

Asymptotic-Preserving Discretization Schemes

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Asymptotic Limits

- An asymptotic limit associated with a PDE is a limit in which certain terms in the PDE are made “small” relative to other terms.
- This ordering in size is achieved via a scaling parameter that goes to zero in the asymptotic limit.
- For instance, denoting the scaling parameter by ϵ , each term is assumed to be scaled by ϵ^n for some integer, $n \geq 0$.
- One generally determines the scaling on a physical basis by first non-dimensionalizing the PDE, scaling the non-dimensional parameters that appear in the PDE, and returning the equation to dimensional form.
- A power-series solution in ϵ is assumed for the scaled PDE.
- Substituting this expansion into the scaled PDE and equating coefficients of each power of ϵ leads to a hierarchical set of equations for the expansion coefficients.

Asymptotic Limits

- One generally finds that the leading-order expansion coefficient satisfies a “simpler” PDE than the original PDE.
- The leading-order coefficient is said to be the asymptotic solution of the original PDE, and the simpler PDE is said to be the asymptotic limit of the original PDE.
- In many instances, the scale lengths associated with solutions of the asymptotic equation are much larger than the smallest scale lengths associated with solutions of the original PDE.
- When this is the case, asymptotic-preserving discretization schemes are necessary for near-asymptotic problems to avoid completely *impractical* mesh resolution requirements.

The Particle Transport Equation

- We begin our discussion with the following particle transport equation:

$$\mu \frac{\partial v N}{\partial x} + (\sigma_a + \sigma_s) v N = \frac{\sigma_s}{2} \int_{-1}^{+1} v N(x, \mu') d\mu' + Q. \quad (1)$$

- This is an equation for a phase-space particle density function, $N(x, \mu)$.
- All particles travel at a single speed, v , in directions characterized by the cosine, $\mu = v_x/v$.
- Particles are randomly absorbed and scattered isotropically within the medium.
- The absorption cross section is σ_a , and the scattering cross section is σ_s . The sum of these two cross sections is the total cross section, $\sigma_t = \sigma_a + \sigma_s$.
- $Q(x)$, is the particle source function.

The Particle Transport Equation

- The boundary conditions for Eq.(1) are defined in terms of the incident particle distributions at the boundaries.
- Equation (1) physically represents a statement of particle conservation within a differential phase-space volume.
- The mean-free-path, which is the mean distance between interactions, is given by $\lambda_t = 1/\sigma_t$.
- λ_t is a basic transport scale length associated with strongly absorbing problems.
- For instance, a particle beam entering a purely absorbing medium will be attenuated after traveling a distance, s , by $\exp(-s/\lambda_t)$.

The Asymptotic Diffusion Limit for Transport

- It is convenient for our purposes to re-express Eq.(1) in terms of the angular flux, ψ :

$$\mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = (\sigma_t - \sigma_a) \phi + Q, \quad (2)$$

where

$$\phi = \frac{1}{2} \int_{-1}^{+1} \psi(x, \mu') d\mu'. \quad (3)$$

- ϕ is called the scalar flux.

The Asymptotic Diffusion Limit for Transport

- Non-dimensionalizing Eq.(2), we get

$$P_1 \mu \frac{\partial \hat{\psi}}{\partial \hat{x}} + \hat{\sigma}_t \hat{\psi} = (\hat{\sigma}_t - P_2 \hat{\sigma}_a) \hat{\phi} + P_3 \hat{Q}, \quad (4)$$

where

$$P_1 = \lambda_t / \ell_\infty, \quad P_2 = \sigma_{a,\infty} / \sigma_{t,\infty}, \quad P_3 = Q_\infty / (\sigma_{t,\infty} \phi_\infty). \quad (5)$$

- P_1 is scaled $O(\epsilon)$, which means that a mean-free-path is small compared to the spatial scalelength of the asymptotic solution.
- P_2 is scaled $O(\epsilon^2)$, which means that the probability of being absorbed per interaction is very small.
- P_3 is scaled $O(\epsilon^2)$, which only relates to normalization, i.e., that the asymptotic solution remains finite and non-zero as $\epsilon \rightarrow 0$.

The Asymptotic Diffusion Limit for Transport

- Returning Eq.(5) to dimensional form and dividing by ϵ , we get

$$\mu \frac{\partial \psi}{\partial x} + \frac{\sigma_t}{\epsilon} \psi = \left(\frac{\sigma_t}{\epsilon} - \epsilon \sigma_a \right) \phi + \epsilon Q. \quad (6)$$

We assume a power-series expansion in ϵ for the asymptotic solution:

$$\psi = \sum_{n=0}^{\infty} \psi^{(n)} \epsilon^n. \quad (7)$$

- Substituting from Eq.(7) into Eq.(6), and equating coefficients of like powers of ϵ , we get a heirarchical set of equations.
- The $O(\epsilon^0)$ equation yields

$$\psi^{(0)} = \phi^{(0)}. \quad (8)$$

- This simply states that the leading-order solution is isotropic.

The Asymptotic Diffusion Limit for Transport

- The $O(\epsilon^1)$ equation plus previous equations yield

$$\psi^{(1)} = -\frac{\mu}{\sigma_t} \frac{\partial \phi^{(0)}}{\partial x} + \phi^{(1)}. \quad (9)$$

- The $O(\epsilon^2)$ equation plus previous equations yield

$$\mu \left(-\frac{\mu}{\sigma_t} \frac{\partial \phi^{(0)}}{\partial x} + \phi^{(1)} \right) + \sigma_t \psi^{(2)} = \sigma_t \phi^{(2)} - \sigma_a \phi^{(0)} + Q. \quad (10)$$

- Averaging Eq.(10) over all μ (via integration), we find that the leading order solution, $\phi^{(0)} = \psi^{(0)}$, satisfies the following diffusion equation:

$$-\frac{\partial}{\partial x} \left(\frac{1}{3\sigma_t} \frac{\partial \phi^{(0)}}{\partial x} \right) + \sigma_a \phi^{(0)} = Q. \quad (11)$$

The Asymptotic Diffusion Limit for Transport

- Boundary conditions for the asymptotic diffusion equation must be determined via a boundary-layer analysis.
- With vacuum boundary conditions for the transport, the asymptotic diffusion boundary conditions are $\phi^{(0)} = 0$ at both boundaries.
- The fundamental scale length for the diffusion equation is the diffusion length, $L = \sqrt{\frac{\lambda_t}{3\sigma_a}}$.
- For instance, homogeneous diffusion solutions are of the form $a \exp(-x/L) + b \exp(x/L)$.
- Note that L can be arbitrarily large with respect to λ_t
- In the diffusion limit, $\lambda_t/L \rightarrow 0$.

Discrete Asymptotic Diffusion Limits

- With one caveat, a discrete asymptotic diffusion limit is completely analogous to the corresponding analytic limit.
- The caveat is that an additional non-dimensional parameter appears.
- This parameter is h/ℓ_∞ , which represents the cell width divided by the scale length of the asymptotic diffusion solution.
- Different scalings of h/ℓ_∞ lead to discrete different discrete diffusion limits.
- The most natural discrete limit, called the thick limit, corresponds to the assumption that h/ℓ_∞ is $O(1)$.
- This limit is referred to as thick because $h\lambda_t \rightarrow \infty$.
- A consistent discretization of the transport equation is asymptotic-preserving with respect to the thick diffusion limit if it asymptotically limits to a consistent discretization of the diffusion equation.

Discrete Asymptotic Diffusion Limits

- This implies that any mesh resolution criteria will depend only upon scale lengths associated with the diffusion equation in diffusive problems.
- If h/ℓ_∞ is scaled $O(\epsilon)$, the associated limit is referred to as the intermediate limit because h/λ_t is $O(1)$.
- If h/ℓ_∞ is scaled $O(\epsilon^2)$, the associated limit is referred to as the thin limit because h/λ_t is $O(\epsilon)$.
- In the intermediate and thin limits, the mesh becomes infinitely resolved with respect to the scale length of the asymptotic solution, so an analytic diffusion equation is obtained rather than a discrete diffusion equation.
- A consistent discretization of the transport equation is asymptotic-preserving with respect to the intermediate and thin diffusion limit if it asymptotically limits to the analytic asymptotic diffusion equation.

Discrete Asymptotic Diffusion Limits

- The performance of discretization schemes in the intermediate and thin limits is strongly related to truncation error.
- Essentially any scheme with second-order (or higher order) truncation error will preserve the intermediate limit.
- Essentially any scheme with first-order (or higher order) truncation error will preserve the thin limit.
- Preserving the thin limit does not imply that a scheme is practical for any class of diffusive problems.
- Preserving the intermediate limit means that a scheme can be efficiently used in weakly diffusive problems, i.e., problems for which the diffusion length is only a few mean-free-paths thick.
- Preserving the thick limit means that a scheme can be efficiently used in highly diffusive problems.

The Upwind Scheme

- The transport equation spatially differenced over the interval, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, using the upwind scheme is

$$\mu(\psi_{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}})/h + \sigma_{t,i}\psi_i = (\sigma_t - \sigma_s)\phi_i + Q_i, \quad (12)$$

where the cell-average value, ψ_i is “upwinded” in accordance with the direction of particle flow:

$$\begin{aligned} \psi_i &= \psi_{i+\frac{1}{2}} \quad \text{for } \mu > 0, \\ &= \psi_{i-\frac{1}{2}} \quad \text{for } \mu < 0. \end{aligned} \quad (13)$$

- This scheme has a first-order truncation error.
- It preserves the thin limit, but not the intermediate or thick limits.

The Upwind Scheme

- In the intermediate limit it converges to an incorrect diffusion equation:

$$-\frac{\partial}{\partial x} \left[\frac{1}{3\sigma_t} \left(1 + \frac{3\sigma_t h}{4} \right) \frac{\partial \phi_i^{(0)}}{\partial x} \right] + \sigma_a \phi_i^{(0)} = Q_i. \quad (14)$$

- In the thick limit it yields a diffusion equation that is unrelated to the physical properties of the transport medium and has no source:

$$\frac{1}{4h} \left(\phi_i^{(0)} - \phi_{i-1}^{(0)} \right) - \frac{1}{4h} \left(\phi_{i+1}^{(0)} - \phi_i^{(0)} \right) = 0. \quad (15)$$

The Diamond Scheme

- The transport equation spatially differenced over the interval, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, using the diamond scheme is

$$\mu(\psi_{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}})/h + \sigma_{t,i}\psi_i = (\sigma_t - \sigma_s)\phi_i + Q_i, \quad (16)$$

where the cell-average value is related to the cell edge values by

$$\psi_i = \frac{1}{2} \left(\psi_{i+\frac{1}{2}} + \psi_{i-\frac{1}{2}} \right). \quad (17)$$

- This scheme has a second-order truncation error.
- It preserves all three limits.

The Diamond Scheme

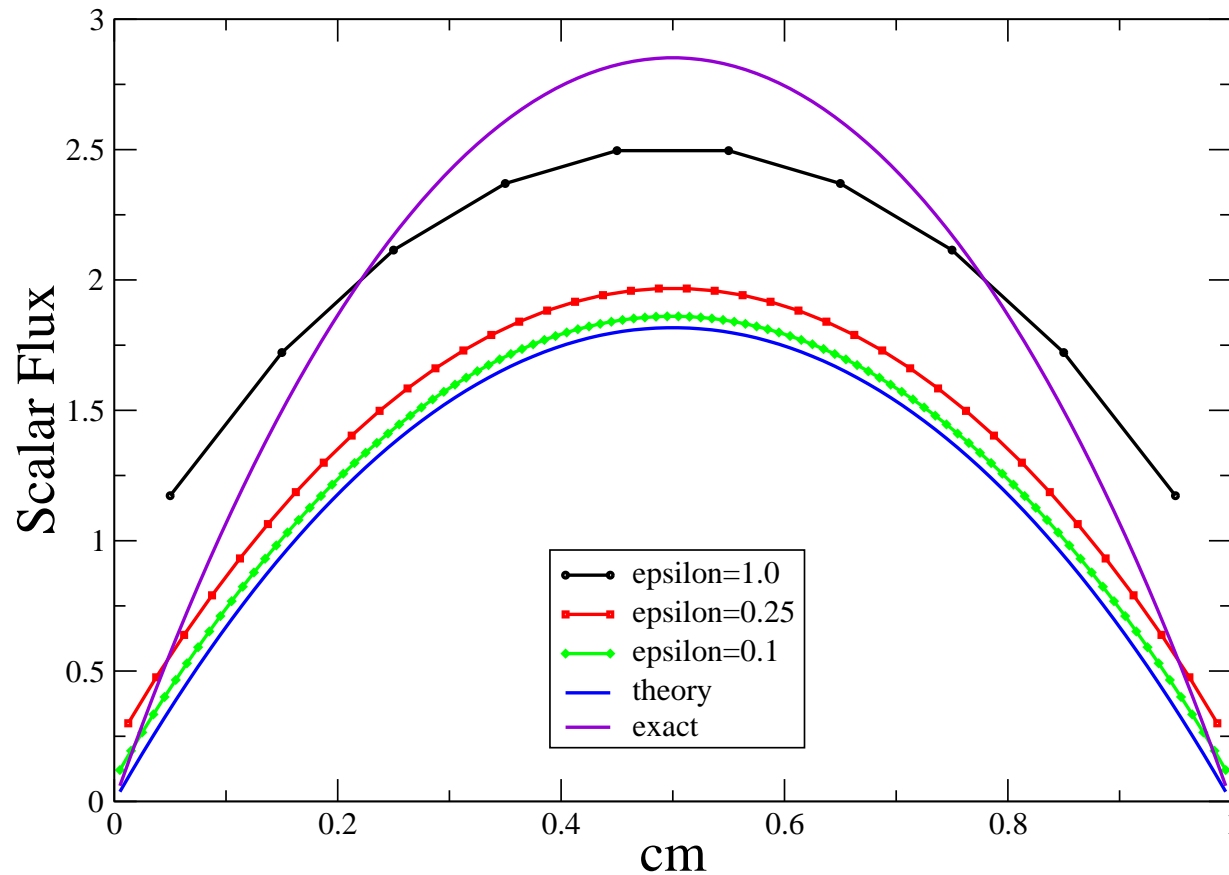
- In the thick limit it yields the following consistent diffusion discretization:

$$-\frac{1}{3\sigma_t} \left(\phi_{i+\frac{3}{2}}^{(0)} - 2\phi_{i+\frac{1}{2}}^{(0)} + \phi_{i-\frac{1}{2}}^{(0)} \right) / h^2 + \frac{\sigma_a}{4} \left(\phi_{i+\frac{3}{2}}^{(0)} + 2\phi_{i+\frac{1}{2}}^{(0)} + \phi_{i-\frac{1}{2}}^{(0)} \right) = \frac{1}{2} (Q_{i+1} + Q_i) . \quad (18)$$

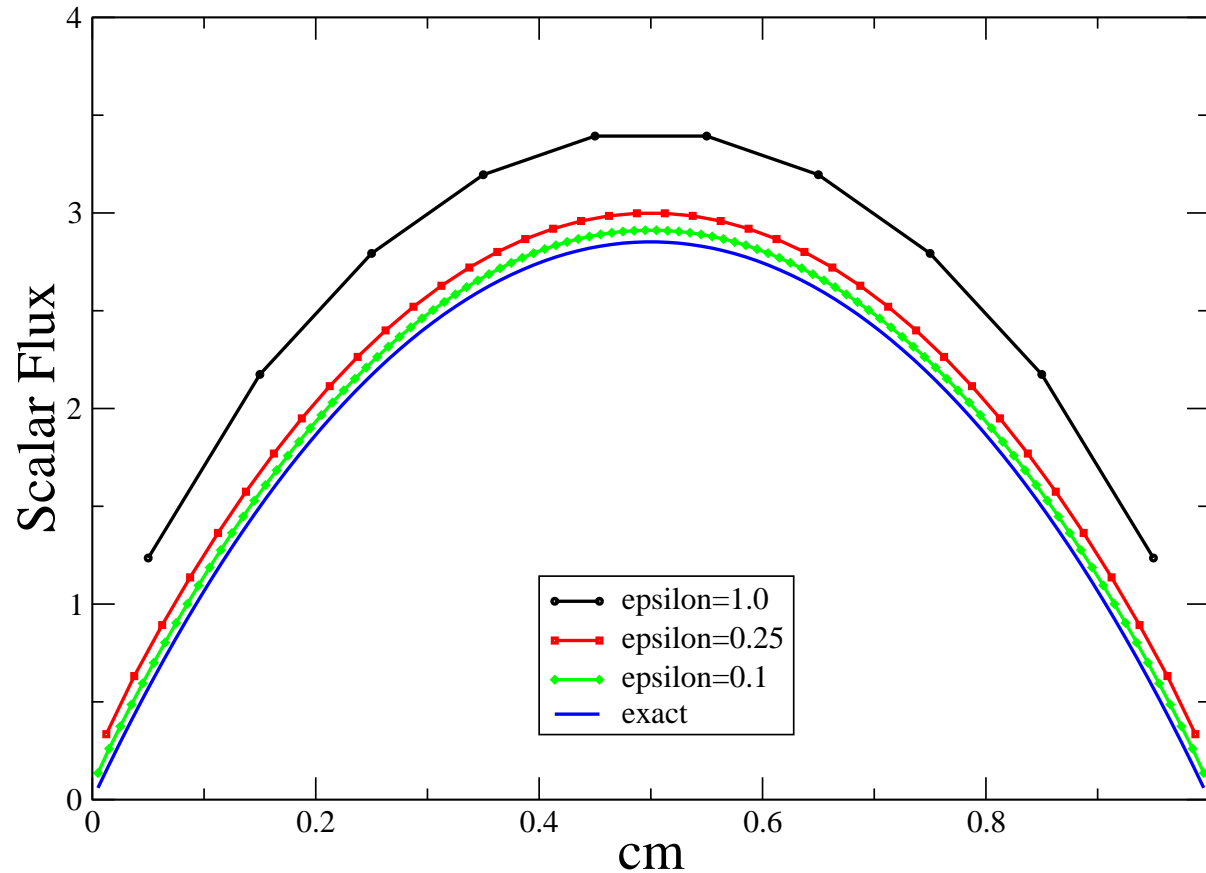
Computational Examples

- The first problem set illustrates the intermediate limit.
- A calculation is performed with each discretization scheme for $\epsilon = 1$, $\epsilon = 0.25$, and $\epsilon = 0.1$.
- As $\epsilon \rightarrow 0$, the analytic transport solution converges to an analytic diffusion solution with zero Dirichlet boundary conditions. This solution is denoted by “exact”.
- The spatial domain is $[0, 1]$, with no incoming particles on the boundaries.
- For $\epsilon = 1$, $\sigma_t = 10 \text{ cm}^{-1}$, $\sigma_a = 0.1 \text{ cm}^{-1}$, $Q = 1 \text{ p}/(\text{cm}^3\text{-sec})$, $h = 0.1 \text{ cm}$.
- For $\epsilon = 0.1$, $\sigma_t = 100 \text{ cm}^{-1}$, $\sigma_a = 0.01 \text{ cm}^{-1}$, $Q = 0.1 \text{ p}/(\text{cm}^3\text{-sec})$, $h = 0.01 \text{ cm}$.
- h/λ_t is fixed at 1 mfp for all ϵ .
- The incorrect diffusion solution predicted for the upwind scheme is denoted by “theory”.

Intermediate Limit - Upwind Differencing



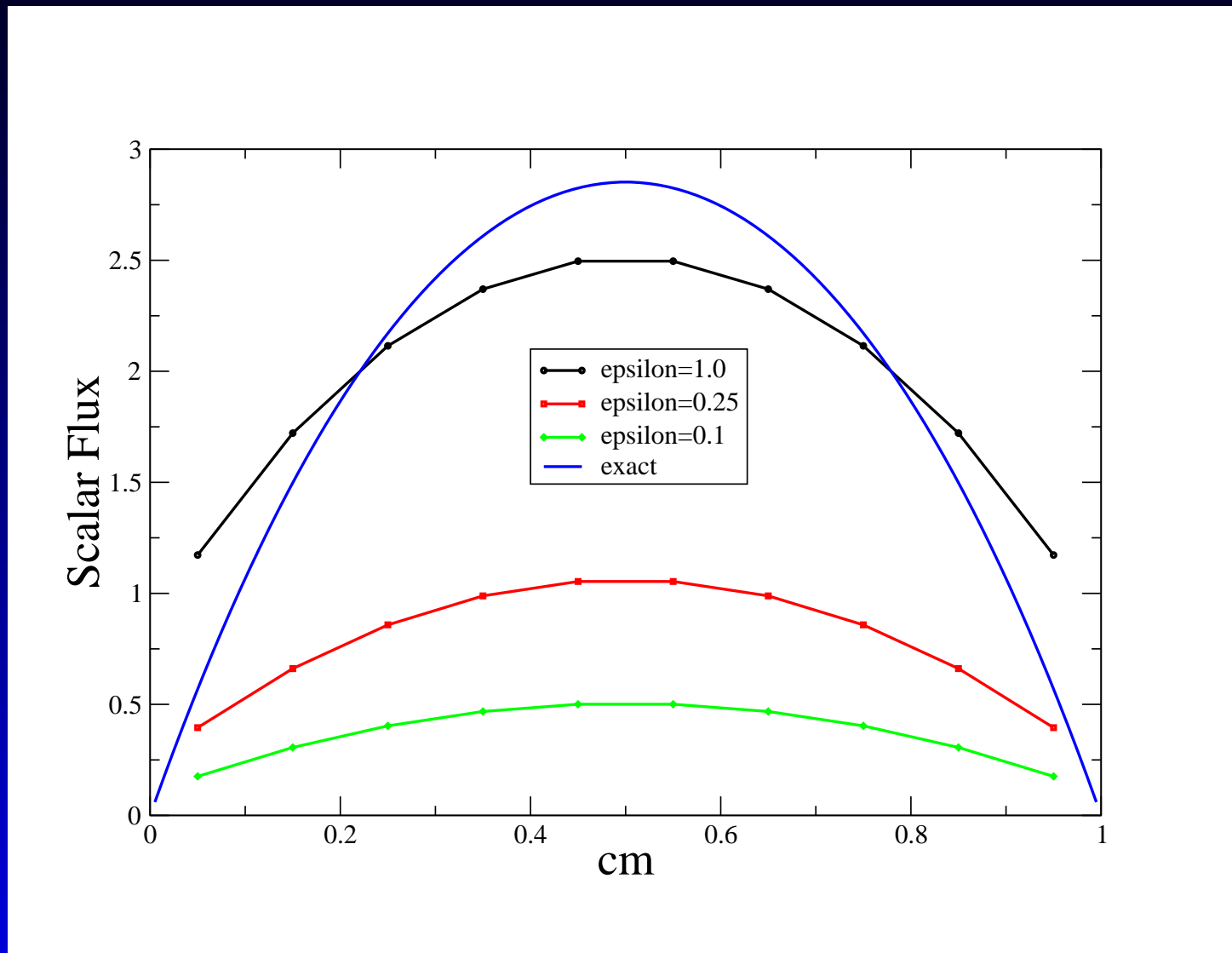
Intermediate Limit - Diamond Differencing



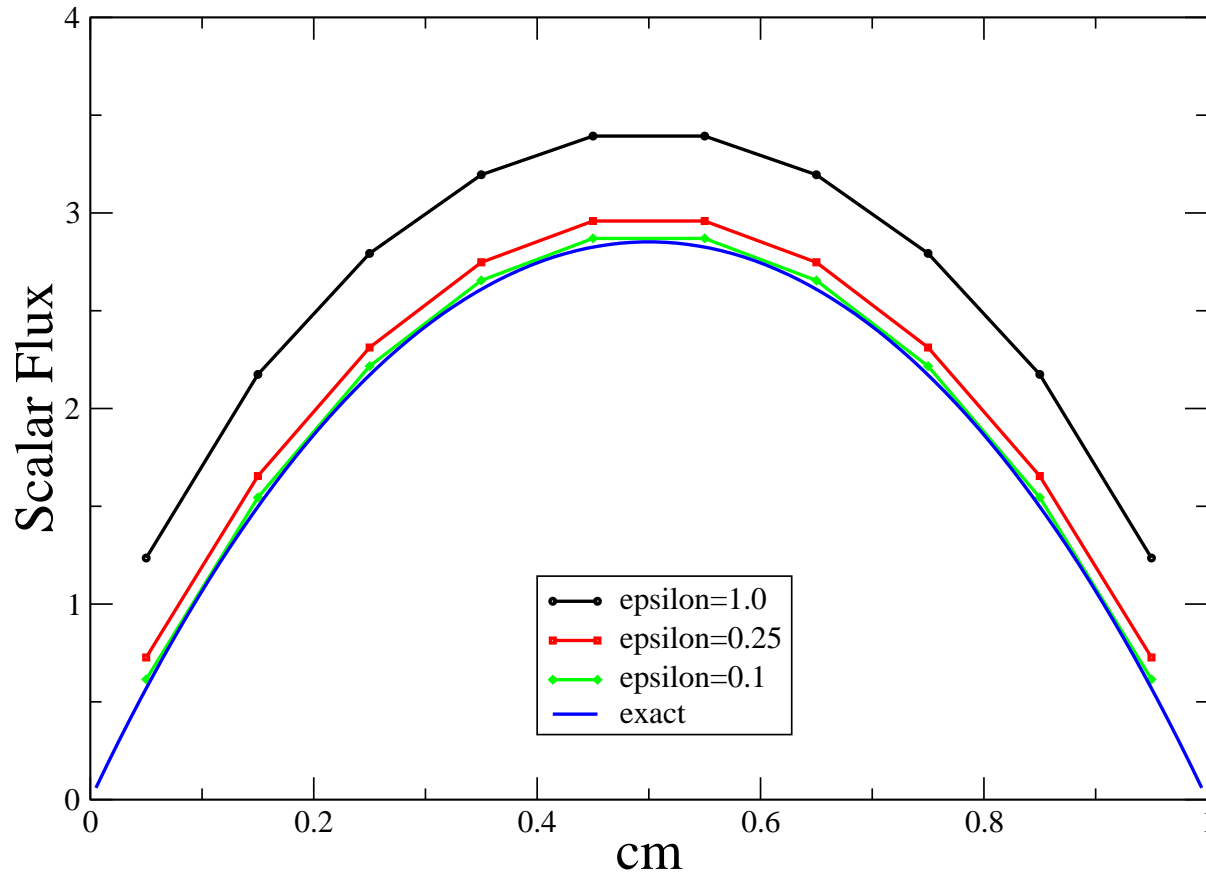
Computational Examples

- The second problem set illustrates the thick limit.
- The procedure is identical to that of the first set except that h remains fixed for all ϵ and h/λ_t increases with ϵ .
- $h = 0.1$ cm.
- For $\epsilon = 1$, $h/\lambda_t = 1$ mfp.
- For $\epsilon = 0.1$, $h/\lambda_t = 10$ mfp
- “Exact” denotes the correct asymptotic diffusion solution.

Thick Limit - Upwind Differencing



Thick Limit - Diamond Differencing



Other Considerations

- Spatial boundary layers generally exist at interfaces between transport and diffusive regions, and at the outer boundaries of diffusive regions.
- Robustness with unresolved spatial boundary layers is a major consideration for schemes that are asymptotic-preserving.
- To obtain “good” (if not accurate) behavior with unresolved spatial boundary layers, one must generally use lumped discontinuous finite-element discretizations.