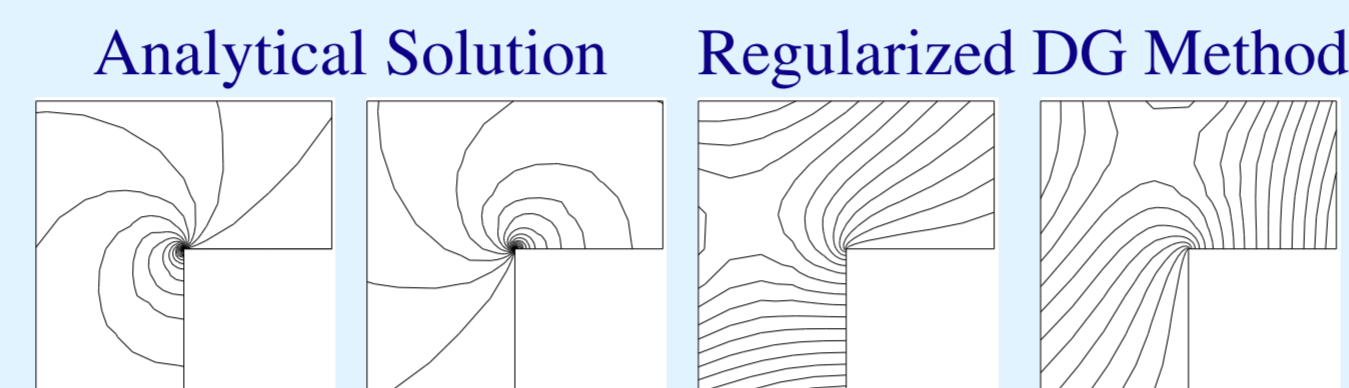
Regularity assumptions are not minimal

\Rightarrow DG method may *not* converge for nonsmooth problems.

3.2 Example

Let $\Omega = (-1, 1)^2 \setminus [0, 1] \times (-1, 0]$ and select appropriate data so that \mathbf{u} is given, in terms of the polar coordinates (r, ϑ) , by

$$\mathbf{u}(x, y) = \nabla(r^{2/3} \sin(2\vartheta/3)) \quad \Rightarrow \quad \mathbf{u} \in H^{2/3-\varepsilon}(\Omega)^2, \quad \varepsilon > 0.$$



3.3 Mixed DG Formulation

Introducing the Lagrange multiplier $p \in H_0^1(\Omega)$ gives:

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \varepsilon \nabla p &= \mathbf{j} & \text{in } \Omega, \\ \nabla \cdot (\varepsilon \mathbf{u}) &= 0 & \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad p = 0 & \text{on } \Gamma. \end{aligned}$$

Mixed DG methods: Find (\mathbf{u}_h, p_h) in $(S_h^\ell)^3 \times S_h^m$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) + s_h(\mathbf{u}_h, \mathbf{v}) + b_h(\bar{\mathbf{v}}, p_h) &= (\mathbf{j}, \mathbf{v})_\Omega, \\ b_h(\mathbf{u}_h, \bar{q}) - c_h(p_h, q) &= 0 \end{aligned}$$

for all $(\mathbf{v}, q) \in (S_h^\ell)^3 \times S_h^m$, where

$$\begin{aligned} s_h(\mathbf{u}, \mathbf{v}) &= \int_{\mathcal{F}_h^I} \mathbf{b} [\mathbf{u}]_N [\bar{\mathbf{v}}]_N ds, \quad c_h(p, q) = \int_{\mathcal{F}_h} \mathbf{c} \varepsilon [p]_N \cdot [\bar{q}]_N ds, \\ b_h(\mathbf{v}, p) &= -(\varepsilon \mathbf{v}, \nabla_h p) + \int_{\mathcal{F}_h} \{\{\varepsilon \mathbf{v}\}\} \cdot [p]_N ds, \end{aligned}$$

and

$$\mathbf{b} := \beta h, \quad \beta \geq 0, \quad \mathbf{c} := \gamma h^{-1}, \quad \gamma > 0.$$

- **Method I (stabilized):** $\beta > 0$ and $m = \ell$.

- **Method II (non-stabilized):** $\beta = 0$ and $m = \ell + 1$.

Theorem 2 ([6, 5]) Assuming that

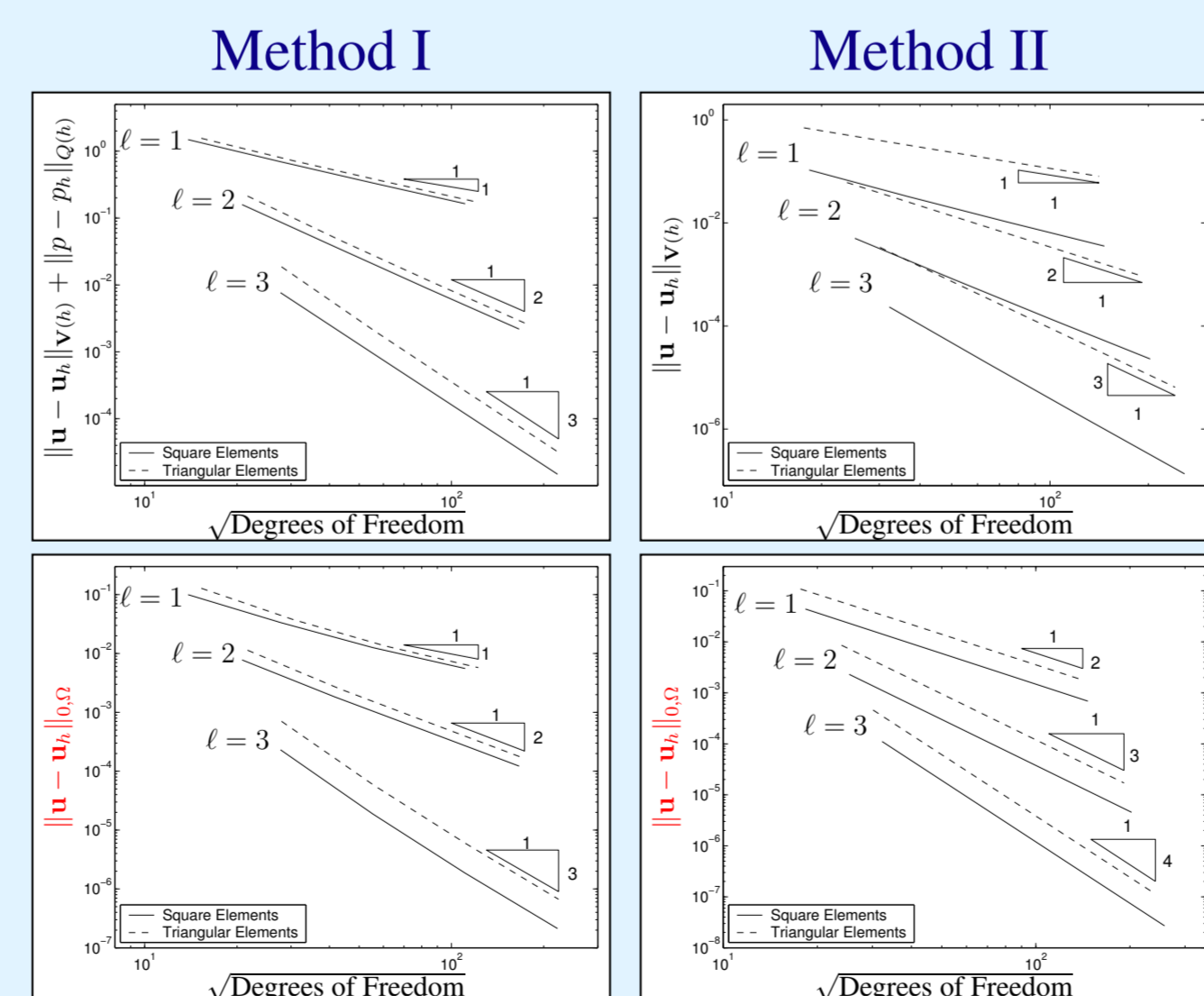
$$\mathbf{u} \in H^s(\mathcal{T}_h)^3, \quad \nabla \times \mathbf{u} \in H^s(\mathcal{T}_h)^3, \quad p \in H^{s+1}(\mathcal{T}_h), \quad s > 1/2,$$

both methods satisfy the following a priori error bound:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{V(h)} + \|p - p_h\|_{Q(h)} \\ \leq C h^{\min\{s,\ell\}} \left[\|\varepsilon \mathbf{u}\|_{s,\mathcal{T}_h} + \|\mu^{-1} \nabla \times \mathbf{u}\|_{s,\mathcal{T}_h} + \|p\|_{s+1,\mathcal{T}_h} \right]. \end{aligned}$$

3.4 Example (Smooth Solution)

We set $\Omega = (-1, 1)^2$ and select appropriate data so that $\mathbf{u} = (-\exp(x)y \cos(y) + \sin(y), \exp(x)y \sin(y))$ and $p = \sin(\pi(x-1)/2) \sin(\pi(y-1)/2)$.



\Rightarrow Method I is *suboptimal* in the $L^2(\Omega)$ norm.

3.5 A Posteriori Error Estimation

Theorem 3 ([3]) Assuming that $\nabla \cdot \mathbf{j} = 0$ holds, so that $p \equiv 0$, Method II satisfies the following a posteriori error bound:

$$\|\mathbf{u} - \mathbf{u}_h\|_{V(h)} + \|p - p_h\|_{Q(h)} \leq C \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2},$$

where the elemental error indicator η_K is given by

$$\begin{aligned} \eta_K^2 &= h_K^{2\sigma} \|\mathbf{j} - \nabla \times (\mu^{-1} \nabla \times \mathbf{u}_h) + \varepsilon \nabla p_h\|_{0,K}^2 \\ &\quad + h_K^{2\sigma-1} \|\hat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h)\|_{0,\partial K}^2 + h_K^{-1} \|\mu^{\frac{1}{2}} [\mathbf{u}_h]_T\|_{0,\partial K}^2 \\ &\quad + h_K \|\varepsilon [\mathbf{u}_h]_N\|_{0,\partial K \setminus \Gamma}^2 + h_K^2 \|\nabla \cdot (\varepsilon \mathbf{u}_h)\|_{0,K}^2 \\ &\quad + \|\varepsilon^{\frac{1}{2}} \nabla p_h\|_{0,K}^2 + h_K^{-1} \|\varepsilon^{\frac{1}{2}} [p_h]_N\|_{0,\partial K}^2, \end{aligned}$$

$\sigma \in (1/2, 1]$ denotes the parameter within the embeddings $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \hookrightarrow H^\sigma(\Omega)^3$ and $H(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega) \hookrightarrow H^\sigma(\Omega)^3$, and $\hat{\tau}_K(\mathbf{v})$ is the numerical flux:

$$\hat{\tau}_K(\mathbf{v}) = \begin{cases} \mathbf{n}_K \times (\{\{\mu^{-1} \nabla \times \mathbf{v}\}\} - \mu^{-1} \mathbf{a} [\mathbf{v}]_T) & \text{on } \partial K \setminus \Gamma, \\ \mathbf{n}_K \times (\mu^{-1} \nabla \times \mathbf{v} - \mu^{-1} \mathbf{a} (\mathbf{n}_K \times \mathbf{v})) & \text{on } \partial K \cap \Gamma. \end{cases}$$

4 High-Frequency Problem

Renaming $(\varepsilon - i\omega^{-1}\sigma)$ by ε , problem (1)–(2) becomes

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \omega^2 \varepsilon \mathbf{u} = \mathbf{j} \quad \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma.$$

Here, we assume that ω^2 is not an eigenvalue of the underlying Maxwell eigenproblem.

4.1 Direct DG Method

Find $\mathbf{u}_h \in (S_h^\ell)^3$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) - \omega^2 (\varepsilon \mathbf{u}_h, \mathbf{v})_\Omega = (\mathbf{j}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in (S_h^\ell)^3.$$

Theorem 4 ([2]) Assuming that $\mathbf{u} \in H^s(\mathcal{T}_h)^3$ and $\nabla \times \mathbf{u} \in H^s(\mathcal{T}_h)^3$, $s > 1/2$, for $0 < h \leq h_0$, we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{V(h)} \leq C h^{\min\{s,\ell\}} \left[\|\varepsilon \mathbf{u}\|_{s,\mathcal{T}_h} + \|\mu^{-1} \nabla \times \mathbf{u}\|_{s,\mathcal{T}_h} \right].$$

Additionally, given that $\mathbf{u} \in H^{\ell+1}(\mathcal{T}_h)^3$ and Ω is convex,

$$\|\varepsilon^{\frac{1}{2}} (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \leq C h^{\ell+1} \|\mathbf{u}\|_{\ell+1,\mathcal{T}_h}, \quad \text{for } 0 < h \leq h_1.$$

4.2 Mixed DG Method

Writing $\mathbf{u} = \mathbf{w} + \nabla \varphi$, with $\varphi \in H_0^1(\Omega)$ and $\mathbf{w} \in H_0(\text{curl}; \Omega)$, $\nabla \cdot \mathbf{w} = 0$, setting $p := \omega^2 \varphi$ and renaming \mathbf{w} by \mathbf{u} gives:

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \omega^2 \varepsilon \mathbf{u} - \varepsilon \nabla p &= \mathbf{j} & \text{in } \Omega, \\ \nabla \cdot (\varepsilon \mathbf{u}) &= 0 & \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad p = 0 & \text{on } \Gamma. \end{aligned}$$

Mixed DG Method: find (\mathbf{u}_h, p_h) in $(S_h^\ell)^3 \times S_h^{\ell+1}$, such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) - \omega^2 (\varepsilon \mathbf{u}_h, \mathbf{v})_\Omega + b_h(\bar{\mathbf{v}}, p_h) &= (\mathbf{j}, \mathbf{v})_\Omega, \\ b_h(\mathbf{u}_h, \bar{q}) - c_h(p_h, q) &= 0 \end{aligned}$$

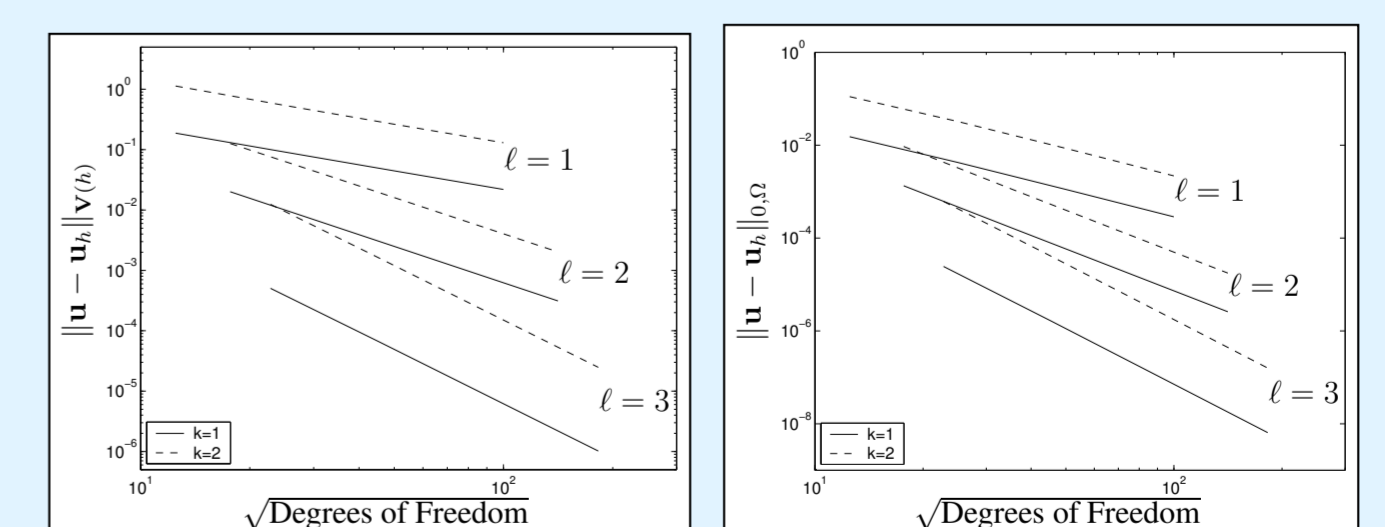
for all $(\mathbf{v}, q) \in (S_h^\ell)^3 \times S_h^{\ell+1}$.

\Rightarrow Optimal energy and $L^2(\Omega)$ -norm error bounds (see [1]).

Main Advantage: Mixed formulation applicable to the full PDE system in *both* low- and high-frequency regimes.

4.3 Example - Mixed DG Method

We set $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $\Omega = (-1, 1)^2$, $\mathbf{j} = \mathbf{0}$, and select appropriate boundary conditions so that $\mathbf{u} = (\sin(ky), \sin(kx))$ and $p = 0$, where $k = \omega \sqrt{\mu_0 \varepsilon_0}$ is the wave number.



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