Compatible discretizations for eigenvalue problems

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Plan of the talk

- Standard finite element approximation of Laplace eigenproblem
- Approximation theory of compact operators
- Laplace eigenproblem in mixed form
- A counterexample (inf-sup OK and spurious eigenvalues)
- Finite element approximation of eigenproblems in mixed form
- Applications:
  - Maxwell's eigenproblem, time harmonic Maxwell's equations
  - Band gap of photonic crystals
  - Evolution problems in mixed form

Variationally posed eigenproblem (Laplace operator)

Abstract framework

$H \quad (= L^2(\Omega)) \quad , V \quad (= H^1_0(\Omega)) \quad \subset H$

Hilbert spaces, V compactly embedded in H

$a(u,v) \quad (= \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x}) \qquad V \times V \to \mathbb{R}$

bilinear, continuous, symmetric, coercive

$b(u,v) \quad (= (u,v)) \qquad H \times H \to \mathbb{R}$

bilinear, continuous, symmetric

Eigenvalue problem

Find $\lambda \in \mathbb{R}$ such that for some $u \in V$ with $u \not\equiv 0$ it holds

$a(u,v) = \lambda b(u,v) \quad \forall v \in V$

Laplace eigenproblem

Strong form

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak form

$\lambda \in \mathbb{R}$, $u \in V$, $u \not\equiv 0$:

$$a(u,v) = \lambda b(u,v) \quad \forall v \in V$$

Resolvent operator

$T : H \to H$, $T(H) \subset V$ implies T is compact

$$a(Tf,v) = b(f,v) \quad \forall v \in V$$

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots$$

$E_i = \mathrm{span}(u_i)$, normalization $b(u_i,u_i) = 1$

$$V = \bigoplus_{i=1}^{\infty} E_i$$

Laplace eigenproblem: approximation

$V_h \subset V$, $\dim V_h = N(h)$

Discrete problem

Find $\lambda_h \in \mathbb{R}$ such that for some $u_h \in V_h$ with $u_h \not\equiv 0$ it holds

$$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h$$

Discrete (compact) resolvent operator

$T_h : H \to H$

$$a(T_h f, v) = b(f, v) \quad \forall v \in V_h$$

$\lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{i,h} \le \cdots \le \lambda_{N(h),h}$

$E_{i,h} = \mathrm{span}(u_{i,h})$, normalization $b(u_{i,h}, u_{i,h}) = 1$

$$V_h = \bigoplus_{i=1}^{N(h)} E_{i,h}$$

Convergence of eigenvalues/eigenfunctions

Some notation

$m : \mathbb{N} \to \mathbb{N}$ such that for $N \in \mathbb{N}$

$\lambda_{m(1)} < \lambda_{m(2)} < \cdots < \lambda_{m(N)} < \dots$

$\hat{\delta}(E, F) = \max(\delta(E, F), \delta(F, E))$, where E, F subspaces of H

$$\delta(E, F) = \sup_{u \in E,\ ||u||_H = 1} \inf_{v \in F} ||u - v||_H$$

Definition of convergence

<Comments>

$\forall \varepsilon > 0,\ \forall N \in \mathbb{N},\ \exists h_0 > 0$ such that $\forall h \le h_0$

- $\displaystyle \max_{i=1,\dots,m(N)} |\lambda_i - \lambda_{i,h}| \le \varepsilon$
- $\displaystyle \hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h}\right) \le \varepsilon$

Convergence of eigenmodes (cont'ed)

Uniform convergence

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

Theorem

If T is selfadjoint and compact

$$\boxed{\text{Uniform convergence}} \iff \boxed{\text{Eigenmodes convergence}}$$

Strategy

1) prove uniform convergence,
2) estimate the order of convergence

Galerkin approximation of compact operators

Bramble–Osborn '73
Osborn '75
Kolata '78

Céa's Lemma

$T_h = P_h T$, with P_h projection w.r.t. bilinear form a

$$T - T_h = (I - P_h)T$$

If $I - P_h$ converges to zero pointwise and T is compact, then $T - T_h$ converges to zero uniformly (consequence of Banach–Steinhaus theorem)

<Proof>

Estimating the order of convergence

If

$$||T - T_h||_{\mathcal{L}(V,V)} = \varepsilon(h)$$
$$||T^* - T_h^*||_{\mathcal{L}(V,V)} = \varepsilon^*(h)$$

Kolata '78
Babuška–Osborn '91

then

$$|\lambda_i - \lambda_{i,h}| \le C(\varepsilon(h)\varepsilon^*(h))^{1/\alpha_i} \quad \alpha_i \text{ ascent of } \lambda_i I - T$$
$$\hat{\delta}(E_i, E_{i,h}) \le C\varepsilon(h)$$

If T symmetric then

$$|\lambda_i - \lambda_{i,h}| \le C\varepsilon(h)^2$$
$$\hat{\delta}(E_i, E_{i,h}) \le C\varepsilon(h)$$

Moreover, one can prove $\lambda_{i,h} \ge \lambda_i$ $\forall i$ (from $V_h \subset V$)

Conclusions on standard Galerkin approximation

Any finite element choice which provides a (pointwise) convergent scheme for the approximation of a problem with compact resolvent can be successfully applied to the approximation of the corresponding eigenvalue problem (uniform convergence)

No extra compatibility!

Eigenproblems in mixed form

Mercier–Osborn–Rappaz–Raviart '81
B.–Brezzi–Gastaldi '97-'00

Let's start with Laplace problem $-\Delta u = g$

Source problem

$\sigma \in H(\mathrm{div};\Omega) = \Sigma,\ \ u \in L^2(\Omega) = U$

$$\begin{cases} (\sigma,\tau) + (\mathrm{div}\,\tau, u) = 0 & \forall\tau\in\Sigma \quad (\sigma = \nabla u) \\ (\mathrm{div}\,\sigma, v) = -(g, v) & \forall v\in U \quad (-\mathrm{div}\,\sigma = g) \end{cases}$$

Matrix form $(\Sigma_h \subset \Sigma,\ U_h \subset U)$

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde\sigma \\ \tilde u \end{pmatrix} = \begin{pmatrix} 0 \\ -\tilde g \end{pmatrix}$$

Laplace eigenproblem in mixed form

Find $\lambda \in \mathbb{R}$ such that for some $(\sigma, u) \in \Sigma \times U$ with $u \not\equiv 0$

$$\begin{cases} (\sigma,\tau) + (\operatorname{div}\tau, u) = 0 & \forall \tau \in \Sigma \\ (\operatorname{div}\sigma, v) = -\lambda(u,v) & \forall v \in U \end{cases} \qquad (-\Delta u = \lambda u)$$

Matrix form $(\Sigma_h \subset \Sigma$, $U_h \subset U)$

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = -\tilde{\lambda} \begin{pmatrix} 0 & 0 \\ 0 & M_U \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix}$$

Remark. Similarly, one can deal with problems of the type

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = \lambda \begin{pmatrix} M_\Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix}$$

Definition of the resolvent operator

A first natural (but wrong) definition

$T_1 : U \to \Sigma \times U$

$T_1(g) = (\sigma, u)$

One would like to compute eigenvalues. . .

$T_2 : (\Sigma \times U)' \to \Sigma \times U$

$T_2(f, g) = (\sigma, u)$ with

$$\begin{cases} (\sigma, \tau) + (\operatorname{div} \tau, u) = < f, v > & \forall \tau \in \Sigma \\ (\operatorname{div} \sigma, v) = -(g, v) & \forall v \in U \end{cases}$$

$$T_{\Sigma U} \quad \boxed{\begin{array}{c} (f, g) \xmapsto{\text{cutoff}} (0, g) \xmapsto{T_2} (\sigma, u) \\ L^2 \times L^2 \longrightarrow L^2 \times L^2 \end{array}} \quad \text{is compact}$$

Uniform convergence?

Let's try to follow Kolata's argument

$T_{\Sigma U} - T_{\Sigma U,h} = (I - Q_h)T_{\Sigma U}$

✓ $||(I - Q_h)(\sigma, u)||_{\Sigma\times U} \to 0$ for all $(\sigma, u) \in \Sigma \times U$

✗ $T_{\Sigma U} : L^2 \times L^2 \to \Sigma \times U$ is not compact

✗ $T_{\Sigma U} : \Sigma \times U \to \Sigma \times U$ is not compact either

Remark: standard mixed estimates don't help

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \leq C \inf_{\tau_h, v_h} (\underbrace{||\sigma - \tau_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

A counterexample

(Fix–Gunzburger–Nicolaides ’81)
B.–Brezzi–Gastaldi ’00

Ω square of side π
Mesh of subsquares divided into 4 triangles (crisscross)

$\Sigma_h = \{$continuous p.w. linears (componentwise)$\}$
$U_h = \operatorname{div} \Sigma_h \subset \{$p.w. constants$\}$

Theorem (compatibility)
For all $g \in L^2(\Omega)$ there exists a unique solution $(\sigma_h, u_h) \in \Sigma_h \times U_h$ such that
$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \to 0$

That is $||(I - Q_h)(\sigma, u)||_{\Sigma\times U} \to 0$ pointwise (inf-sup condition)

A counterexample (cont'ed)

exact	computed			
1.00000	1.00428	1.00190	1.00107	1.00068
1.00000	1.00428	1.00190	1.00107	1.00069
2.00000	2.01711	2.00761	2.00428	2.00274
4.00000	4.06804	4.03037	4.01710	4.01095
4.00000	4.06804	4.03037	4.01710	4.01095
5.00000	5.10634	5.04748	5.02674	5.01712
5.00000	5.10634	5.04748	5.02674	5.01712
	5.92293	5.96578	5.98074	5.98767
8.00000	8.27128	8.12151	8.06845	8.04383
9.00000	9.34085	9.15309	9.08640	9.05537
9.00000	9.34085	9.15309	9.08640	9.05537
d.o.f.	254	574	1022	1598

Spurious eigenvalues

<See them>

Definition of the resolvent operator (cont'ed)

A better definition

$T_U : U \to U$

$\sigma \in \Sigma$, $T_U g \in U$ such that

$$\begin{cases} (\sigma, \tau) + (\operatorname{div} \tau, T_U g) = 0 & \forall \tau \in \Sigma \\ (\operatorname{div} \sigma, v) = -(g, v) & \forall v \in U \end{cases}$$

Remark: operator is now compact, but standard mixed estimates don't help again

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \le C \inf_{\tau_h, v_h} (\underbrace{||\sigma - \tau_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

Remark: we need an estimate for u_h which does not involve $\operatorname{div} \sigma$

Uniform convergence $||T_U - T_{U,h}|| \to 0$

- Ellipticity in the kernel
  $||\tau_h||^2_{L^2} \geq \alpha||\tau_h||^2_{\Sigma}$ for all $\tau_h \in \Sigma_h$ s.t. $\{(\operatorname{div}\tau_h, v) = 0,\ \forall v \in U_h\}$
- Fortin operator $\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.
  $(\operatorname{div}(\sigma - \Pi_h\sigma), v) = 0 \quad \forall v \in U_h$
  $||\Pi_h\sigma||_\Sigma \leq C||\sigma||_{\Sigma^+}$

$$||\sigma - \sigma_h||_{L^2} \leq C\left(||\sigma - \Pi_h\sigma||_{L^2} + (1/\sqrt{\alpha}) \inf_{v_h\in U_h} ||u - v_h||_U\right)$$

$$||u - u_h||_U \ \leq C\left(\inf_{v_h\in U_h} ||u - v_h||_U + ||\sigma - \sigma_h||_{L^2}\right)$$

<Proof>

Fortid condition

Definition

The spaces Σ_h, U_h satisfy the Fortid condition if there exists a Fortin operator which converges strongly to the identity operator, namely

$\Pi_h : \Sigma^+ \longrightarrow \Sigma_h$ s.t.

$$(\mathrm{div}(\sigma - \Pi_h\sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h\sigma||_\Sigma \leq C||\sigma||_{\Sigma^+}$$

$$\boxed{||I - \Pi_h||_{\mathcal{L}(\Sigma^+,L^2)} \longrightarrow 0}$$

Final convergence result

Theorem

Assume ellipticity in the kernel and Fortid condition

For any $N \in \mathbb{N}$ define $\rho_N(h) :]0,1] \to \mathbb{R}$ as

$$\rho_N(h) = \sup_{u \in \bigoplus_{i=1}^{m(N)} E_i} \left(\inf_{v_h} ||u - v_h||_U + || \nabla u - \Pi_h \nabla u||_{L^2} \right)$$

Then $||T_U - T_{U,h}||_{\mathcal{L}(U,U)} \to 0$ and the following estimates hold true

$$\sum_{i=1}^{m(N)} |\lambda_i - \lambda_{i,h}| \leq C(\rho_N(h))^2$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h} \right) \leq C \rho_N(h)$$

Back to the counterexample

Crisscross mesh

$\Sigma_h = \{$continuous p.w. linears (componentwise)$\}$

$U_h = \operatorname{div} \Sigma_h \subset \{$p.w. constants$\}$

Theorem

With the above choice of spaces, there exist a sequence $\{g_h\} \subset U$ with $\|g_h\|_0 = 1$ s.t.

$$\boxed{\|u - u_h\|_U \not\to 0}$$

that is $\|T_U - T_{U,h}\|_{\mathcal{L}(U,U)} \not\to 0$

Proof

Use inequality by Qin '94 based on idea of Boland–Nicolaides '85

<More detail>

A positive example

General mesh (triangles, parallelograms, tetrahedrons, parallelepipeds)
Σ_h : Raviart–Thomas space of order k
U_h : $\mathcal{P}_{k-1}$ or tensor product polynomials $\mathcal{Q}_{k-1}$

Fortid: the interpolant is a Fortin operator.

See also Falk–Osborn '80

Convergence: $O(h^{2k})$ eigenvalues, $O(h^k)$ eigenfunctions.

General theory

$\begin{pmatrix}\mathbf{0}\\ \mathbf{g}\end{pmatrix}$-type problems

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & 0 \\ 0 & M_U \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix}$$

Laplace, biharmonic, Maxwell (1), . . .

$\begin{pmatrix}\mathbf{f}\\ \mathbf{0}\end{pmatrix}$-type problems

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = \lambda \begin{pmatrix} M_\Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix}$$

Stokes, Laplace with Lagrange multipliers, Maxwell (2), . . .

<More detail>

Applications: Maxwell

Maxwell system for vector phasors

$$\begin{aligned}
&\operatorname{curl}\mathcal{E} = -j\omega\mu\mathcal{H} && \text{in } \Omega\\
&\operatorname{div}(\varepsilon\mathcal{E}) = 0 && \text{in } \Omega\\
&\operatorname{curl}\mathcal{H} = j\omega\varepsilon\mathcal{E} && \text{in } \Omega\\
&\operatorname{div}(\mu\mathcal{H}) = 0 && \text{in } \Omega\\
&\mathcal{E}\times\mathbf{n} = 0 && \text{on } \partial\Omega\\
&(\mu\mathcal{H})\cdot\mathbf{n} = 0 && \text{on } \partial\Omega
\end{aligned}$$

Eliminating $\mathcal{H}$ gives

$$\begin{aligned}
&\operatorname{curl}(\mu^{-1}\operatorname{curl}\mathcal{E}) = \omega^2\varepsilon\mathcal{E} && \text{in } \Omega\\
&\operatorname{div}(\varepsilon\mathcal{E}) = 0 && \text{in } \Omega\\
&\mathcal{E}\times\mathbf{n} = 0 && \text{on } \partial\Omega
\end{aligned}$$

The model problem

Remark: for simplicity of notation $\varepsilon, \mu = 1$, set $\mathbf{u} = \mathcal{E}$, $\lambda = \omega^2$ (see also Fernandes–Gilardi '97 and Caorsi–Fernandes–Raffetto '00)

Find $\lambda \in \mathbb{R}$ s.t. for some $\mathbf{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ with $\mathbf{u} \not\equiv 0$

$$(\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$$

$\lambda = 0$ is not an eigenvalue

Drop the divergence free constraint: $\mathbf{u} = \nabla\varphi + \mathrm{curl}\,\psi$

- $\nabla\varphi \rightsquigarrow \lambda = 0$
- $\mathrm{curl}\,\psi \rightsquigarrow \lambda > 0$ are the same as before

The model problem (cont'ed)

Find $\lambda \in \mathbb{R}$ s.t. $\lambda \neq 0$ and for some $\mathbf{u} \in H_0(\mathrm{curl};\Omega) = \Sigma$ with $\mathbf{u} \not\equiv 0$

$$(\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \Sigma$$

Same eigenvalues as before!

Finite element approximation

$\Sigma_h \subset \Sigma$

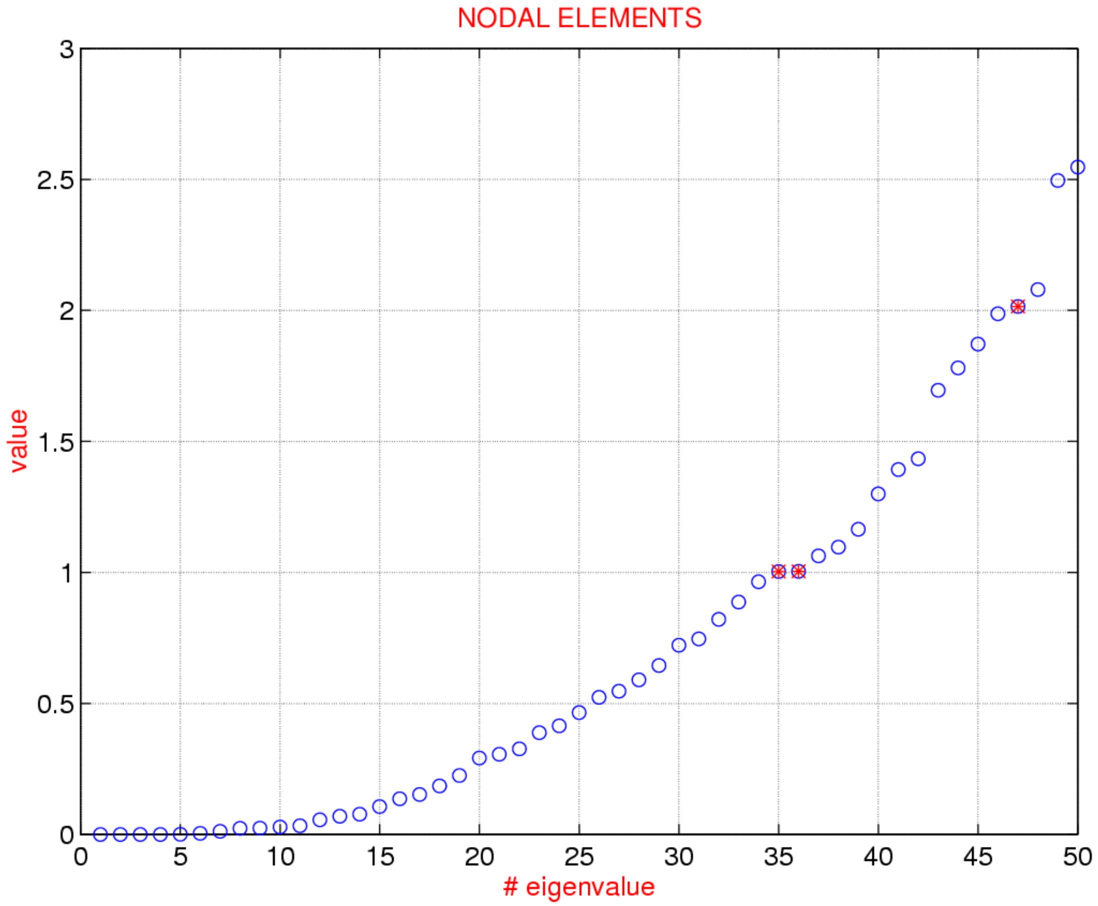
Find $\lambda_h \in \mathbb{R}$ s.t. $\lambda_h \neq 0$ and for some $\mathbf{u}_h \in \Sigma_h$ with $\mathbf{u}_h \not\equiv 0$

$$(\mathrm{curl}\,\mathbf{u}_h, \mathrm{curl}\,\mathbf{v}) = \lambda_h(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in \Sigma_h$$

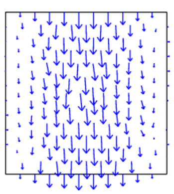
Numerical tests: nodal elements

B.–Fernandes–Gastaldi–Perugia ’99

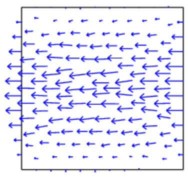
It is well-known that the use of nodal elements is not practicable



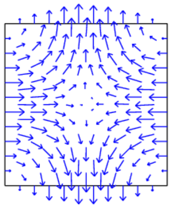
1st eig.



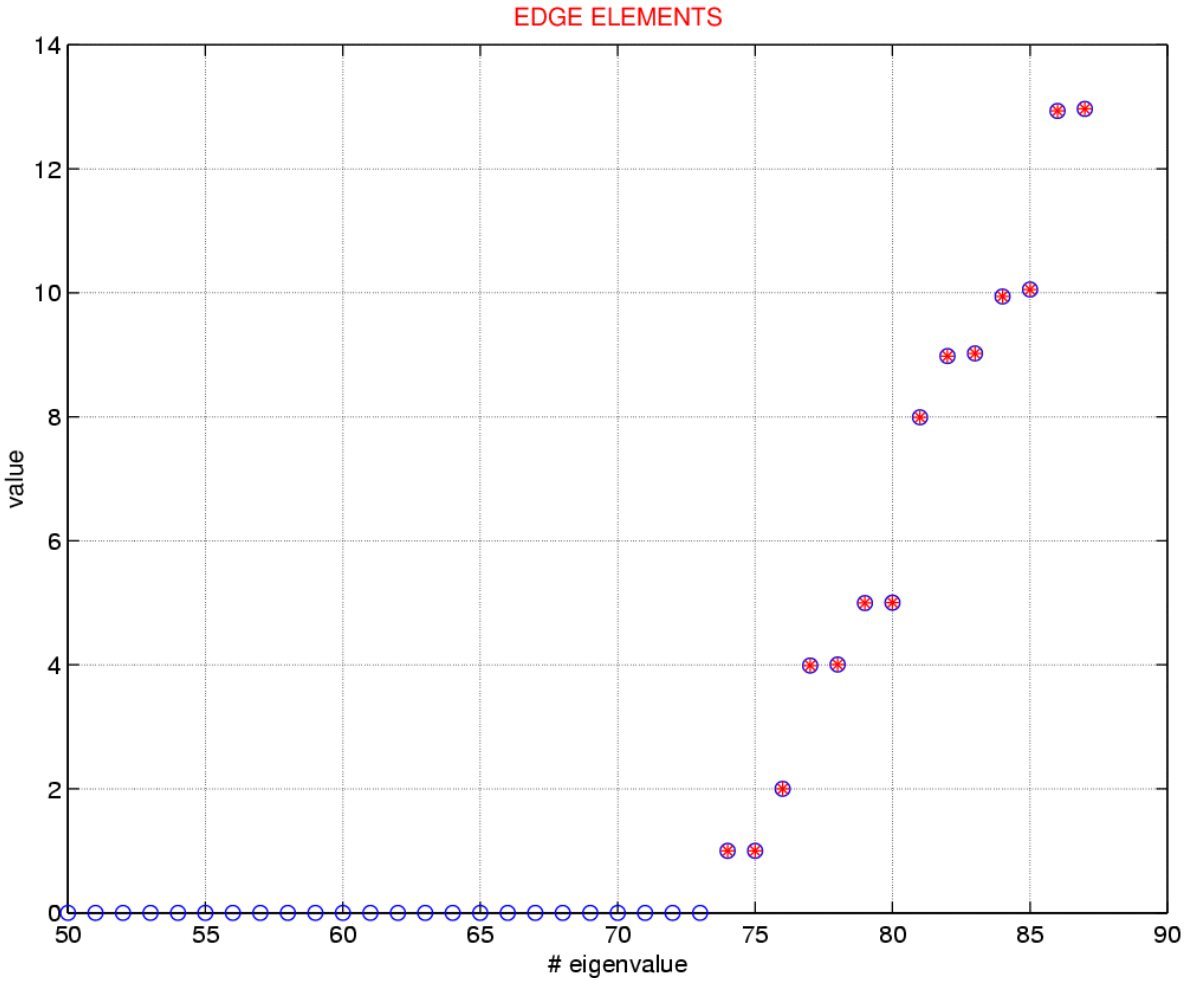
2nd eig.

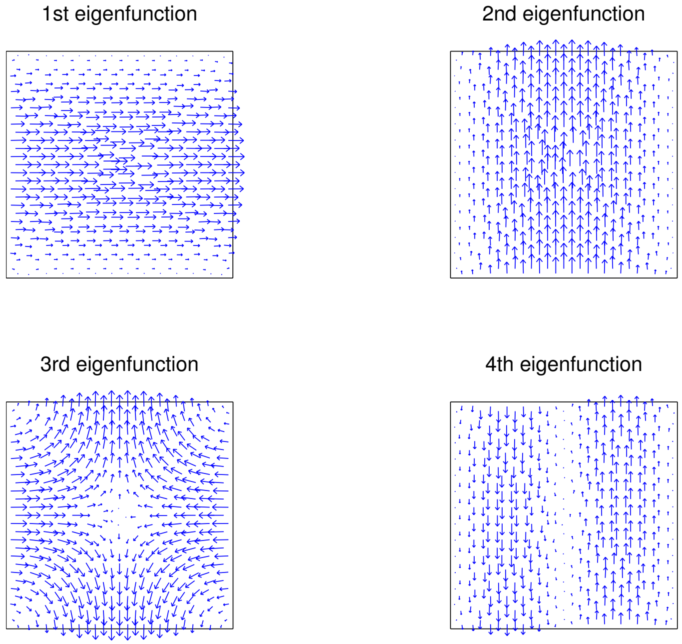


3rd eig.



Numerical tests (cont'ed): edge elements



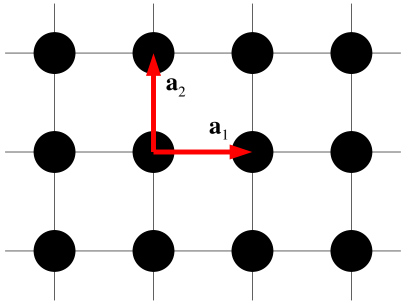


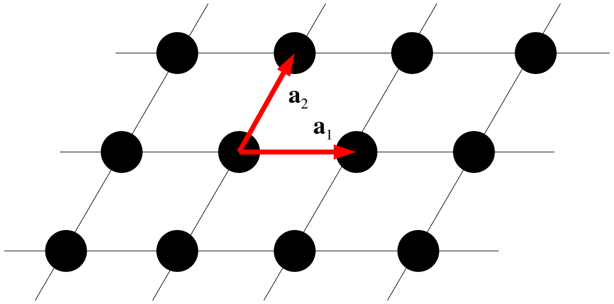
<More detail>

Applications: photonic crystals

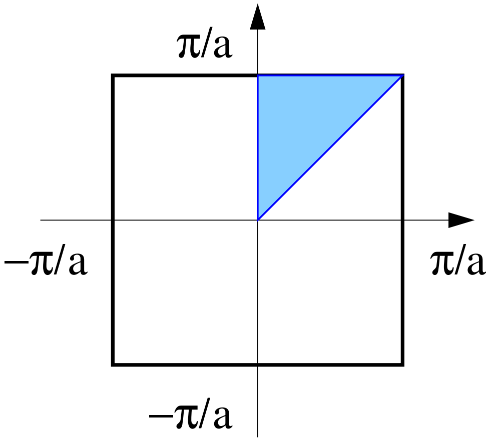
Periodic structures

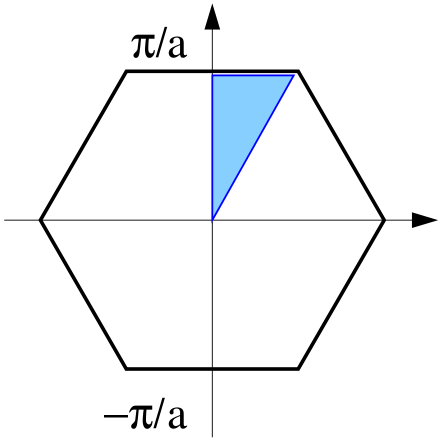
B.–Conforti–Gastaldi ’04





Bloch Flochet analysis, first Brillouin zone





Photonic crystals (cont'd)

Bloch waves $H(x) = e^{i\alpha\cdot\mathbf{x}}\mathbf{u}(\mathbf{x})$, where $\mathbf{u}$ periodic and $\alpha \in K$

$$\begin{aligned} &\nabla_\alpha \times \varepsilon^{-1}\nabla_\alpha \times \mathbf{u} = \omega^2\mathbf{u} && \text{in } \Omega \\ &\nabla_\alpha \cdot \mathbf{u} = 0 && \text{in } \Omega \end{aligned}$$

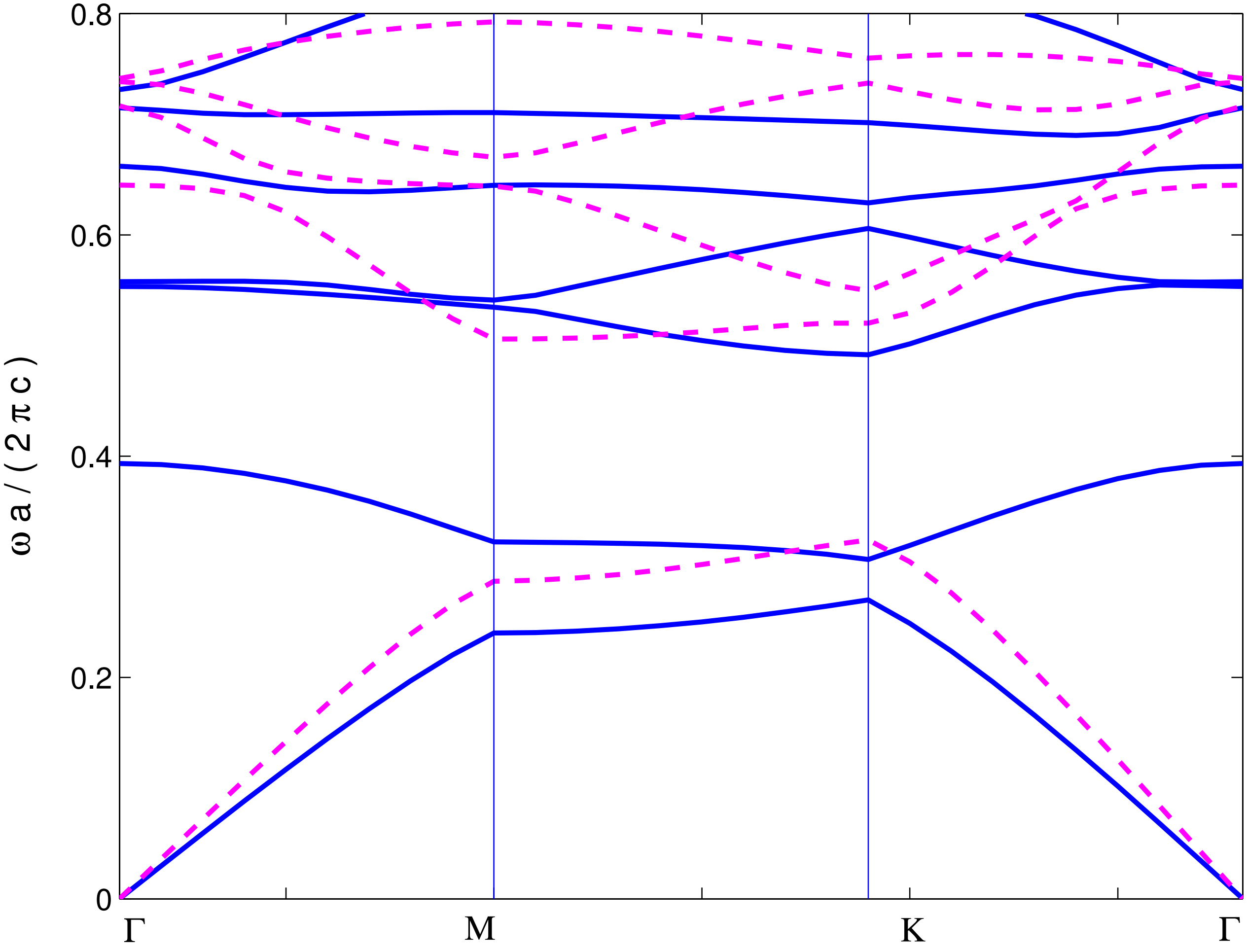
with periodic boundary conditions and with

$$\nabla_\alpha = \nabla + i\alpha$$

Natural idea: de Rham complex for the operator ∇_α and extend analysis performed for Maxwell (see also Dobson–Pasciak '01)

<More detail>

Numerical tests (modified edge elements)



Applications: evolution problem in mixed form

Johnson–Thomée '81
B.–Gastaldi '02

Conditions for a good approximation (?)

Numerical test: heat equation in mixed form.

Space semidiscretization:

$\sigma_h(t) \in \Sigma_h \subset H(\mathrm{div};\Omega)$, $u_h(t) \in V_h \subset L^2$

$$\begin{aligned} &(\sigma_h(t),\tau) + (\operatorname{div}\tau, u_h(t)) = 0 && \forall\tau \in \Sigma_h \\ &(\operatorname{div}\sigma_h(t), v) - \frac{d}{dt}(u_h(t), v) = -(f, v) && \forall v \in V_h \end{aligned}$$

Approximation of the time derivative: implicit Euler

Choice of the discrete spaces

$\Omega =]0,\pi[^2$, mesh of triangles

☛ RT: $\Sigma_h =$ lowest order Raviart–Thomas, $V_h =$ p.w. constants

- ▶ OK for steady problems Raviart–Thomas '77
- ▶ OK for eigenvalue problems Falk–Osborn '80

☛ P1: $\Sigma_h =$ continuous p.w. linears, $V_h = \operatorname{div}\Sigma_h$

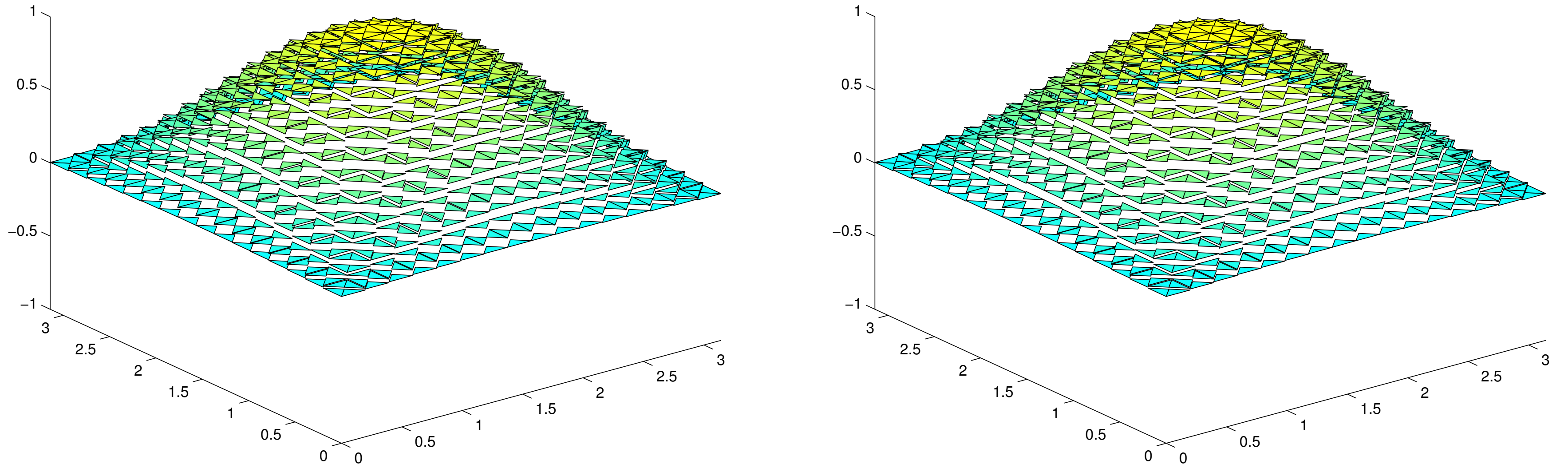
- ▶ OK for steady problems B.–Brezzi–Gastaldi '00
- ▶ Spurious eigenvalues B.–Brezzi–Gastaldi '00

Numerical results: $\mathbf{g = 2\sin x \sin y}$

$T = 10$, $dt = 0.1$, 16×16 mesh

L.=RT

R.=P1

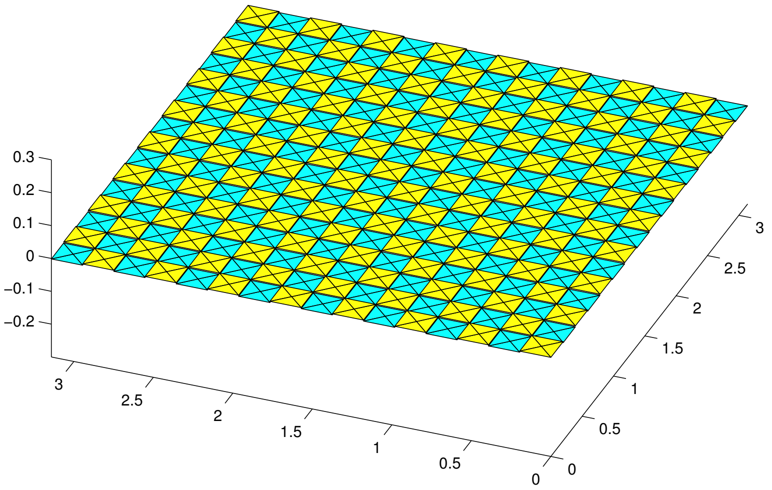


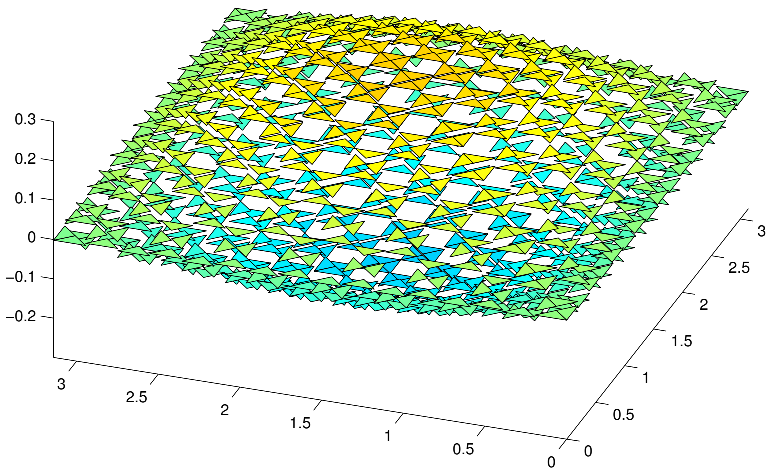
A more sophisticated example

$T = 10$, $dt = 0.1$, 16×16 mesh

Now $f = f(h)$ is a 16-by-16 checkerboard with values ± 1

L.=RT R.=P1

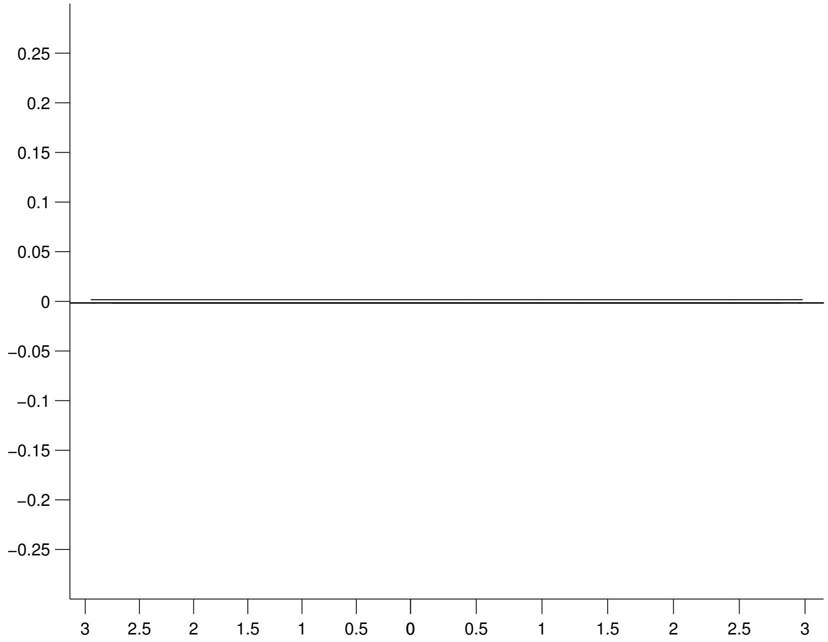


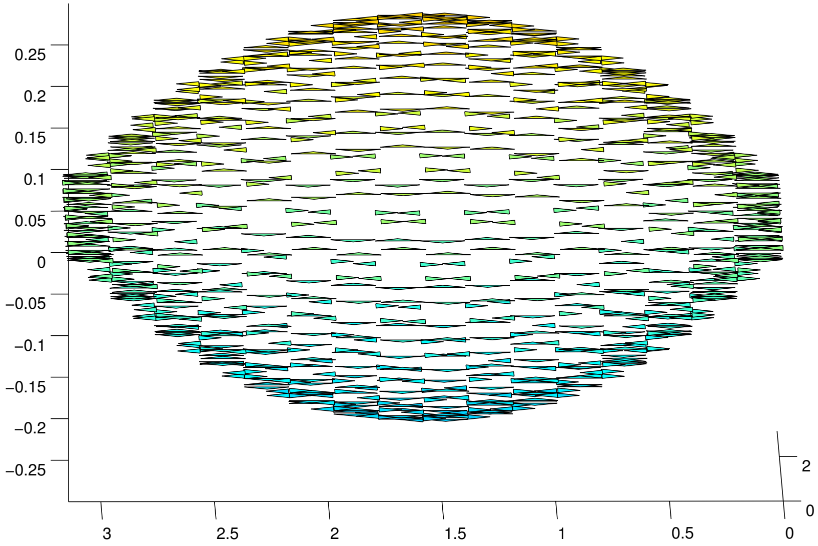


Same pictures from a different angle

L.=RT

R.=P1





<More detail>

Theorems of convergence

Continuous problem: $\sigma(t) \in \Sigma$, $u(t) \in V$ s.t.

$$
\begin{aligned}
&a(\sigma(t),\tau) + b(\tau, u(t)) = 0 && \forall \tau \in \Sigma \\
&b(\sigma(t), v) - \frac{d}{dt}(u(t), v) = -_{V'}\!< g(t), v >_V && \forall v \in V \\
&u(0) = u_0
\end{aligned}
$$

Discrete problem $\sigma_h(t) \in \Sigma_h$, $u_h(t) \in V_h$ s.t.

$$
\begin{aligned}
&a(\sigma_h(t),\tau) + b(\tau, u_h(t)) = 0 && \forall \tau \in \Sigma_h \\
&b(\sigma_h(t), v) - \frac{d}{dt}(u_h(t), v) = -_{V'}\!< g(t), v >_V && \forall v \in V_h \\
&u_h(0) = u_{0,h}
\end{aligned}
$$

First convergence theorem: $\boxed{\mathbf{B}(\mathbf{\Sigma_h}) \subset \mathbf{V_h}}$

$V \subset H \simeq H' \subset V'$

Π Fortin operator

$\Pi : \Sigma \to \Sigma_h$

$b(\tau - \Pi\tau, v) = 0 \quad \forall v \in V_h$

P projection operator

$P : V \to V_h$

$(w - Pw, v) = 0 \quad \forall v \in V_h$

Then:

$$\max_{t\in[0,T]} ||u(t) - u_h(t)||_H^2 + \int_0^T ||\sigma(t) - \sigma_h(t)||_a^2 \, dt \leq$$

$$||Pu_0 - u_{0,h}||_H^2 + \max_{t\in[0,T]} ||u(t) - Pu(t)||_H^2 + 2\int_0^T ||\sigma(t) - \Pi\sigma(t)||_a^2 \, dt$$

Second convergence theorem

Π and P *elliptic projection*

$$\max_{t\in[0,T]} ||u(t) - u_h(t)||_H^2 + \int_0^T ||\sigma(t) - \sigma_h(t)||_a^2 \, dt \le$$
$$||Pu_0 - u_{0,h}||_H^2 + \max_{t\in[0,T]} ||u(t) - Pu(t)||_H^2 +$$
$$2\int_0^T ||\sigma(t) - \Pi\sigma(t)||_a^2 \, dt + C\int_0^T \left\| \frac{\partial}{\partial t}(u(t) - Pu(t)) \right\|_H^2 dt$$

Conclusions

- Standard Galerkin approximation of eigenvalue problems is straightforward extension of usual theory

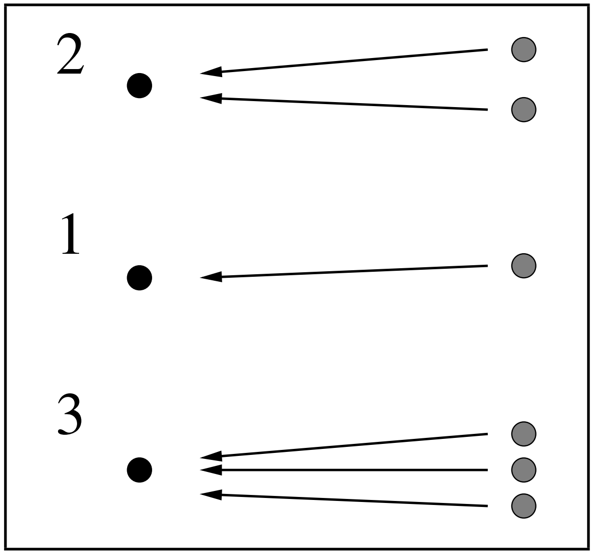
- Mixed problems require special theory (two different theories for $\binom{f}{0}$ and $\binom{0}{g}$ problems)

- Theory applies to several problems (Maxwell, photonic crystals)

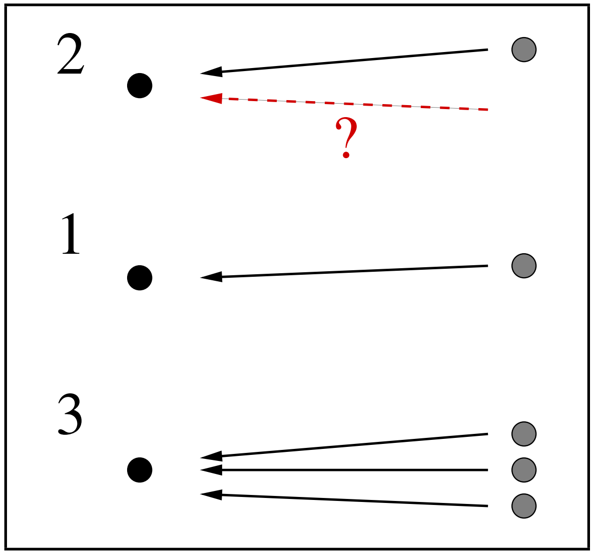
- Evolution problems in mixed form

The end

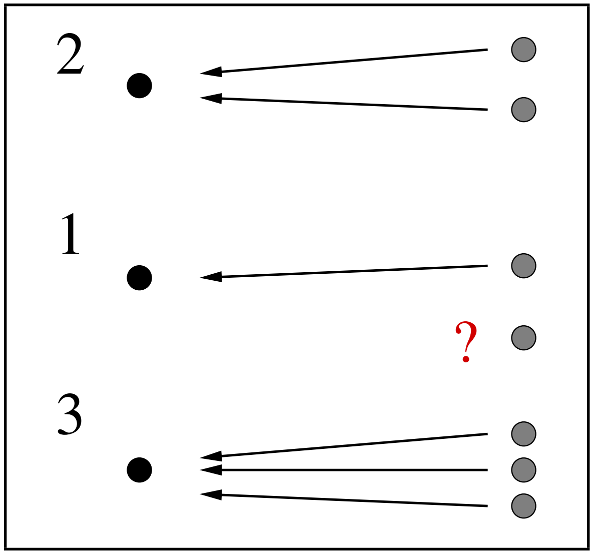
Examples of good or bad convergence



Good



Bad (missing)



Bad (spurious)

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$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

$||(I - P_h)(u)||_H \to 0$ for any $u \in V$
Banach-Steinhaus thm. implies $||I - P_h||_{\mathcal{L}(V,H)} \leq C$

By contradiction $||(I - P_h)T||_{\mathcal{L}(H,H)} \not\to 0$
Then (up to subsequences) $||(I - P_h)Tf_h||_H \not\to 0$ with $||f_h||_H \leq 1$

If $T : H \to V$ is compact then $Tf_h \to w$ in V

Take h small enough s.t. $||Tf_h - w||_V + ||(I - P_h)w||_H \leq \varepsilon$, then
$||(I-P_h)Tf_h||_H \leq ||(I-P_h)(Tf_h-w)||_H + ||(I-P_h)w||_H \leq (C+1)\varepsilon$

Remark. Same argument shows $||T - T_h||_{\mathcal{L}(V,V)} \to 0$ under the hypotheses $T : V \to V$ compact and $||I - P_h||_{\mathcal{L}(V,V)} \leq C$

$||T - T_h||_{\mathcal{L}(H,H)} \to 0$ (cont'd)

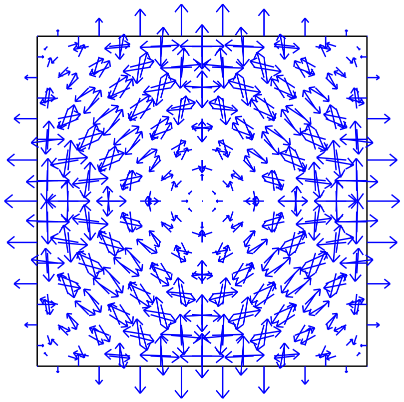
A direct proof using a priori estimates

$$||(T - T_h)f||_H = ||u - u_h||_H \leq Ch^{\alpha}||f||_H$$

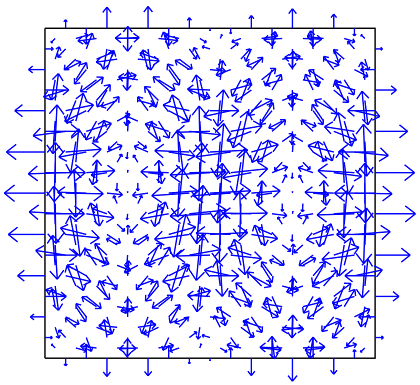
<Back>

Spurious eigenfunctions

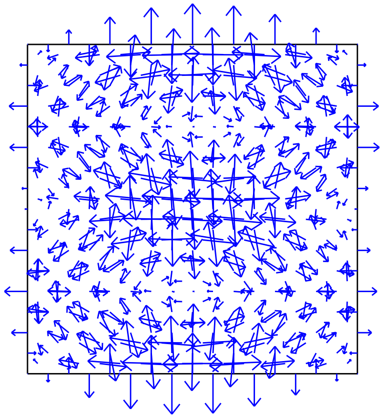
1st spurious eigenfunction



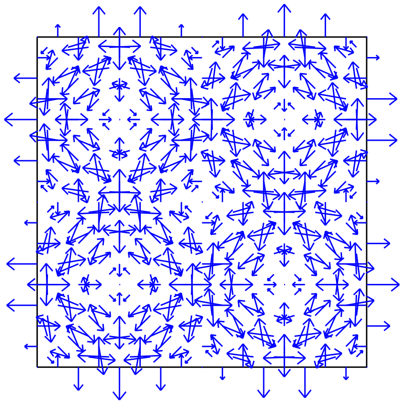
2nd spurious eigenfunction



3rd spurious eigenfunction



4th spurious eigenfunction



<Back>

$P = L^2$-projection onto U_h

$$||\Pi_h\sigma - \sigma_h||^2_{L^2} = (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) + (\sigma - \sigma_h, \Pi_h\sigma - \sigma_h)$$

$$= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) - (\mathrm{div}(\Pi_h\sigma - \sigma_h), u - Pu)$$

$$\leq ||\Pi_h\sigma - \sigma||_{L^2}||\Pi_h\sigma - \sigma_h||_{L^2} + ||\,\mathrm{div}(\Pi_h\sigma - \sigma_h)||_{L^2}||u - Pu||_U$$

$$\leq ||\Pi_h\sigma - \sigma_h||_{L^2}\left(||\Pi_h\sigma - \sigma||_{L^2} + (1/\sqrt{\alpha})||u - Pu||_U\right)$$

$$||Pu - u_h||_U \leq C \sup_{\tau_h} \frac{(Pu - u_h, \mathrm{div}\,\tau_h)}{||\tau_h||_\Sigma}$$

$$\leq C \sup_{\tau_h} \frac{(Pu - u, \mathrm{div}\,\tau_h) + (u - u_h, \mathrm{div}\,\tau_h)}{||\tau_h||_\Sigma}$$

$$\leq C\left(||Pu - u||_U + \sup_{\tau_h} \frac{-(\sigma - \sigma_h, \tau_h)}{||\tau_h||_\Sigma}\right)$$

$$\leq C\left(||Pu - u||_U + ||\sigma - \sigma_h||_{L^2}\right)$$

<Back>

Inf-sup condition for our problem

$$\inf_{v\in U_h}\sup_{\tau\in\Sigma_h}\frac{(\operatorname{div}\tau, v)}{\|\tau\|_{H(\operatorname{div})}\|v\|_{L^2}}\ge\beta>0$$

cannot be improved: there exists e sequence (properly chosen linear combination of checkerboards on the macroelements) $\{\tilde{v}_h\}$ such that

$$|(\operatorname{div}\tau,\tilde{v}_h)|\le C\|\tau\|_{L^2}\|\tilde{v}_h\|_{L^2}$$

Take $g_h=\tilde{v}_h/\|\tilde{v}_h\|_{L^2}$, then $g_h\rightharpoonup 0$ weakly in L^2 and $Tg_h\to 0$ strongly in L^2.

Let's prove that $\|T_hg_h\|_{L^2}$ is bounded below away from zero (uniformly).

$$
\begin{aligned}
&(\sigma_h, \tau) + (\operatorname{div} \tau, u_h) = 0 \quad && \forall \tau \in \Sigma_h \\
&(\operatorname{div} \sigma_h, v) = -(g_h, v) && \forall v \in U_h
\end{aligned}
$$

$$|(\operatorname{div} \sigma_h, u_h)| = |(g_h, u_h)| \leq \|u_h\|_{L^2}$$

$$|(\operatorname{div} \sigma_h, u_h)| = \|\sigma_h\|_{L^2}^2$$

$$\|g_h\|_{L^2} \|\sigma_h\|_{L^2} \geq \frac{1}{C} |(\operatorname{div} \sigma_h, g_h)| = \frac{1}{C} (g_h, g_h) = \frac{1}{C}$$

$$\|u_h\|_{L^2} \geq \frac{1}{C^2}$$

<Back>

$\binom{\mathbf{0}}{\mathbf{g}}$ **case**

$U \subset H \simeq H' \subset U'$

Weak approximability of U_H^0 with respect to $a(\cdot,\cdot)$.

$$b(\tau_h, u) \le \rho(h)\|u\|_{U_H^0}\|\tau_h\|_a \quad \forall u \in U_H^0,\ \forall \tau_h \in \ker_h(B)$$

Strong approximability of U_H^0.
For each $u \in U_H^0$ there exists $u^I \in U_h$ such that

$$\|u - u^I\| \le \rho(h)\|u\|_{U_H^0}$$

$+$ Fortid

$\binom{f}{0}$ case

$\Sigma \subset H \simeq H' \subset \Sigma'$

Weak approximability of U_0^H.

$$\sup_{\tau_h \in \ker_h(B)} \frac{b(\tau_h, u)}{\|\tau_h\|_\Sigma} \leq \rho(h) \|u\|_{U_0^H}$$

Strong approximability of Σ_0^H.
For each $\sigma \in \Sigma_0^H$ there exists $\sigma^I \in \ker_h(B)$ such that

$$\|\sigma - \sigma^I\|_\Sigma \leq \rho(h) \|\sigma\|_{\Sigma_0^H}$$

$+$ DEK

<Back>

Mixed formulations

Kikuchi's formulation Kikuchi '89

Find $\lambda \in \mathbb{R}$ s.t. for some $\mathbf{u} \in \Sigma$ with $\mathbf{u} \not\equiv 0$ and $p \in H^1_0(\Omega) = Q$

$$(M_1) \quad \begin{array}{ll} (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) + (\mathbf{v}, \nabla p) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \Sigma \\ (\mathbf{u}, \nabla q) = 0 & \forall q \in Q \end{array}$$

Alternative mixed formulation B.–Fernandes–Gastaldi–Perugia '99

Find $\lambda \in \mathbb{R}$ s.t. for some $u \in \operatorname{curl}\Sigma = U$ with $u \not\equiv 0$ and $\sigma \in \Sigma$

$$(M_2) \quad \begin{array}{lll} (\sigma, \tau) + (\operatorname{curl}\tau, u) = 0 & \forall \tau \in \Sigma & \sigma = -\operatorname{curl} u \\ (\operatorname{curl}\sigma, v) = -\lambda(u, v) & \forall v \in U & \operatorname{curl}\sigma = -\lambda u \end{array}$$

Discrete mixed formulations

Kikuchi's formulation

Find $\lambda_h \in \mathbb{R}$ s.t. for some $\mathbf{u}_h \in \Sigma_h$ with $\mathbf{u}_h \not\equiv 0$ and $p_h \in Q_h$

$$(M_{1,h}) \quad \begin{cases} (\operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}) + (\mathbf{v}, \nabla p_h) = \lambda_h (\mathbf{u}_h, \mathbf{v}) & \forall \mathbf{v} \in \Sigma_h \\ (\mathbf{u}_h, \nabla q) = 0 & \forall q \in Q_h \end{cases}$$

Alternative mixed formulation

Find $\lambda_h \in \mathbb{R}$ s.t. for some $u_h \in U_h$ with $u_h \not\equiv 0$ and $\sigma_h \in \Sigma_h$

$$(M_{2,h}) \quad \begin{cases} (\sigma_h, \tau) + (\operatorname{curl} \tau, u_h) = 0 & \forall \tau \in \Sigma_h \\ (\operatorname{curl} \sigma_h, v) = -\lambda_h (u_h, v) & \forall v \in U_h \end{cases}$$

Theorem

$M_{1,h} \iff M_{2,h}$ and equiv. to original problem $(\lambda_h \neq 0)$ provided

$$\boxed{\nabla Q_h \subset \Sigma_h \text{ and } \operatorname{curl} \Sigma_h \subset U_h}$$

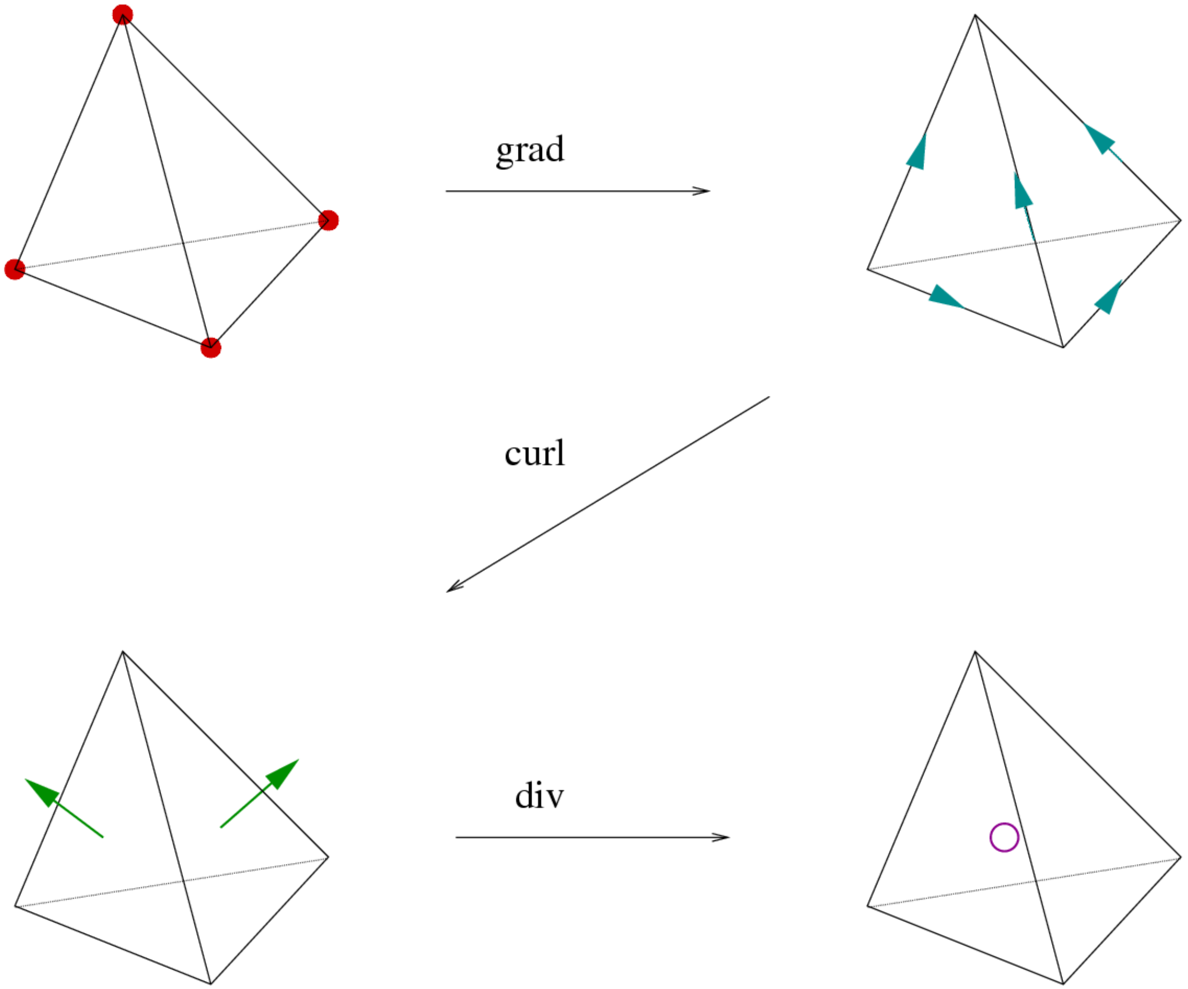
de Rham complex (commuting diagram property)

Douglas–Roberts '82
Bossavit '88

$Q \subset H^1_0(\Omega),\ \Sigma \subset H_0(\mathrm{curl}),\ U \subset H_0(\mathrm{div}),\ S \subset L^2(\Omega)$

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\nabla"] \arrow[d, "\Pi_h^Q"] & \Sigma \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi_h^\Sigma"] & U \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_h^U"] & S/\mathbb{R} \arrow[r] \arrow[d, "\Pi_h^S"] & 0 \\
0 \arrow[r] & Q_h \arrow[r, "\nabla"] & \Sigma_h \arrow[r, "\mathrm{curl}"] & U_h \arrow[r, "\mathrm{div"] & S_h/\mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

Lowest order edge elements



Conditions for good approximation of eigenvalues

- Ellipticity in the kernel

$$||\tau_h||^2_{L^2} \geq \alpha ||\tau_h||^2_\Sigma$$

for all $\tau_h \in \Sigma_h$ s.t. $\{(\mathrm{curl}\,\tau_h, v) = 0, \ \forall v \in U_h\}$

Always satisfied if $\mathrm{curl}\,\Sigma_h \subset U_h$

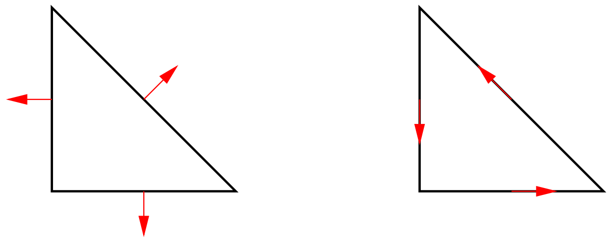
- Fortid condition $\Pi_h : \Sigma^+ \longrightarrow \Sigma_h$ s.t.

$$\begin{aligned} &(\mathrm{curl}(\sigma - \Pi_h\sigma), v) = 0 \quad \forall v \in U_h \\ &||\Pi_h\sigma||_\Sigma \leq C||\sigma||_{\Sigma^+} \\ &||I - \Pi_h||_{\mathcal{L}(\Sigma^+, L^2)} \longrightarrow 0 \end{aligned}$$

Fortid condition for 2D edge elements

Falk–Osborn '80

Two-dimensional triangular edge element is Raviart–Thomas element rotated by an angle of $\pi/2$



- The interpolant is a Fortin operator

Fortid condition for 3D edge elements

B. '00

The interpolant is no longer a Fortin operator. Indeed, one would need the following commuting diagram

$$\begin{tikzcd}
\Sigma \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi_h^\Sigma"] & U \arrow[d, "\Pi_h^U"] \\
\Sigma_h \arrow[r, "\mathrm{curl}"] & U_h
\end{tikzcd}$$

with Π_h^U equal to L^2-projection onto U_h, which is not the case in 3D

- $(\mathrm{curl}(\sigma - \Pi_h^\Sigma \sigma), v) = 0 \quad \forall v \in U_h$ is false in general

Fortid condition for 3D edge elements (cont'ed)

Definition of Fortin operator as solution of a mixed problem $\Pi_h\sigma \in \Sigma_h$, $u_h \in U_h$ such that

$$\begin{cases} (\Pi_h\sigma, \tau) + (\operatorname{curl}\tau, u_h) = 0 & \forall\tau \in \Sigma_h \\ (\operatorname{curl}\Pi_h\sigma, v) = (\operatorname{curl}\sigma, v) & \forall v \in U_h \end{cases}$$

It is clear that Π_h is Fortin (the problem above is stable in the usual sense for mixed problems)

We need Fortid, i.e.,

$$||\sigma - \Pi_h\sigma||_{L^2} \leq \rho(h)||\sigma||_{\Sigma^+}$$

See B. '00 (use inequality from Arnold–Falk–Winther '00)

<Back>

What result is correct?

$f = f(h) \rightharpoonup 0$ weakly in $L^2(\Omega)$, so that (by compactness)
$u(h) \to 0$ strongly in $L^2(\Omega)$

u_h computed with RT
$||u_h(T)||_{L^2} = 0.0050$
u_h computed with P1
$||u_h(T)||_{L^2} = 0.4274$

N	RT	P1
4	0.080756	0.451091
8	0.020186	0.430821
16	0.005047	0.427416
32	0.001262	0.426687

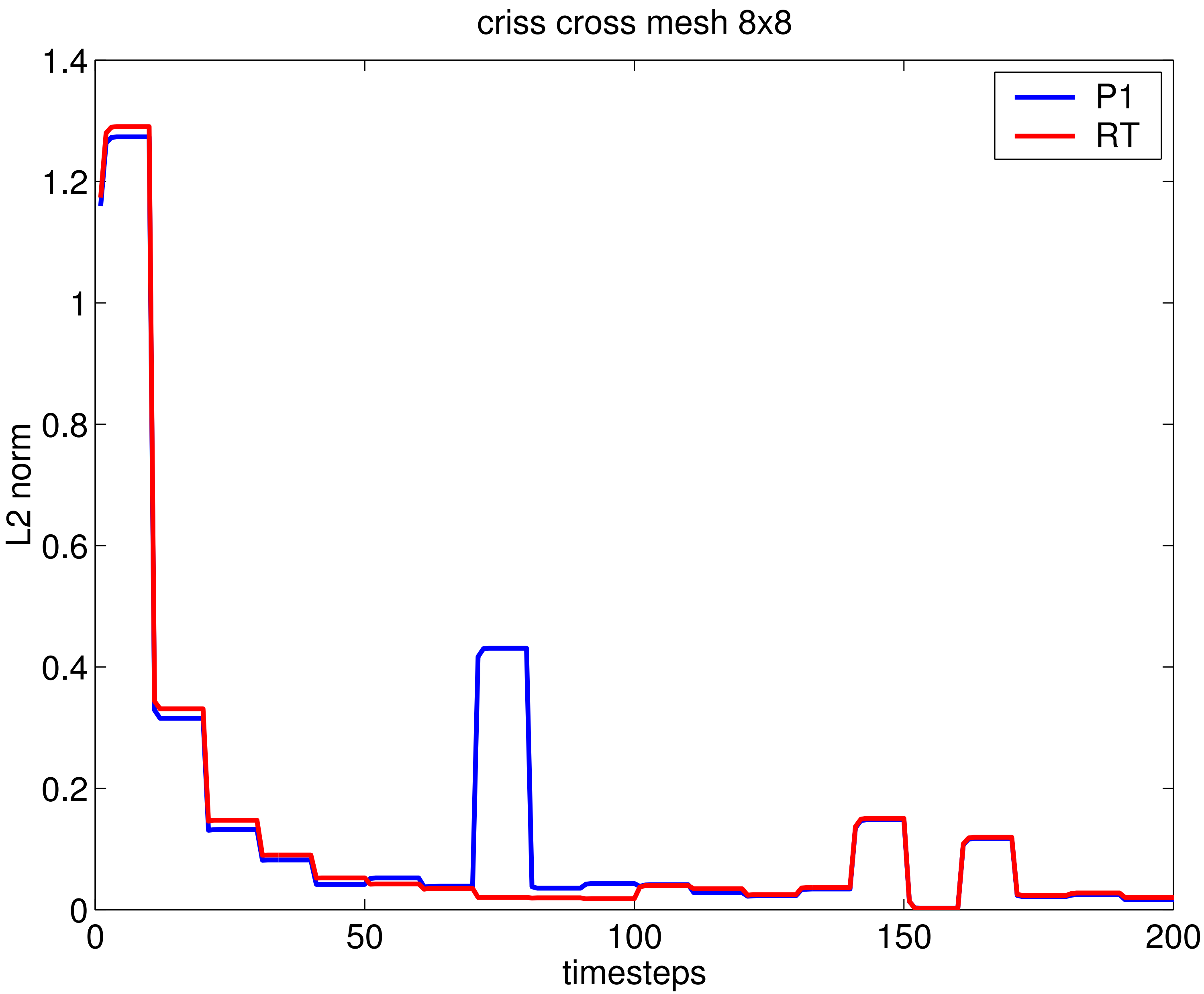
Why does the P1 method behave so badly?

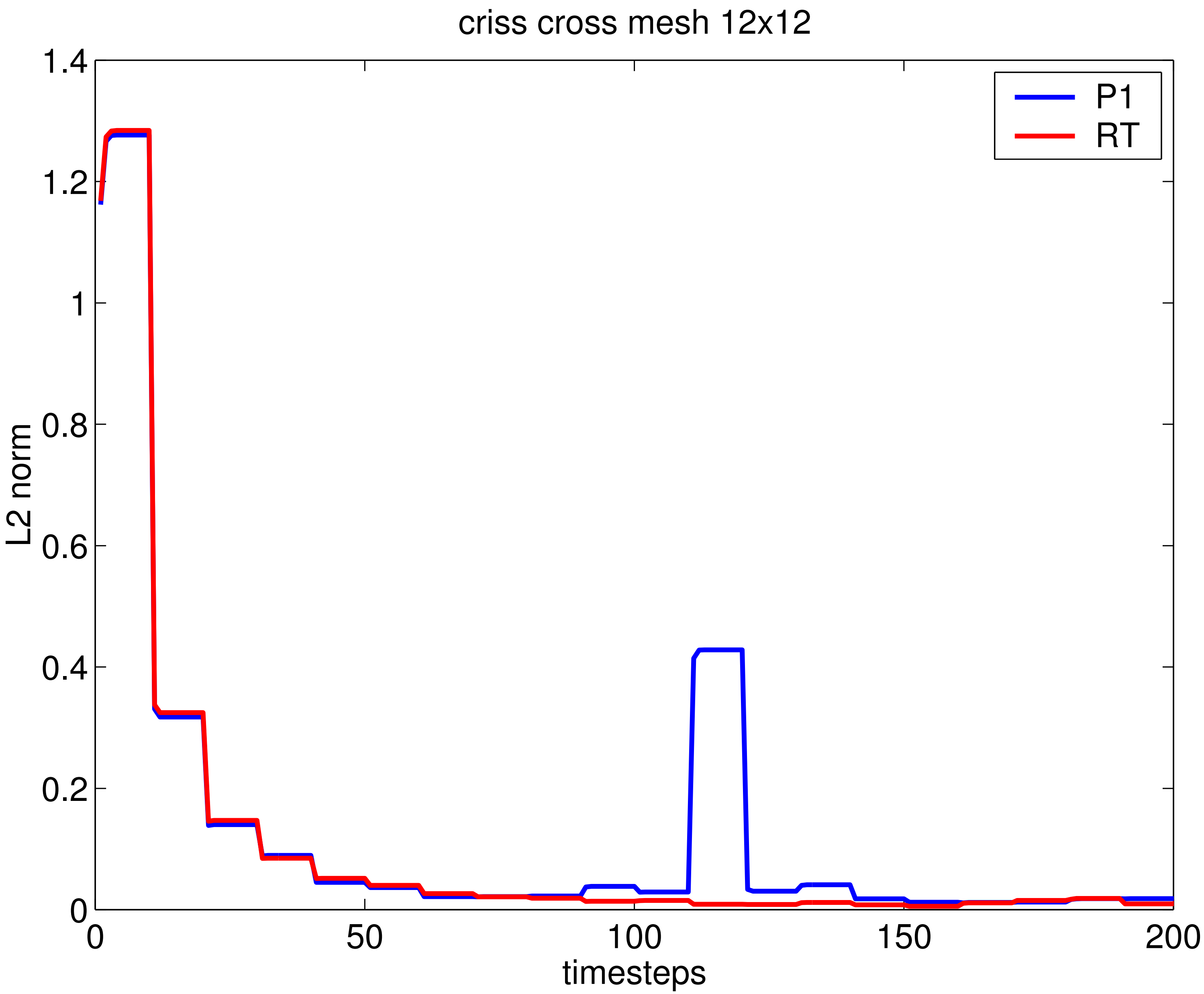
The P1 method presents spurious modes in the computation of eigenvalues. Such spurious modes have a checkerboard-like pattern.

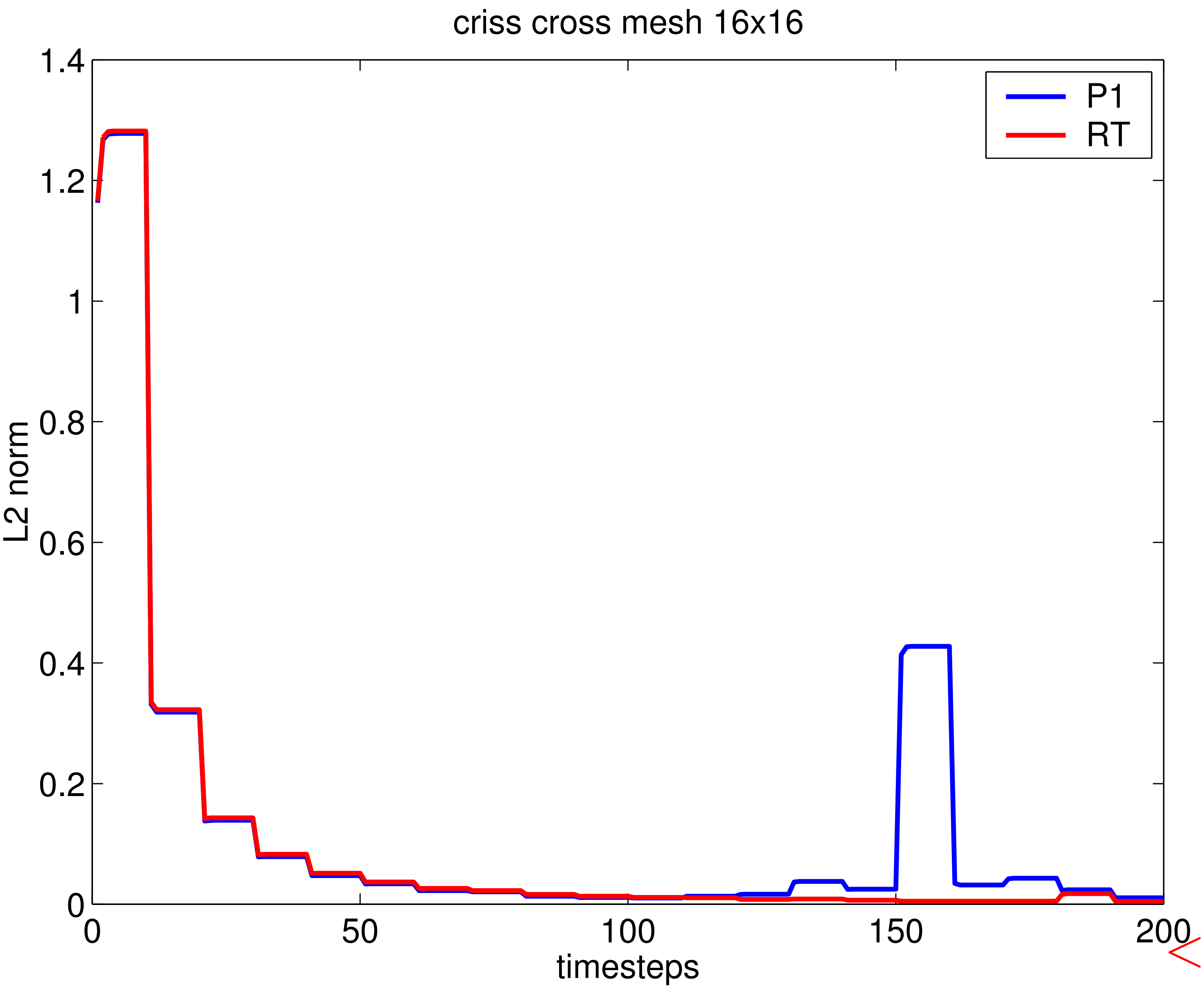
In particular, the first spurious eigenmode is excited by our choice of $g(h)$.

Another test

We solve the same problem with $u_0 = 0$ and a RHS depending on t in such a way that its oscillations get more and more frequent as t increases


criss cross mesh 8x8
L2 norm
timesteps
P1
RT
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
50
100
150
200


criss cross mesh 12x12
P1
RT
L2 norm
timesteps
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
50
100
150
200


criss cross mesh 16x16
L2 norm
timesteps
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
P1
RT

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← → ↑ ↓

Modified edge elements

Definition of spaces

$$
\begin{aligned}
Q_h =&\{q \in H^1_p : q|_K = e^{-i\alpha\cdot\mathbf{x}}\bar{q},\ \ \bar{q} \in \mathcal{P}_{k+1}(K)\}\\
E_h =&\{\mathbf{v} \in H_p(\mathrm{curl}) : \mathbf{v}|_K = e^{-i\alpha\cdot\mathbf{x}}\bar{\mathbf{v}},\ \ \bar{\mathbf{v}} \in \mathcal{E}_k\}\\
F_h =&\{\mathbf{w} \in H_p(\mathrm{div}) : \mathbf{w}|_K = e^{-i\alpha\cdot\mathbf{x}}\bar{\mathbf{w}},\ \ \bar{\mathbf{w}} \in \mathcal{F}_k\}\\
Q_h =&\{v \in L^2 : v|_K = e^{-i\alpha\cdot\mathbf{x}}\bar{v},\ \ \bar{v} \in \mathcal{P}_k(K)\}
\end{aligned}
$$

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