

Higher Order Whitney Forms

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These slides have been prepared to be used during a talk. Therefore, any information given on these slides has to be taken with a pinch of salt: statements may be incomplete, sloppy or unduly terse, because further interpretation and explanation was to be supplied by the speaker.

Rationales

Discrete differential forms = Right tool to discretize a large variety of “fields”
(*cf.* talks by D. Arnold, A. Bossavit, . . .)

Superior resolution of smooth solutions
→ exponential convergence of hp-FEM
(*cf.* work of L. Demkowicz)

Significantly reduced numerical dispersion (for wave propagation)

M. AINSWORTH, *Dispersive properties of high order nedelec/edge element approximation of the time-harmonic maxwell equations*, Phil. Trans. Roy. Soc. Series A, 362 (2004), pp. 471–493.

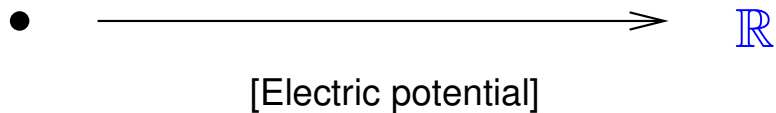
Differential forms

Forms and “Fields”

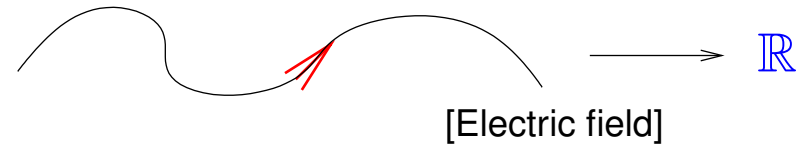
An l -form, $l \in \mathbb{N}$, on an n -manifold Σ is an additive mapping of oriented piecewise smooth l -dimensional submanifolds of Σ into \mathbb{R} (Space \mathcal{F}^l):

$$l\text{-form} : \{\text{oriented } l\text{-submanifolds}\} \xrightarrow{\int} \mathbb{R}$$

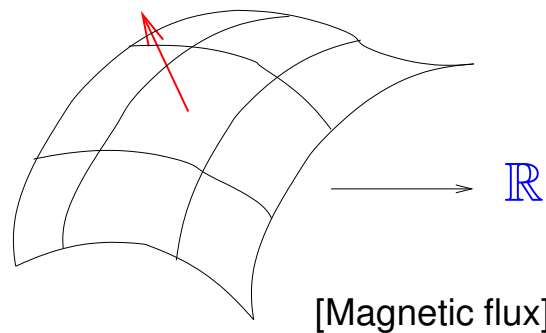
0-form \blacktriangleright point values



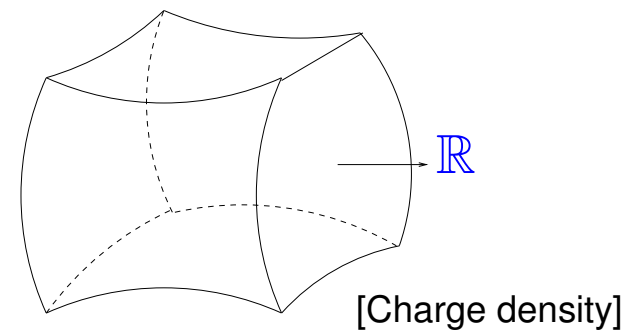
1-form \blacktriangleright path integral



2-form \blacktriangleright flux integral



3-form \blacktriangleright volume integral



Differential Forms

Σ : k -dimensional C^1 -manifold $\subset \mathbb{R}^n$

$T_\Sigma(\mathbf{x})$: Tangential space in $\mathbf{x} \in \Sigma$

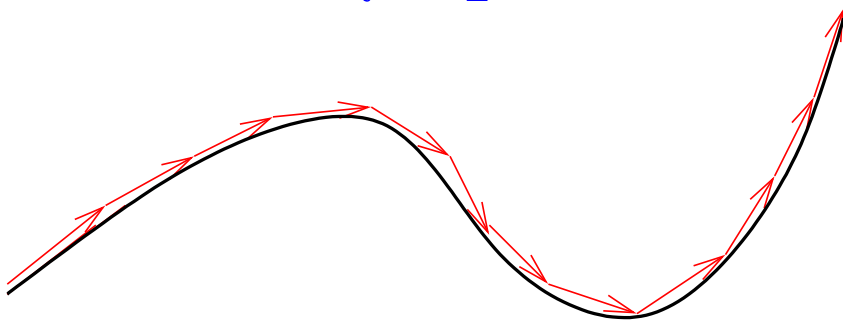
$\Delta^l(T_\Sigma(\mathbf{x}))$: Space of alternating l -multilinear forms on $T_\Sigma(\mathbf{x})$, $\dim \Delta^l = \binom{k}{l}$

+ Bilinear **exterior product** $\wedge : \Delta^l \times \Delta^m \mapsto \Delta^{m+l}$

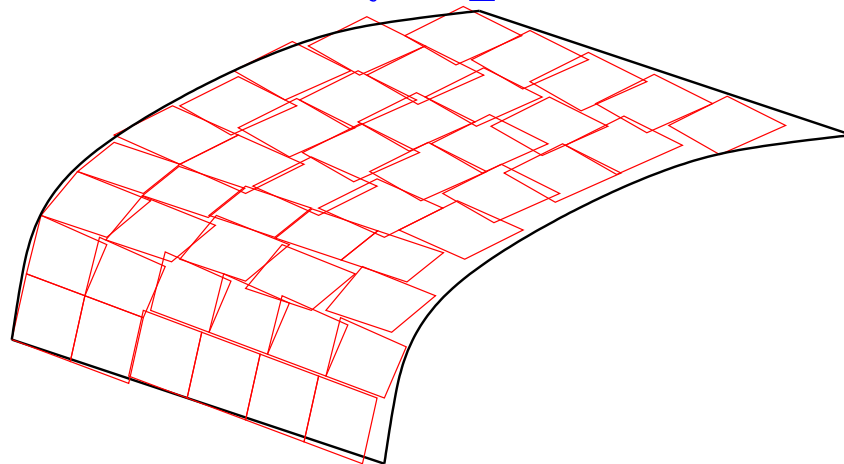
A differential form ω of degree l is a continuous mapping $\omega : \Sigma \mapsto \Delta^l(T_\Sigma(\cdot))$

Embedding into (generalized) l -forms by “Riemann integration”

$l = 1$



$l = 2$



▶ Exterior derivative, trace, and pullback for differential forms

Vector Proxies

Basis representation (in the case $\Sigma = \Omega \subset \mathbb{R}^n$, i.e. $T_\Sigma(\mathbf{x}) = \mathbb{R}^n$):

$$\omega \in \mathcal{DF}^l(\Omega) \Rightarrow \omega = \sum_{(i_1, \dots, i_l) \in I} \varphi_{i_1, \dots, i_l} dx_{i_1} \wedge \dots \wedge dx_{i_l},$$

where $I := \{(i_1, \dots, i_l), 1 \leq i_k \leq n, i_1 < \dots < i_l\}$, $\{dx_i\}$ is a dual basis of \mathbb{R}^n .

► Coordinate functions $\varphi_{i_1, \dots, i_l} : \Omega \mapsto \mathbb{R}$ (“vector-field”) describe ω .

“Canonical coordinate representations” in 3D **Euklidian space**:

Differential form	Related function/vectorfield $\vec{\omega}$
$\omega(\mathbf{x})$	$\omega(\mathbf{x}) = \vec{\omega}(\mathbf{x})$
$\{\mathbf{v} \mapsto \omega(\mathbf{x})(\mathbf{v})\}$	$\omega(\mathbf{x})(\mathbf{v}) = \langle \vec{\omega}(\mathbf{x}), \mathbf{v} \rangle$
$\{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2) = \langle \vec{\omega}(\mathbf{x}), \mathbf{v}_1 \times \mathbf{v}_2 \rangle$
$\{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$\omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \vec{\omega}(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Coordinate representations employ artificial additional structures!

Whitney Forms

Focus: Discrete differential forms on **simplicial** meshes

Perspective: Whitney forms = **finite elements** for differential forms

- **Local trial spaces** on n -simplex T , $n \in \mathbb{N}$: $(\{i_0, \dots, i_l\} \subset \{0, \dots, n\})$

$$\mathcal{W}_0^l = \text{Span} \left\{ \sum_{j=0}^l (-1)^{i_j} \lambda_{i_j} d\lambda_{i_0} \wedge \dots \wedge \cancel{d\lambda_{i_j}} \wedge \dots \wedge d\lambda_{i_l} \right\},$$

where λ_i , $i = 0, \dots, n$ = barycentric coordinate functions of T

- **Local degrees of freedom**: $\mathcal{S}_m(T)$ = set of m -dimensional, $m = 0, \dots, n$, sub-simplices of T

$$\left\{ \omega \mapsto \int_S \omega, S \in \mathcal{S}_m(T) \right\}$$

- ▶ **Local shape functions**: with $I := \{i_0, \dots, i_l\} \subset \{0, \dots, n\}$

$$\beta_I^l := \sum_{j=0}^l (-1)^{i_j} \lambda_{i_j} d\lambda_{i_0} \wedge \dots \wedge \cancel{d\lambda_{i_j}} \wedge \dots \wedge d\lambda_{i_l}.$$

Construction of Higher Order Local Trial Spaces

Requirements

We consider only **simplicial** triangulations (tetrahedral in 3D)

$\mathcal{W}_p^l(\hat{T})$: Local space of discrete l -forms of polynomial degree p on simplex T

- $\mathcal{W}_p^l(T)$ is “**polynomial**”: $\mathcal{DP}_p^l(T) \subset \mathcal{W}_p^l(T) \subset \mathcal{DP}_{p+1}^l(T)$

($\mathcal{DP}_p^l(T)$: polynomial l -forms of degree p , i.e. the coefficient functions in the basis representation are polynomials of degree $\leq p \leftrightarrow$ polynomial vector proxies)

- **Local discrete potentials**: $\{\omega \in \mathcal{W}_p^l(T), d\omega = 0\} = d\mathcal{W}_p^{l-1}(T)$
- **Recursion in dimension**: for all $S \in \mathcal{S}_m(T)$, $l \leq m \leq n$,

$$\mathbf{t}_S \mathcal{W}_p^l(T) \subset \mathcal{W}_p^l(S) .$$

(\mathbf{t}_S = trace of differential form onto S)

▶ The requirements offer no unique characterization.

▶ All these properties are invariant under affine transformation.

Dimension of Local Spaces

$$\dim \mathcal{W}_p^l(T) = \sum_{i=l}^n \binom{n+1}{i+1} \binom{i}{i-l} \binom{p+l}{i}$$

Proof (by induction w.r.t. l, p)

[Notation: $\widetilde{\mathcal{DP}}_p^l(T)$ = Homogeneous polynomial l -forms of degree p]

• Direct sums: $\mathcal{W}_p^l(T) = \mathcal{DP}_p^l(T) \oplus \mathcal{Y}_p^l(T)$, $\mathcal{Y}_p^l(T) \subset \widetilde{\mathcal{DP}}_{p+1}^l(T)$,
 $\widetilde{\mathcal{DP}}_{p+1}^l(T) = \mathcal{Y}_p^l(T) \oplus \widetilde{\mathcal{HN}}_{p+1}^l(T)$, $\widetilde{\mathcal{HN}}_{p+1}^l(T) := \widetilde{\mathcal{DP}}_{p+1}^l(T) \cap \text{Ker}(d)$

• Monomial projections $\mathcal{DP}_p^l(T) \mapsto \widetilde{\mathcal{DP}}_p^l(T)$ and d commute & Koszul exact sequence

• $\widetilde{\mathcal{HN}}_{p+1}^l(T) = d(\widetilde{\mathcal{DP}}_{p+2}^{l-1}(T))$ and $\dim \mathcal{W}_p^l(T) = \dim \mathcal{DP}_{p+1}^l(T) - \dim \widetilde{\mathcal{HN}}_{p+1}^l(T)$

► $\dim \mathcal{W}_p^l(T) = \dim \mathcal{DP}_{p+1}^l(T) - \dim \widetilde{\mathcal{DP}}_{p+2}^{l-1}(T) + \dim \mathcal{DP}_{p+2}^{l-1}(T) - \dim \mathcal{X}_{p+1}^{l-1}(T)$ □

Strategies

Construction of $\mathcal{W}_p^l(T)$, $T = n$ -simplex

“one fell swoop”

- formula for $\mathcal{W}_p^l(T)$
- verification of requirements

Sub-simplex based

- spaces for sub-simplices
- extension into T

Example: Nedélec’s construction for $n = 3$ (Euklidian vector proxies)

[J. NÉDÉLEC, *Mixed finite elements in \mathbb{R}^3* , Numer. Math., 35 (1980), pp. 315–341.]

$$\mathcal{W}_p^0(T) = \mathcal{P}_{p+1}(T) ,$$

$$\mathcal{W}_p^1(T) = (\mathcal{P}_p(T))^3 \oplus \{ \mathbf{q} \in (\tilde{\mathcal{P}}_{p+1}(T))^3 : \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0 \forall \mathbf{x} \in T \} ,$$

$$\mathcal{W}_p^2(T) = (\mathcal{P}_p(T))^3 \oplus \{ \mathbf{q} \in (\tilde{\mathcal{P}}_{p+1}(T))^3 : \mathbf{q}(\mathbf{x}) \times \mathbf{x} = 0 \forall \mathbf{x} \in T \} ,$$

$$\mathcal{W}_p^3(T) = \mathcal{P}_p(T) .$$

“One Fell Swoop” Approach

$\widetilde{\mathcal{DP}}_p^l(T) \subset \mathcal{DP}_p^l(T) :=$ homogeneous polynomial l -forms on simplex T ,
 $\widetilde{\mathcal{HN}}_p^l(T) \subset \widetilde{\mathcal{DP}}_p^l(T) := \{\omega \in \widetilde{\mathcal{DP}}_p^l(T) : d\omega = 0\}$ (closed forms).

$$\begin{aligned}
 \mathcal{Y}_p^l(T) : \quad & \text{direct sum} \quad \widetilde{\mathcal{DP}}_p^l(T) = \mathcal{Y}_p^l(T) \oplus \widetilde{\mathcal{HN}}_p^l(T) \\
 \blacktriangleright \quad & \mathcal{W}_p^l(T) = \mathcal{DP}_p^l(T) \oplus \mathcal{Y}_{p+1}^l(T) .
 \end{aligned}$$

Choice: $\mathcal{Y}_p^l(T) = k_a(\widetilde{\mathcal{HN}}_p^{l+1}(T))$ (\rightarrow generalizes Nedélec’s construction)

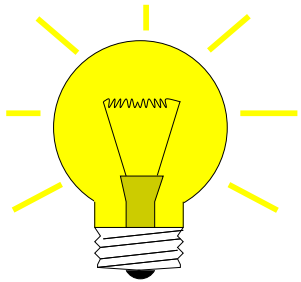
Poincaré mapping $k_a : \{l\text{-forms}\} \mapsto \{l - 1\text{-forms}\}$, ($\bullet = \mathbf{v}_1, \dots, \mathbf{v}_l, \mathbf{v}_i \in \mathbb{R}^n$)

$$k_a(\omega)(\mathbf{x})(\bullet) := \int_0^1 t^{l-1} \omega(t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a}, \bullet) dt \Rightarrow d(k_a(\omega)) + k_a(d\omega) = \omega$$

R.H., *Canonical construction of finite elements*, Math. Comp., 68 (1999), pp. 1325–1346.

R.H., *Higher order Whitney forms*, in Geometric Methods for Computational Electromagnetics, F. Teixeira, ed., vol. 32 of PIER, EMW Publishing, Cambridge, MA, 2001, pp. 271–299.

Sub-simplex Based Approach (I)



[Idea borrowed from p-version of Lagrangian FEM]

- Polynomial l -forms on each $S \in \mathcal{S}_m(T)$, $l \leq m \leq n$, with *vanishing trace* on $\partial S \Rightarrow$ space $\mathcal{X}_p^l(S)$
- Polynomial extension, preserving vanishing trace on $\mathcal{S}_m(T) \setminus \{S\}$



Exploit **recursion in dimension** !

$$S \in \mathcal{S}_m(T): \quad \mathcal{X}_p^l(S) \subset \mathcal{Z}_{p+1}^l(S) := \begin{cases} \{\omega \in \mathcal{DP}_{p+1}^l(S), \mathbf{t}_{\partial S} \omega = 0\}, & \text{for } l < m, \\ \{\omega \in \mathcal{DP}_{p+1}^l(S), \int_S \omega = 0\}, & \text{for } l = m. \end{cases}$$

$$\text{Required: } \mathcal{Z}_p^l(S) \subset \mathcal{X}_p^l(S) \wedge \{\omega \in \mathcal{X}_p^l(S), d\omega = 0\} = d\mathcal{X}_p^{l-1}(S), \quad 1 \leq l \leq m.$$

$$\dim \mathcal{X}_p^l = \binom{m}{l} \binom{p+l}{m}, \quad \text{if } l < n, \quad \dim \mathcal{X}_p^l = \binom{p+l}{m} - 1, \quad \text{if } l = m.$$

Sub-simplex Based Approach (II)

Construction uses **some complement** \mathcal{Y}_p^l : $\mathcal{Z}_p^l(S) = \mathcal{Y}_p^l(S) \oplus \text{Kern}(d)$


$$\mathcal{X}_p^l(S) = d\mathcal{Y}_{p+1}^{l-1}(S) \oplus \mathcal{Y}_p^l(S).$$

$\lambda_{s_0}, \dots, \lambda_{s_m}$ = barycentric coordinate functions associated with $S \in \mathcal{S}_m(T)$:

$$\omega \in \mathcal{Z}_p^l(S) \Rightarrow \omega = \sum_{I \subset \{s_0, \dots, s_m\}, |I|=l} \alpha_I(\lambda_{s_0}, \dots, \lambda_{s_m}) \lambda_{I'} d\lambda_I,$$

for $l \leq m$, $p > m - l$, with $\alpha_I \in \tilde{\mathcal{P}}_q(\mathbb{R}^{m+1})$, $q := p - (m + 1 - l)$,

$I' := \{s_0, \dots, s_m\} \setminus I$, $d\lambda_I := d\lambda_{i_1} \wedge \dots \wedge d\lambda_{i_l}$.

Barycentric representation  **barycentric extension** $E_{S,p}^l$
 (Ensures continuity of traces across interelement boundaries)

Exact Sequence Property

$$\mathcal{W}_p^l(\mathcal{T}) = \mathcal{W}_0^l(\mathcal{T}) + \sum_{m=l}^n \sum_{S \in \mathcal{S}_m(\mathcal{T})} \mathbb{E}_{S,p}^l(\mathcal{X}_p^l(S)) .$$

$$S \in \mathcal{S}_m(\mathcal{T}): \quad \{0\} \xrightarrow{d} \mathcal{X}_p^0(S) \xrightarrow{d} \mathcal{X}_p^1(S) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{X}_p^m(S) \xrightarrow{d} \{0\} .$$

Since $d \circ \mathbb{E}_{S,p}^l = \mathbb{E}_{S,p}^{l+1} \circ d$:

$$\{0\} \xrightarrow{d} \mathbb{E}_{S,p}^0(\mathcal{X}_p^0(S)) \xrightarrow{d} \mathbb{E}_{S,p}^1(\mathcal{X}_p^1(S)) \xrightarrow{d} \dots \xrightarrow{d} \mathbb{E}_{S,p}^m(\mathcal{X}_p^m(S)) \xrightarrow{d} \{0\} .$$

Local discrete potentials in p -hierarchical surplus: on simplicial mesh \mathcal{T}

$$\omega \in \mathcal{W}_p^l(\mathcal{T}) \cap \text{Kern}(d), \text{ supp } \omega \subset \mathcal{U}_S \Rightarrow \begin{cases} \exists \eta \in \mathcal{W}_p^{l-1}(\mathcal{T}), \text{ supp } \eta \subset \mathcal{U}_S, \\ d\eta = (Id - \Pi^l)\omega . \end{cases}$$

$\Pi^l = \mathcal{W}_0^l(\mathcal{T})$ -projection (“Whitney interpolation”), $\mathcal{U}_S := \{T \in \mathcal{T} : S \subset \bar{T}\}$.

Variable Order Whitney Forms

Assign degree p_S to every $S \in \mathcal{S}_m(T)$, $l \leq m \leq n$ \triangleright “degree vector” \mathbf{p}

$$\mathcal{W}_{\mathbf{p}}^l(T) = \mathcal{W}_0^l(T) + \sum_{m=l}^n \sum_{S \in \mathcal{S}_m(T)} \mathbb{E}_{S, p_S}^l(\mathcal{X}_{p_S}^l(S)) .$$

Application: hp-FE Galerkin discretization

$\mathcal{S}_m(\mathcal{T}) = m$ -dimensional simplices of a simplicial mesh \mathcal{T} , $l \leq m \leq n$

$p_T \in \mathbb{N}_0 =$ polynomial degree assigned to every $T \in \mathcal{T}$

minimum degree rule

$$p_S := \min\{p_T : S \in \bar{T}\}, S \in \mathcal{S}_m(T)$$

$$\mathcal{W}_{\mathbf{p}}^l(T) := \{\omega \in l\text{-forms} : \omega|_T \in \mathcal{W}_{\mathbf{p}}^l(T)\} .$$

Local Shape Functions

Barycentric Formulas

Local shape functions for $n = 3, l = 1, p = 2$ (Nedélec's space)

Interior	Face $\langle \mathbf{a}_i, \mathbf{a}_m, \mathbf{a}_k \rangle$	Edge $\langle \mathbf{a}_k, \mathbf{a}_m \rangle$
$\lambda_2\lambda_3\lambda_4d\lambda_1 - \lambda_1\lambda_2\lambda_3d\lambda_4$	$\lambda_m^2\lambda_k d\lambda_i - \lambda_i\lambda_m\lambda_k d\lambda_m$	$\lambda_i^2\lambda_m d\lambda_i - \lambda_i^3 d\lambda_m$
$\lambda_3\lambda_4\lambda_1d\lambda_2 - \lambda_2\lambda_3\lambda_4d\lambda_1$	$\lambda_k^2\lambda_i d\lambda_m - \lambda_m\lambda_k\lambda_i d\lambda_k$	$\lambda_m^2\lambda_i d\lambda_i - \lambda_m^3 d\lambda_i$
$\lambda_4\lambda_1\lambda_2d\lambda_3 - \lambda_3\lambda_4\lambda_1d\lambda_2$	$\lambda_i^2\lambda_m d\lambda_k - \lambda_k\lambda_i\lambda_m d\lambda_i$	$\lambda_i^2\lambda_m d\lambda_i - \lambda_i^2\lambda_m^3 d\lambda_m$
	$\lambda_i\lambda_m\lambda_k d\lambda_i - \lambda_m^2\lambda_k d\lambda_m$	
	$\lambda_m\lambda_k\lambda_i d\lambda_m - \lambda_k^2\lambda_i d\lambda_k$	
	$\lambda_k\lambda_i\lambda_m d\lambda_k - \lambda_i^2\lambda_m d\lambda_i$	

See (for general formula):

J. GOPALAKRISHNAN, L. GARCIA-CASTILLO, AND L. DEMKOWICZ, *Nédélec spaces in affine coordinates*, Report 03-48, TICAM, University of Texas at Austin, Austin, TX, 2003.

J. WEBB, *Hierarchical vector basis functions of arbitrary order for triangular and tetrahedral finite elements*, IEEE Trans. Antennas & Propagation, 47 (1999), pp. 1244–1253.

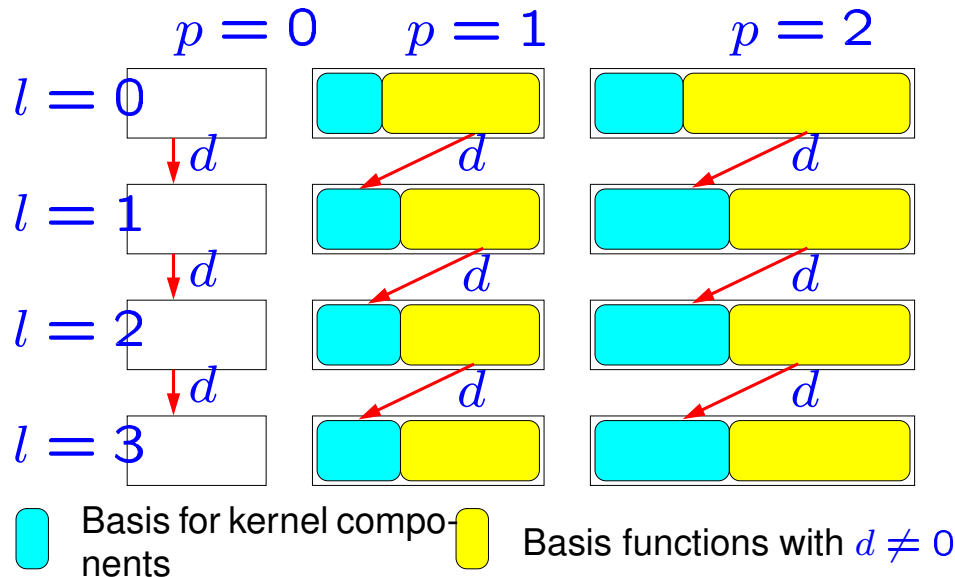
Hierarchical Local Shape Functions

For $S \in \mathcal{S}_m(T)$ choose $\widetilde{\mathfrak{B}}_p^l(S) \subset \mathcal{X}_p^l(S)$ such that

$$\mathfrak{B}_p^l(S) := \bigcup_{q=0}^p \widetilde{\mathfrak{B}}_q^l(S) \quad \text{is basis of } \mathcal{X}_p^l(S)$$

Hierarchical local shape functions:

$$\mathfrak{B}_p^l(T) = \mathfrak{B}_0^l(T) \cup \bigcup_{m=l}^n \bigcup_{S \in \mathcal{S}_m(T)} E_{S,p}^l(\mathfrak{B}_p^l(S)) .$$



$$\mathcal{X}_p^l(S) = d\mathcal{Y}_{p+1}^{l-1}(S) \oplus \mathcal{Y}_{p+1}^l(S) .$$

Separation of kernel:

$$\widetilde{\mathfrak{B}}_p^0(S) = \{\beta_0^0, \dots, \beta_{N_0}^0\} ,$$

$$\widetilde{\mathfrak{B}}_p^l(S) = \{\beta_i^l\}_{i=1}^{N_p} \cup \{d\beta_i^{l-1}\}_{i=0}^{N_p^{l-1}} .$$

“Optimized” Local Shape Functions

“Optimized” = yielding low condition numbers for FE stiffness matrix
(given a particular linear variational problem)

For $n = 2, 3$, $l = 1$ see:

M. AINSWORTH AND J. COYLE, *Conditioning of hierarchic p -version Nédélec elements on meshes of curvilinear quadrilaterals and hexahedra*, Report 26, Department of Mathematics, University of Strathclyde, Glasgow, UK, 2001.

M. AINSWORTH AND J. COYLE, *Hierarchic finite element bases on unstructured tetrahedral meshes*, Int. J. Numer. Meth. Engr., 58 (2003), pp. 2103–2130.

Optimized local shape functions convenient,
but is it worth the fuss?

NO

Static condensation
+
Block preconditioning



will accelerate iterative solvers
anyway

Discrete Poincaré-Friedrichs-type Inequality

Poincaré-Friedrichs-Type Inequality

For domain $\Omega \subset \mathbb{R}^3$:

$$H_0(\text{curl}; \Omega) := \{w \in L^2(\Omega) : \text{curl } w \in L^2(\Omega), w \times n = 0 \text{ on } \partial\Omega\} .$$

$L^2(\Omega)$ -orthogonal Helmholtz decomposition

$$H_0(\text{curl}; \Omega) = Z(\Omega) \oplus (H_0(\text{curl}; \Omega) \cap \text{Kern}(\text{curl})) .$$

Since $Z(\Omega) \hookrightarrow L^2(\Omega)$ compact

$$\exists C = C(\Omega) > 0: \quad \|u\|_{L^2(\Omega)} \leq C \|\text{curl } u\|_{L^2(\Omega)} \quad \forall u \in Z(\Omega) .$$

Discrete Complement Space

Discrete 1-forms on \mathcal{T} = tetrahedral mesh of Ω (Nedélec's space):

$$\mathcal{W}_p^1(\mathcal{T}) := \{\mathbf{w}_p \in H_0(\text{curl}; \Omega) : \mathbf{w}_p|_T \in \mathcal{W}_p(T) \forall T \in \mathcal{T}\} .$$

$L^2(\Omega)$ -orthogonal **discrete Helmholtz decomposition**:

$$\mathcal{W}_p^1(\mathcal{T}) = Z_p(\mathcal{T}) \oplus (\mathcal{W}_p^1(\mathcal{T}) \cap \text{Kern}(\text{curl})) .$$

Note:

$$Z_p(\mathcal{T}) \not\subset Z(\Omega) !$$



Is there $C = C(\mathcal{T}) > 0$ independent of p such that

$$\|\mathbf{u}_p\|_{L^2(\Omega)} \leq C \|\text{curl } \mathbf{u}_p\|_{L^2(\Omega)} \quad \forall \mathbf{u}_p \in Z_p(\mathcal{T}) ?$$

Local Stable Discrete Potentials

Use **Poincaré mapping** $k_a : \{l\text{-forms}\} \mapsto \{l-1\text{-forms}\}$, ($\bullet = \mathbf{v}_1, \dots, \mathbf{v}_l, \mathbf{v}_i \in \mathbb{R}^n$)

$$k_a(\omega)(\mathbf{x})(\bullet) := \int_0^1 t^{l-1} \omega(t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a}, \bullet) dt \Rightarrow d(k_a(\omega)) + k_a(d\omega) = \omega$$

For $l = 2, n = 3$ & vector proxies:

$$(k_0 \mathbf{w})(\mathbf{x}) = \int_0^1 t(\mathbf{w}(t\mathbf{x})) \times \mathbf{x} dt \quad \triangleright \quad (\operatorname{div} \mathbf{w} = 0 \Rightarrow \operatorname{curl}(k_0 \mathbf{w}) = \mathbf{w}).$$

$$D \subset \mathbb{R}^3: \quad \|k_0 \mathbf{w}\|_{L^2(D)} \leq \operatorname{diam}(D) \|\mathbf{w}\|_{L^2(D)} \quad \forall \mathbf{w} \in C^1(\bar{D})$$

[Apply to $\operatorname{curl} \mathbf{w}_p, \mathbf{w}_p \in \mathcal{W}_p^1(T), T \in \mathcal{T}$]

For all $\mathbf{w}_p \in \mathcal{W}_p^1(T)$ there is $\mathbf{u}_p \in \mathcal{W}_p^1(T)$ such that

$$\operatorname{curl} \mathbf{u}_p = \operatorname{curl} \mathbf{w}_p \quad \wedge \quad \|\mathbf{u}_p\|_{L^2(T)} \leq \operatorname{diam}(T) \|\operatorname{curl} \mathbf{w}_p\|_{L^2(T)}$$

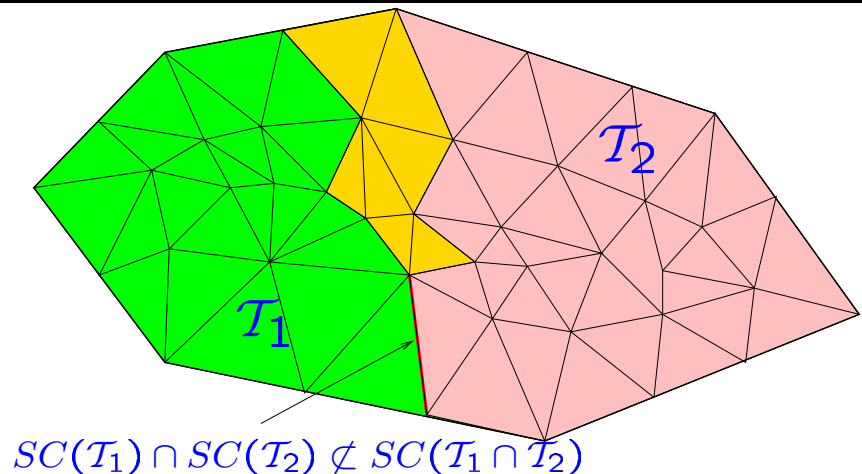
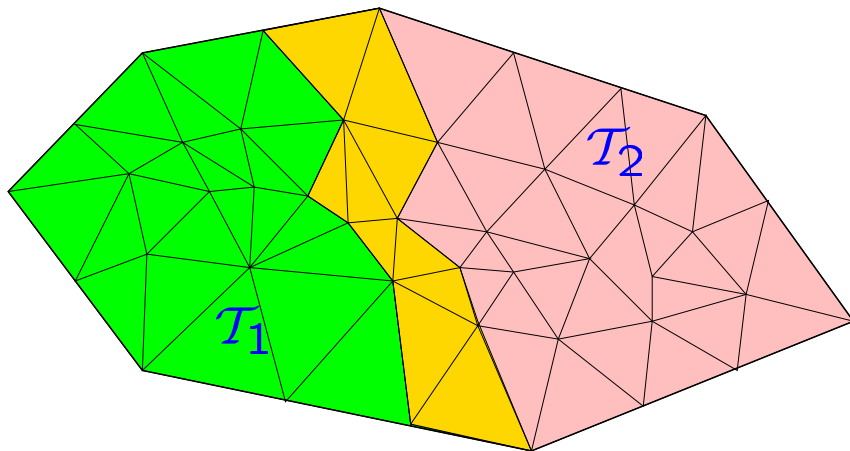
J. GOPALAKRISHNAN AND L. DEMKOWICZ, *Quasioptimality of some spectral mixed method*, Report 03-32, TICAM, University of Texas at Austin, Austin, TX, 2003.

Fragmentable Meshes (I)

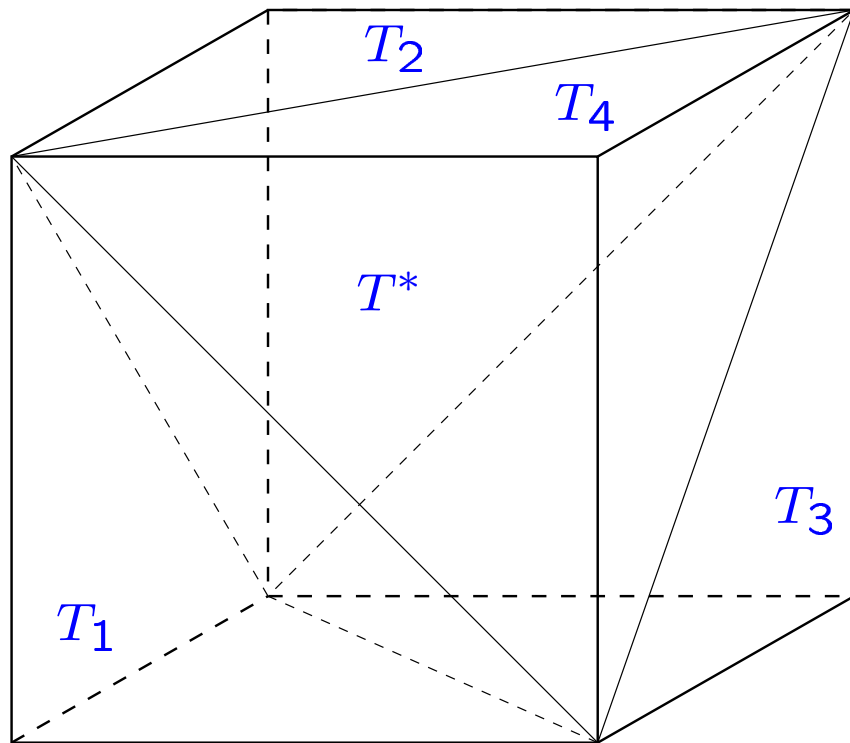
$SC(\mathcal{T})$ = simplicial complex generated by simplicial mesh \mathcal{T}
(set of all vertices/edges/faces/cells of simplicial mesh \mathcal{T})

Simplicial mesh \mathcal{T} is **fragmentable**, if one of the following is true:

1. $\bigcup\{\bar{T} : T \in \mathcal{T}\} \neq \emptyset$,
2. $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, $\mathcal{T}_1, \mathcal{T}_2 \neq \mathcal{T}$ fragmentable, and
 - $SC(\mathcal{T}_1 \cap \mathcal{T}_2)$ can be partitioned into sub-complexes with vanishing first Betti number (“simply connected”),
 - $SC(\mathcal{T}_1) \cap SC(\mathcal{T}_2) \subset SC(\mathcal{T}_1 \cap \mathcal{T}_2)$.

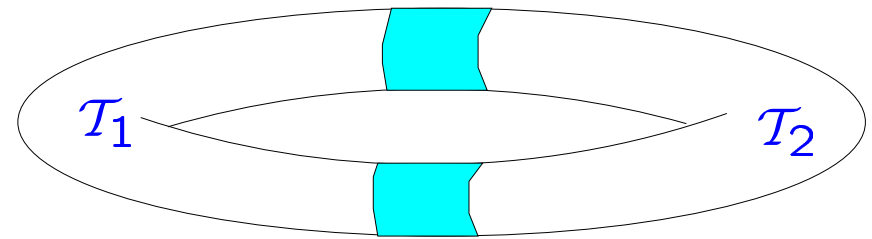


Fragmentable Meshes (II)



$$\mathcal{T}_1 = \{T_1, T_2, T^*\}, \mathcal{T}_2 = \{T_3, T_4, T^*\},$$

overlap T^*



Splitting of a triangulated torus: overlap consists of two disjoint parts with vanishing first Betti number each

Note: If $SC(\mathcal{T})$ has non-zero second Betti number $\Rightarrow \mathcal{T}$ not fragmentable
(Example: Spherical shell)

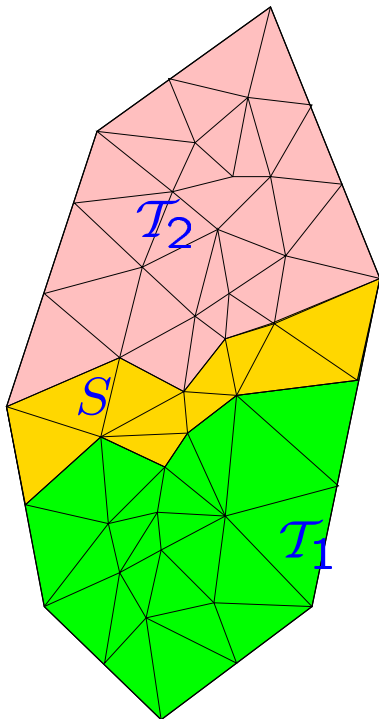
Inductive Argument

Given : $\mathbf{v}_p \in \mathbf{curl} \mathcal{W}_p^1(\mathcal{T}) \subset \mathcal{W}_p^2(\mathcal{T}) \quad \triangleright \quad \mathbf{div} \mathbf{v}_p = 0.$

Sought : $\mathbf{w}_p \in \mathcal{W}_p^1(\mathcal{T}), \mathbf{curl} \mathbf{w}_p = \mathbf{v}_p, \quad \|\mathbf{w}_p\|_{L^2(\Omega)} \leq C(\mathcal{T}) \|\mathbf{v}_p\|_{L^2(\Omega)}$

Assume : simplicial mesh \mathcal{T} **fragmentable**

I: If $\bigcup \{\bar{T} : T \in \mathcal{T}\} \neq \emptyset$ use stable local discrete vector potential



II: Decompose $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$

Induction hypothesis: for $i = 1, 2$

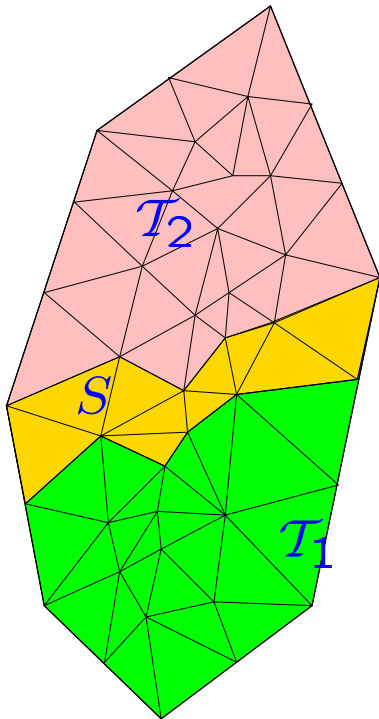
$$\exists \mathbf{w}_p^i \in \mathcal{W}_p^1(\mathcal{T}_i), \mathbf{curl} \mathbf{w}_p^i = \mathbf{v}_p|_{\mathcal{T}_i},$$

$$\|\mathbf{w}_p^i\|_{L^2(\Omega_i)} \leq C(\mathcal{T}_i) \|\mathbf{v}_p\|_{L^2(\Omega_i)}.$$

In $S = \Omega_1 \cap \Omega_2$: ($\mathcal{T}_S = \mathcal{T}_1 \cap \mathcal{T}_2$)

$$\mathbf{curl}(\mathbf{w}_p^1 - \mathbf{w}_p^2) = 0 \Rightarrow \mathbf{w}_p^1 - \mathbf{w}_p^2 = \mathbf{grad} \psi_p, \psi_p \in \mathcal{W}_p^0(\mathcal{T}_S).$$

Welding by Polynomial Extension



With $C = C(\mathcal{T}_1) > 0$ independent of p

$$\begin{aligned} \exists \tilde{\psi}_p \in \mathcal{W}_p^0(\mathcal{T}_1): \quad & \tilde{\psi}_p|_S = \psi_p, \\ & |\tilde{\psi}_p|_{H^1(\Omega_1)} \leq C |\psi_p|_{H^1(S)}. \end{aligned}$$

$$\mathbf{w}_p = \begin{cases} \mathbf{w}_p^1 - \mathbf{grad} \tilde{\psi}_p & \text{in } \Omega_1, \\ \mathbf{w}_p^2 & \text{in } \Omega_2. \end{cases}$$

III: Enforcing homogeneous boundary conditions by polynomial extension:

$$\mathbf{t}_{\partial\Omega} \mathbf{w}_p \in \mathbf{grad}_\Gamma \mathcal{W}_p^0(\mathcal{T}_{|\partial\Omega}).$$

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