Differential Complexes and Stability of Finite Element Methods

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with help from R. Falk and R. Winther
and inspiration from many

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Stability is subtle

$$u_h \in W_h : \quad B_h(u_h, v) = F_h(v) \quad \forall v \in V_h$$

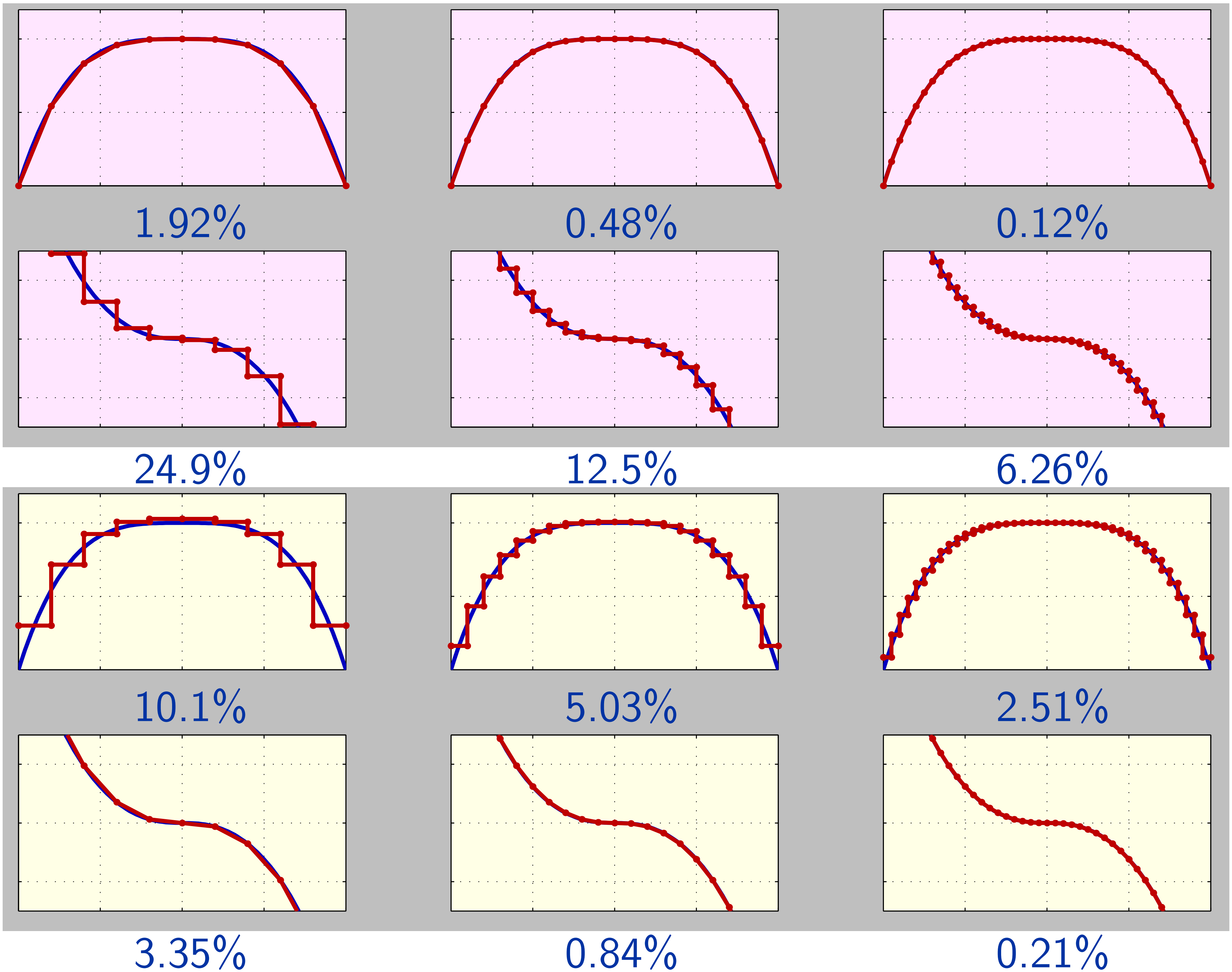
If $W_h = V_h$ and $B_h(w, w) \approx \|w\|^2$, stability in $\|\cdot\|$ is trivial.

If $V_h \neq W_h$, or B_h is not positive definite, or a different norm is of interest, stability is subtle even for simple problems!

$\sigma - u' = 0, \quad \sigma' = f$ on $(-1, 1)$, $\quad u(\pm 1) = 0$ 1D "mixed Laplacian"

$$(\sigma, u) \in L^2 \times \mathring{H}^1 : \langle \sigma, \tau \rangle - \langle u', \tau \rangle - \langle \sigma, v' \rangle = \langle f, v \rangle \quad \forall (\tau, v) \in L^2 \times \mathring{H}^1$$

$$(\sigma, u) \in H^1 \times L^2 : \langle \sigma, \tau \rangle + \langle u, \tau' \rangle + \langle \sigma', v \rangle = \langle f, v \rangle \quad \forall (\tau, v) \in H^1 \times L^2$$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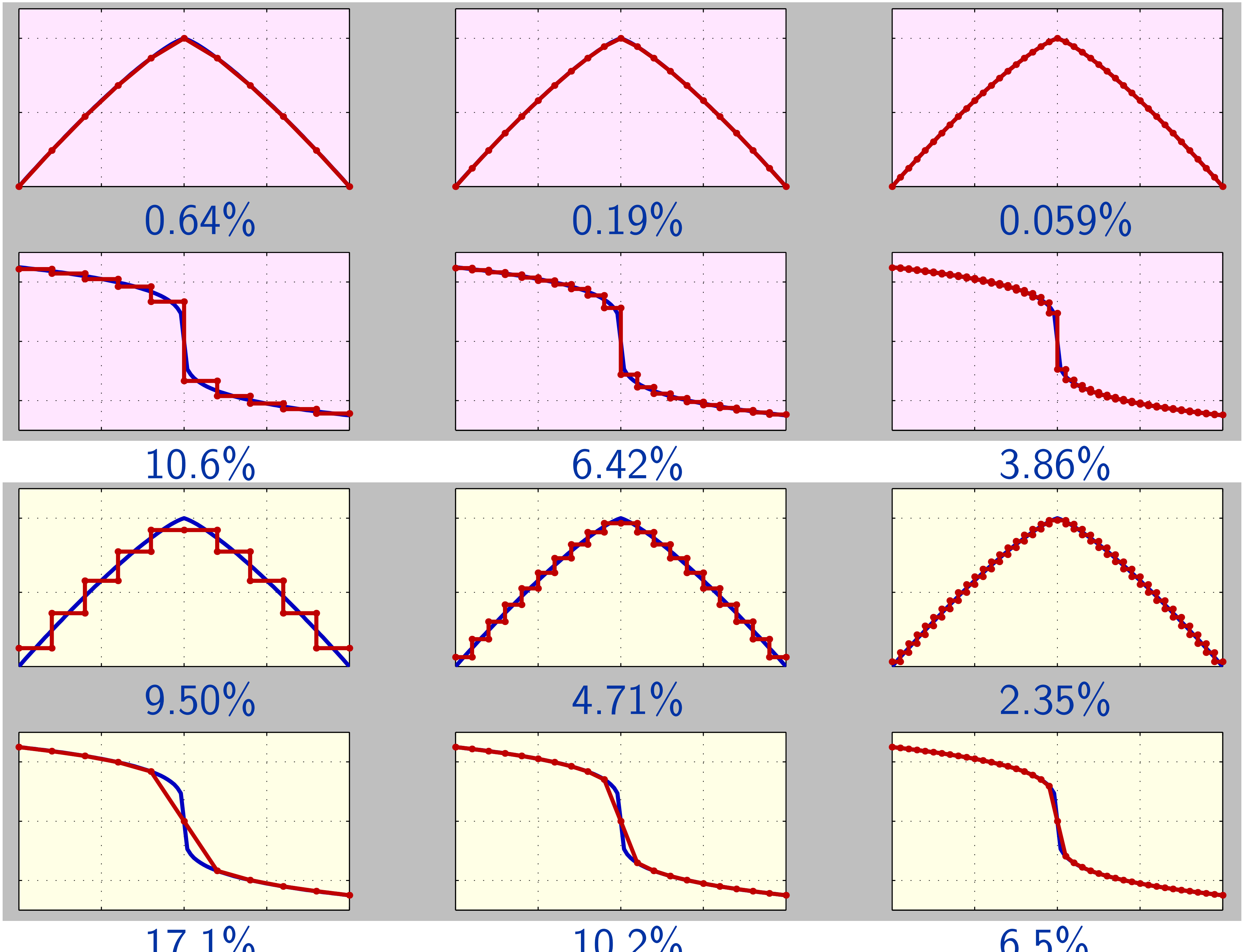
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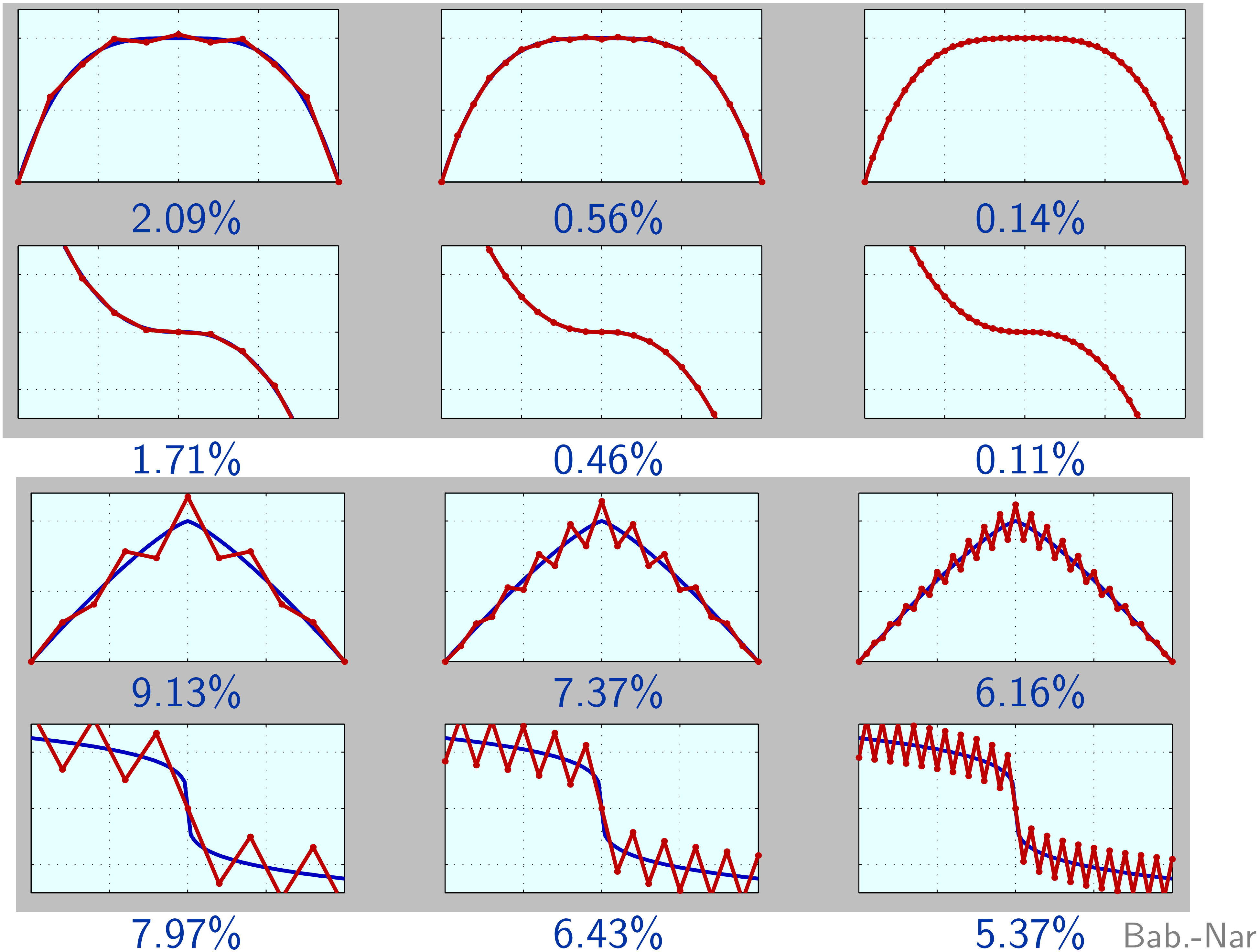
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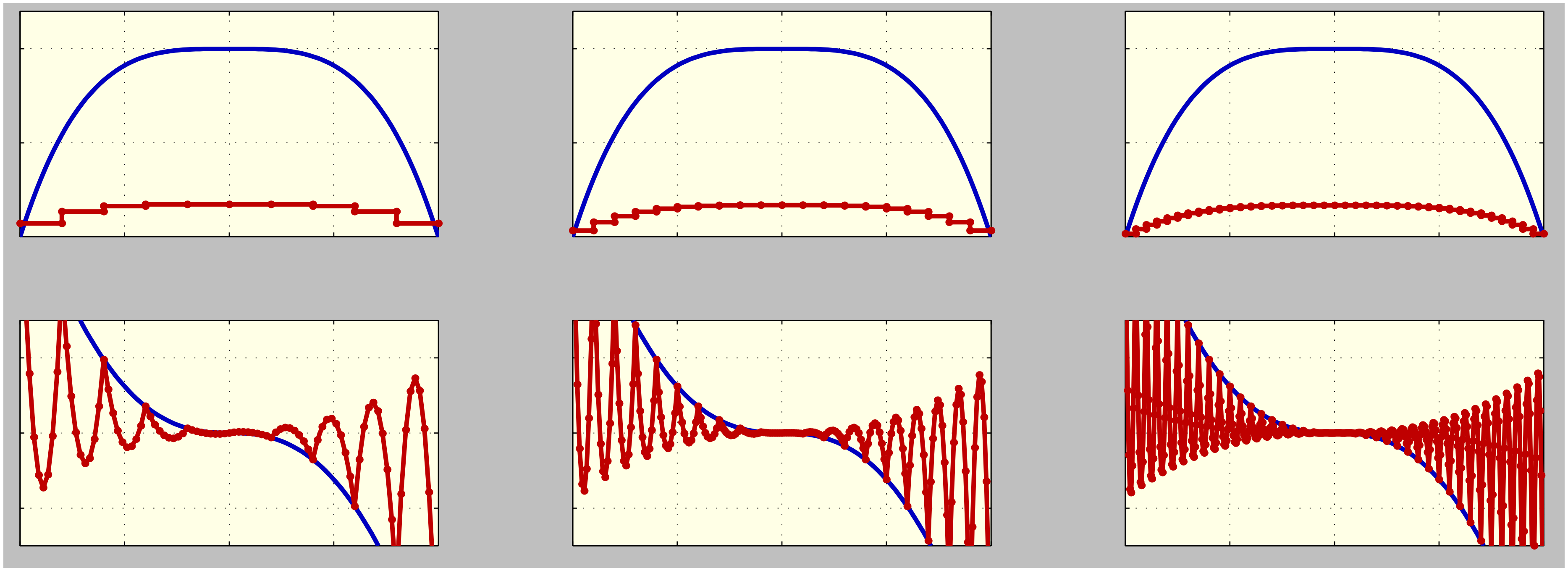
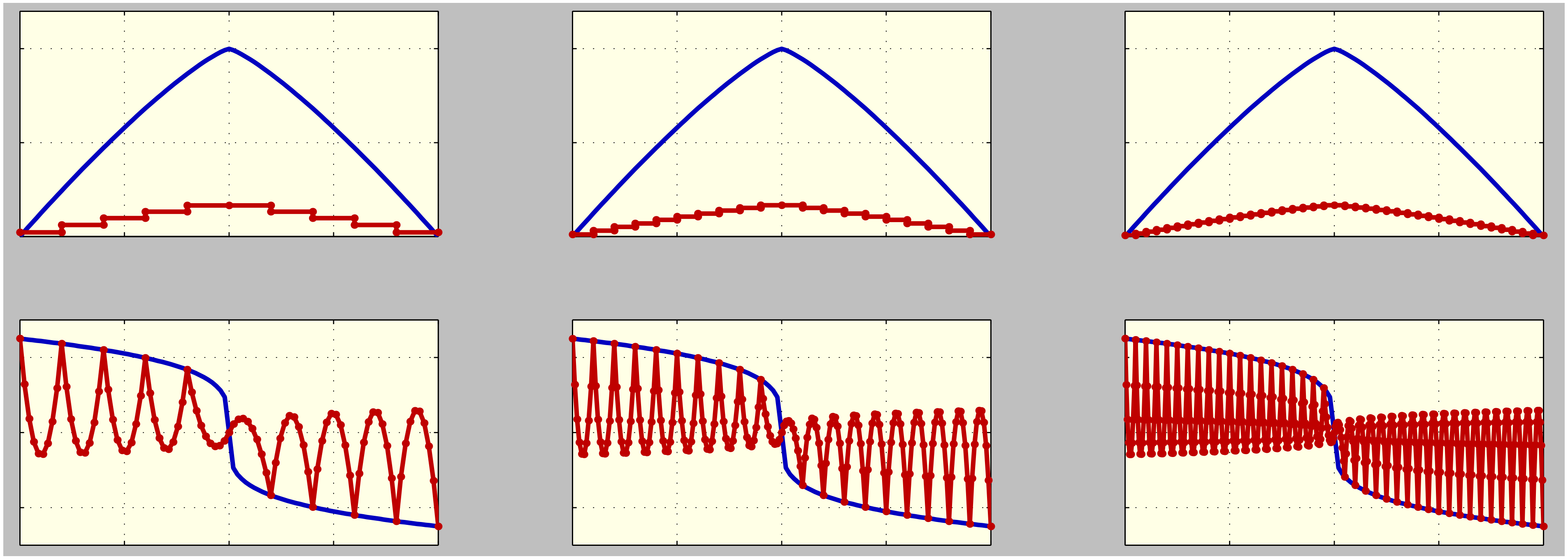
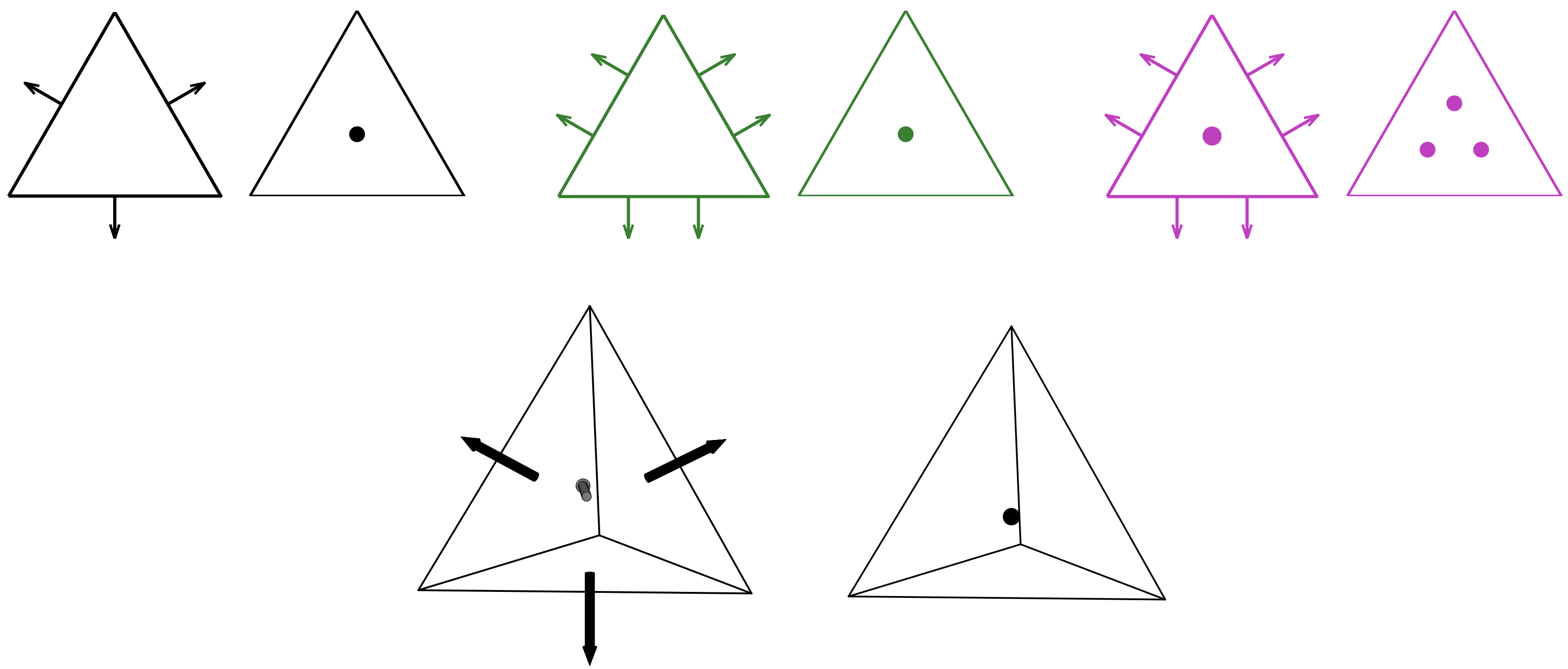




Bathe–Brezzi

$$(\sigma, u) \in H(\mathrm{div}) \times L^2 :$$

$$\langle \sigma, \tau \rangle + \langle u, \operatorname{div} \tau \rangle + \langle \operatorname{div} \sigma, v \rangle = \langle f, v \rangle \quad \forall (\tau, v) \in H(\mathrm{div}) \times L^2$$

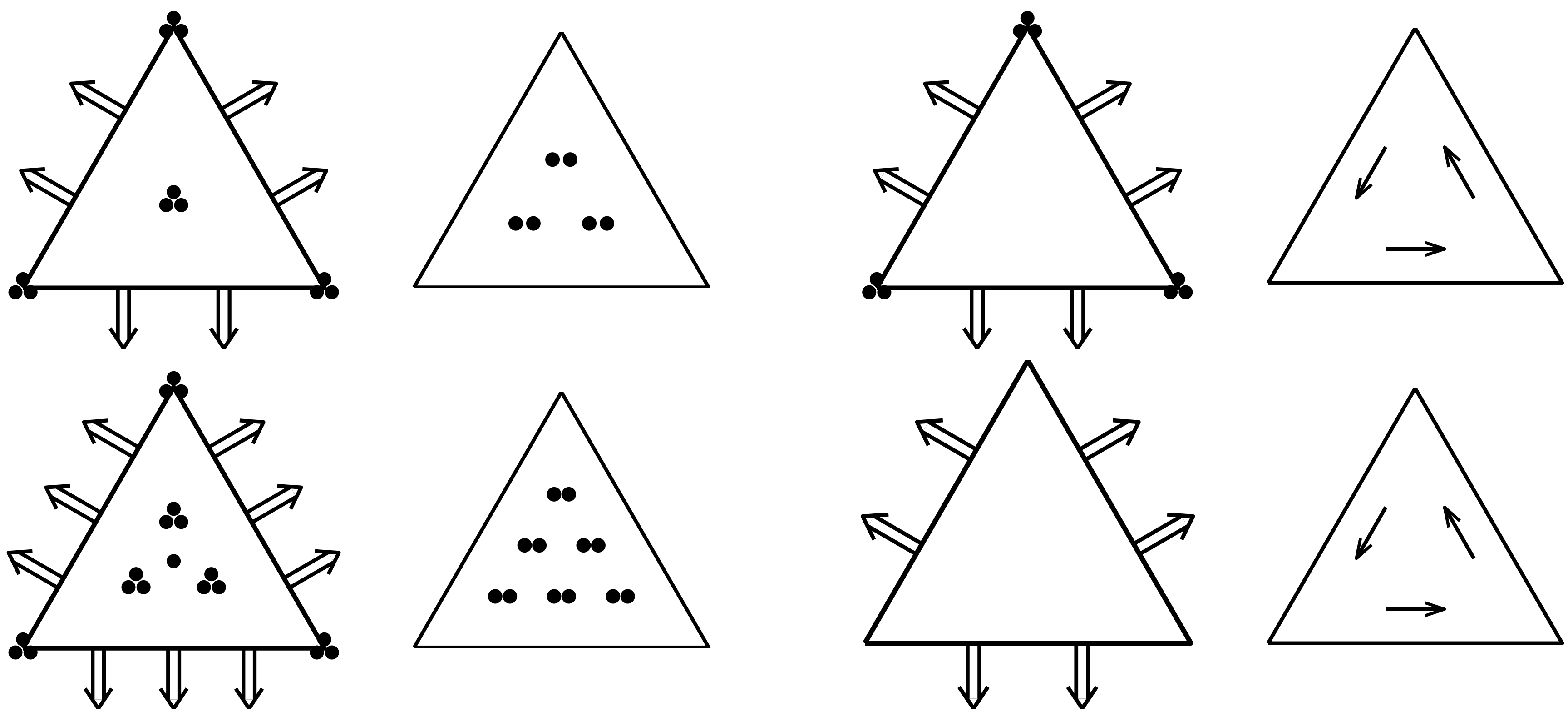


Raviart–Thomas, Brezzi–Douglas–Marini

$$(\sigma, u) \in H(\mathrm{div}, \mathsf{Sym}) \times L^2(\Omega, \mathbb{R}^n):$$

$$\langle \sigma, \tau \rangle + \langle u, \mathrm{div}\, \tau \rangle + \langle \mathrm{div}\, \sigma, v \rangle = \langle f, v \rangle$$

$$\forall (\tau, v) \in H(\mathrm{div}, \mathsf{Sym}) \times L^2(\Omega, \mathbb{R}^n)$$

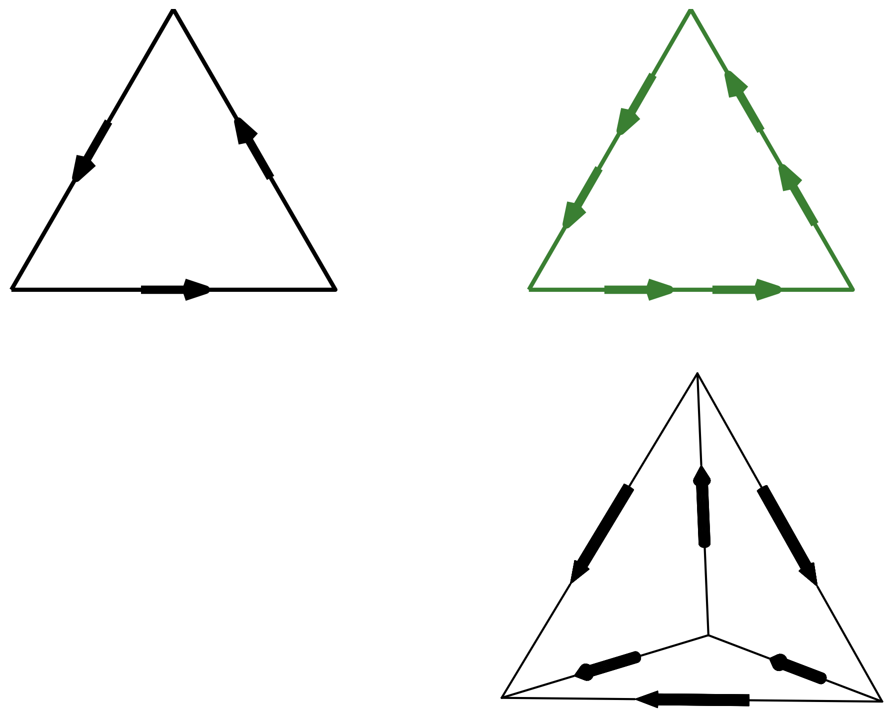


Arnold–Winther

Curl–curl problems (electromagnetics)

$$0 \neq E \in H(\text{curl}), \quad \lambda \in \mathbb{R}$$

$$\langle \text{curl}\, E, \text{curl}\, F \rangle = \lambda \langle E, F \rangle \quad \forall F \in H(\text{curl})$$



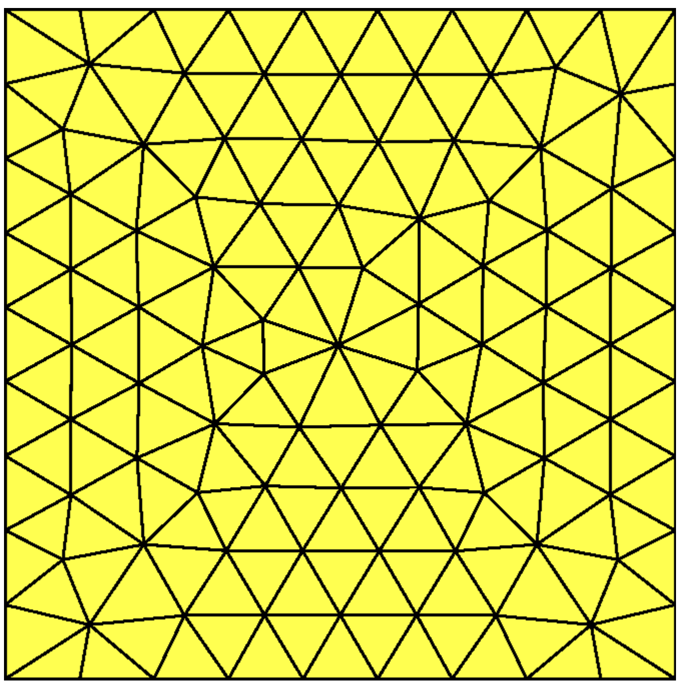
Nedelec

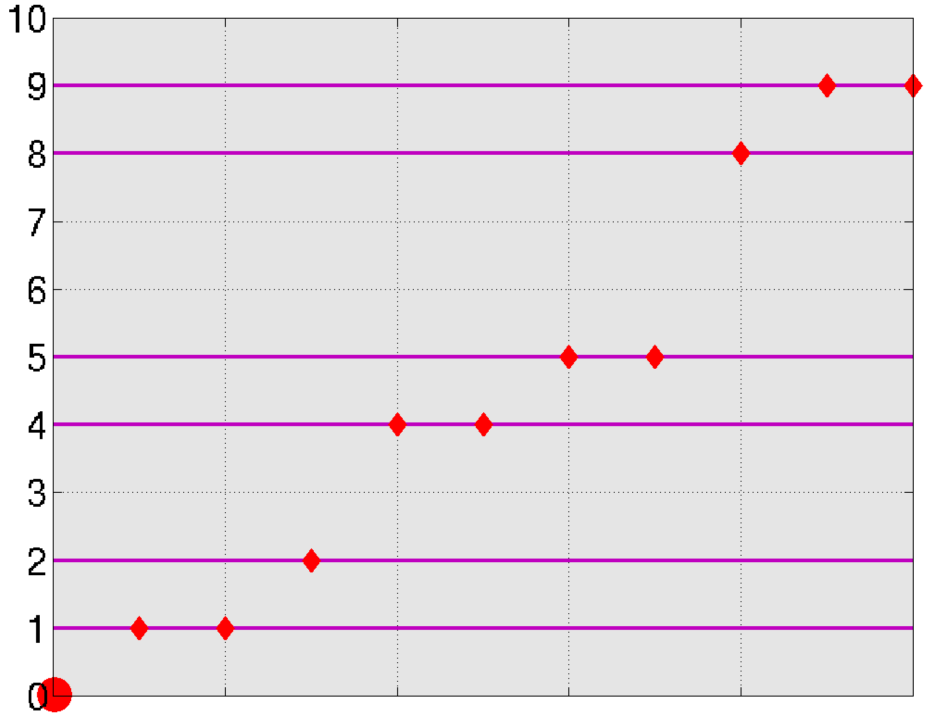
Example

$$0 \neq E_h \in W_h, \quad \lambda_h \in \mathbb{R}$$

$$\langle \operatorname{curl} E_h, \operatorname{curl} F \rangle = \lambda_h \langle E_h, F \rangle \quad \forall F \in W_h$$

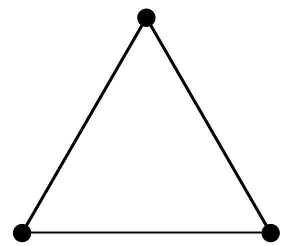
On a square eigenvalues are known: $\lambda = m^2 + n^2, \quad 0 \leq m, n \in \mathbb{Z}$



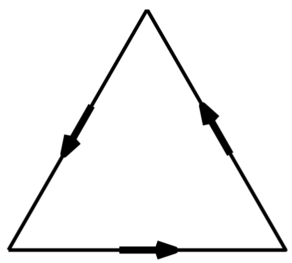


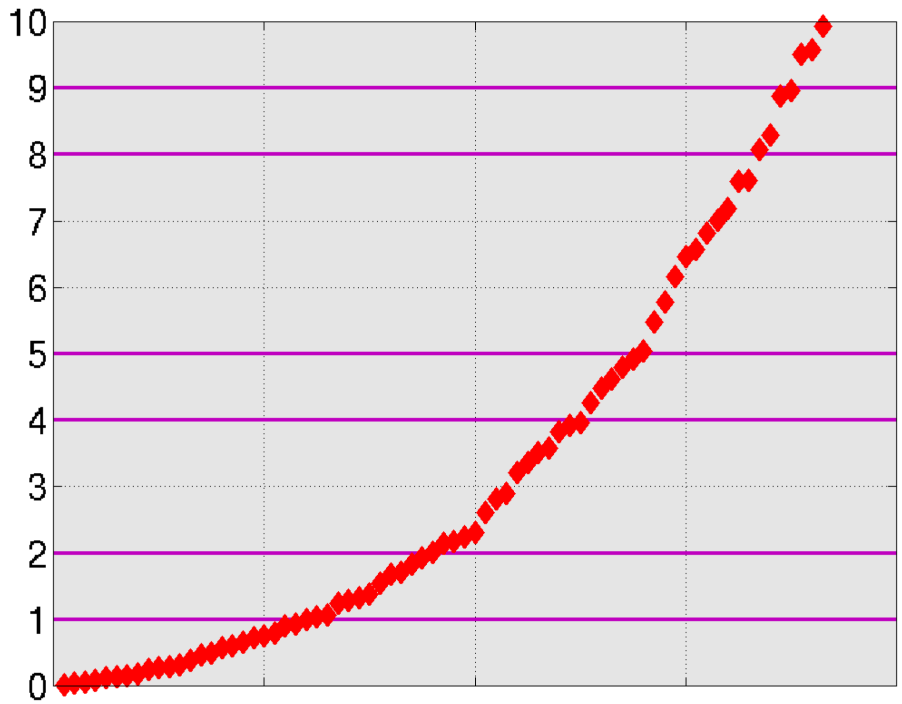
Computed eigenvalues

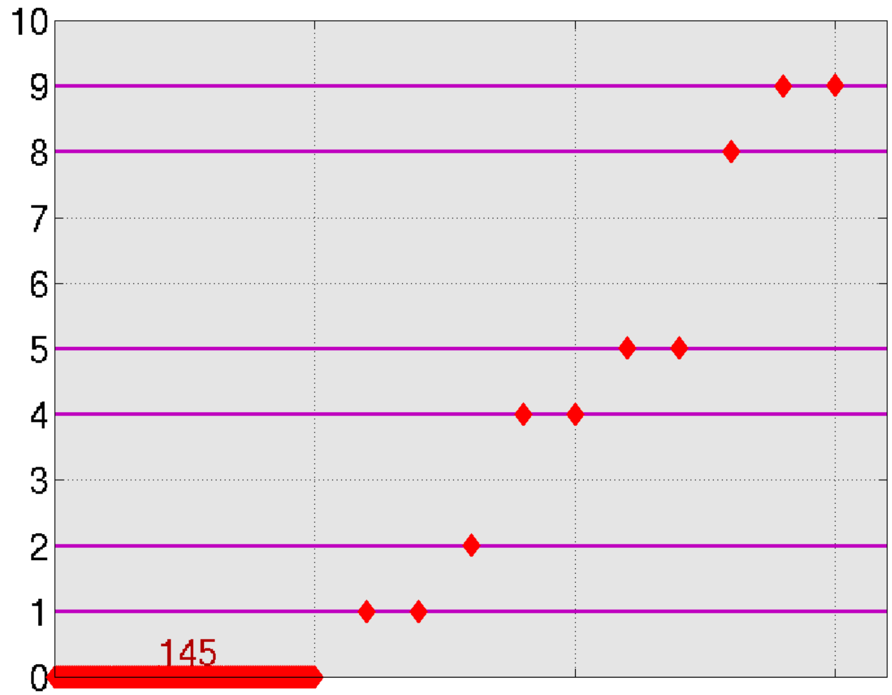
Lagrange elts



Edge elts







V a vectorspace, $\dim V = n$

- $\mathrm{Alt}^k V = \{\, \omega : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R} \,|\, \text{alternating} \,\}$
- $\mathrm{Alt}^0 V = \mathbb{R}$, $\quad \mathrm{Alt}^1 V = V^*$, $\quad \dim \mathrm{Alt}^k V = \binom{n}{k}$
- Exterior product: $\wedge$: $\mathrm{Alt}^k V \times \mathrm{Alt}^m V \to \mathrm{Alt}^{k+m} V$
- An inner product on V induces an inner product on $\mathrm{Alt}^k V$:

$$\langle \omega, \eta \rangle = \sum_{1 \le \sigma_1 < \cdots < \sigma_k \le n} \omega(v_{\sigma_1}, \ldots, v_{\sigma_k}) \eta(v_{\sigma_1}, \ldots, v_{\sigma_k}),$$

for an orthonormal basis $v_1, \ldots, v_n$

- If V is also oriented, $\mathrm{Alt}^k V \approx \mathrm{Alt}^{n-k} V$ (Hodge $\star$)

Ω a smooth n-manifold

- $\Lambda^k(\Omega) = \{\, x \in \Omega \mapsto \omega_x \in \mathrm{Alt}^k(T_x\Omega) \mid \text{smooth} \,\}$
- Also $C^0\Lambda^k(\Omega)$, $L^2\Lambda^k(\Omega)$, $L^p\Lambda^k(\Omega)$, . . .
- $\displaystyle\int_f \omega \in \mathbb{R}$ for $\dim f = k$, $\omega \in \Lambda^k V$
- Exterior derivative $d_k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$
- $d_{k+1} d_k = 0$

De Rham complex:

$$0 \to \mathbb{R} \xrightarrow{\subset} \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$$

Exact if Ω is contractible

Each $T_x\Omega$ endowed with an inner product

$L^2\Lambda^k(\Omega)$ is a Hilbert space

$$H\Lambda^k(\Omega) = \{\,\omega \in L^2\Lambda^k(\Omega)\,|\,d\omega \in L^2\Lambda^{k+1}(\Omega)\,\}$$

$$0 \to \mathbb{R} \xrightarrow{\subset} H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

L^2 de Rham complex, same cohomology

$d^* : \Lambda^{k+1}(\Omega) \to \Lambda^k(\Omega)$: $\langle d^*\omega, \eta\rangle_{L^2\Lambda^k} = \langle \omega, d\eta\rangle_{L^2\Lambda^{k+1}}$

Hodge Laplacian: $d^*d + dd^* : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$

Polynomial forms and the Koszul complex

For $\Omega \subset \mathbb{R}^n$, degree $r \geq 0$

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n(\Omega) \to 0$$

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\Omega) \to 0$$

Koszul differential $\kappa : \Lambda^{k+1} \to \Lambda^k$:

$$(\kappa\omega)_x(v^1, \dots, v^k) = \omega_x(x, v^1, \dots, v^k)$$

- $\kappa : \mathcal{H}_r\Lambda^k \to \mathcal{H}_{r+1}\Lambda^{k-1}$ (c.f. $d : \mathcal{H}_{r+1}\Lambda^{k-1} \to \mathcal{H}_r\Lambda^k$)
- $(d\kappa + \kappa d)\omega = (r + k)\omega \quad \forall\omega \in \mathcal{H}_r\Lambda^k(\Omega)$

$$0 \leftarrow \mathbb{R} \leftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \leftarrow 0$$

Koszul complex

$$\mathcal{P}_r^+\Lambda^k := \mathcal{P}_r\Lambda^k + \kappa\mathcal{P}_r\Lambda^{k+1}$$

The case $\Omega \subset \mathbb{R}^3$

$$0 \to \mathbb{R} \xrightarrow{\subset} \Lambda^0(\Omega) \xrightarrow{\text{grad}} \Lambda^1(\Omega) \xrightarrow{\text{curl}} \Lambda^2(\Omega) \xrightarrow{\text{div}} \Lambda^3(\Omega) \to 0$$
$$0 \to \mathbb{R} \xrightarrow{\subset} C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0$$

smooth de Rham complex

$$0 \to \mathbb{R} \xrightarrow{\subset} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

L^2 de Rham complex

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_r(\Omega) \xrightarrow{\text{grad}} \mathcal{P}_{r-1}(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} \mathcal{P}_{r-2}(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} \mathcal{P}_{r-3}(\Omega) \to 0$$

polynomial de Rham complex

$$0 \leftarrow \mathbb{R} \longleftarrow \mathcal{P}_r(\Omega) \xleftarrow{\cdot x} \mathcal{P}_{r-1}(\Omega, \mathbb{R}^3) \xleftarrow{\times x} \mathcal{P}_{r-2}(\Omega, \mathbb{R}^3) \xleftarrow{x} \mathcal{P}_{r-3}(\Omega) \leftarrow 0$$

Koszul complex

Some PDEs related to the de Rham complex

$\Omega \subset \mathbb{R}^n$, $0 \le k \le n$, $f \in L^2\Lambda^k(\Omega)$

$$
\begin{aligned}
&\sigma \in H\Lambda^{k-1}(\Omega), \quad u \in H\Lambda^k(\Omega) : \\
&\qquad \langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 \qquad \forall \tau \in H\Lambda^{k-1}(\Omega) \\
&\qquad \langle d\sigma, v \rangle + \langle du, dv \rangle = \langle f, v \rangle \quad \forall v \in H\Lambda^k(\Omega)
\end{aligned}
$$

This is a mixed form of the Hodge Laplacian, well-posed.

- $k = n = 3$: mixed Laplacian
- $k = 0$, $n = 3$: ordinary Laplacian
- $k = 1$ or 2, $n = 3$, two different curl–curl problems

$\sigma \in H(\mathrm{curl}), \quad u \in H(\mathrm{div}):$

$$\langle \sigma, \tau \rangle - \langle \mathrm{curl}\, \tau, u \rangle = 0 \qquad \forall \tau \in H(\mathrm{curl})$$

$$\langle \mathrm{curl}\, \sigma, v \rangle + \langle \mathrm{div}\, u, \mathrm{div}\, v \rangle = \langle f, v \rangle \quad \forall v \in H(\mathrm{div}))$$

$$\sigma = \mathrm{curl}\, u, \quad \mathrm{curl}\, \sigma - \mathrm{grad}\, \mathrm{div}\, u = f \quad \text{in } \Omega$$

$$u \times n = 0, \quad \mathrm{div}\, u = 0 \quad \text{on } \partial\Omega$$

If $\mathrm{div}\, f = 0$, then

$$\mathrm{curl}\, \mathrm{curl}\, u = f, \quad \mathrm{div}\, u = 0 \quad \text{in } \Omega, \quad u \times n = 0 \quad \text{on } \partial\Omega$$

$$\sigma \in H^1, \quad u \in H(\mathrm{curl}):$$

$$\langle \sigma, \tau \rangle - \langle \mathrm{grad}\, \tau, u \rangle = 0 \qquad \forall \tau \in H^1$$

$$\langle \mathrm{grad}\, \sigma, v \rangle + \langle \mathrm{curl}\, u, \mathrm{curl}\, v \rangle = \langle f, v \rangle \quad \forall v \in H(\mathrm{curl}))$$

$$\sigma = -\mathrm{div}\, u, \quad \mathrm{grad}\, \sigma + \mathrm{curl}\,\mathrm{curl}\, u = f \quad \text{in } \Omega$$

$$u \cdot n = 0, \quad \mathrm{curl}\, u \times n = 0 \quad \text{on } \partial\Omega$$

If $f \perp \mathrm{grad}\, H^1$, then $\sigma = 0$ and

$$\mathrm{curl}\,\mathrm{curl}\, u = f, \quad \mathrm{div}\, u = 0 \quad \text{in } \Omega, \quad \mathrm{curl}\, u \times n = 0 \quad \text{on } \partial\Omega$$

Piecewise polynomial differential forms

$\mathcal{T}$ a triangulation of $\Omega \subset \mathbb{R}^n$ by n-simplices

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) := \{\,\omega \in H\Lambda^k(\Omega) \mid \omega_{|_T} \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T}\,\}$$
$$\mathcal{P}_r^+\Lambda^k(\mathcal{T}) := \{\,\omega \in H\Lambda^k(\Omega) \mid \omega_{|_T} \in \mathcal{P}_r^+\Lambda^k(T) \quad \forall T \in \mathcal{T}\,\}$$

- $\mathcal{P}_r^+\Lambda^0(\mathcal{T}) = \mathcal{P}_{r+1}\Lambda^0(\mathcal{T}) \subset H^1$ usual Lagrange finite elements
- $\mathcal{P}_r^+\Lambda^n(\mathcal{T}) = \mathcal{P}_r\Lambda^n(\mathcal{T}) \subset L^2$ usual discontinuous piecewise polys.
- $n = 3$: $\mathcal{P}_r^+\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec elements
- $\mathcal{P}_r\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 2nd family
- $\mathcal{P}_r^+\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Raviart–Thomas elements
- $\mathcal{P}_r\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Brezzi–Douglas–Marini elements

Degrees of freedom

T an n-simplex, $\Delta_d(T)$ = set of faces of dimension d, $0 \le d \le n$

$$\dim \mathcal{P}_r\Lambda^k(T_n) = \binom{n+r}{n}\binom{n}{k}$$

DOF:

$$u \mapsto \int_f u \wedge v, \quad v \in \mathcal{P}^+_{r-d-1+k}\Lambda^{d-k}(f), \quad f \in \Delta_d(T), \quad k \le d \le n$$

$$\dim \mathcal{P}^+_r\Lambda^k(T_n) = \binom{n+r}{n}\binom{n}{k} + \binom{n+r}{n-k-1}\binom{r+k}{k}$$

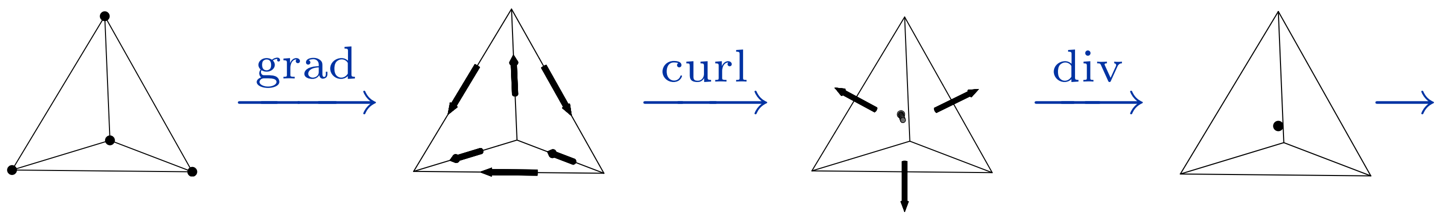
DOF:

$$u \mapsto \int_f u \wedge v, \quad v \in \mathcal{P}_{r-d+k}\Lambda^{d-k}(f), \quad f \in \Delta_d(T_n), \quad k \le d \le n$$

For every $r \ge 0$, an exact piecewise polynomial subcomplex:

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_r^+\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^+\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^+\Lambda^n(\mathcal{T}) \to 0$$

For $n = 3$, $r = 0$ these are the Whitney elements:

$\mathbb{R} \hookrightarrow$  $\to 0$

For all r, the natural projections $\Pi_{r+}^k : \Lambda^k(\Omega) \to \mathcal{P}_r^+\Lambda^k(\mathcal{T})$ relate this to the de Rham sequence commutatively:

$$\begin{array}{ccccccccccc}
0 \to \mathbb{R} & \xrightarrow{\subset} & \Lambda^0(\Omega) & \xrightarrow{d} & \Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda^n(\Omega) \to 0 \\
& & \downarrow \Pi_{r+}^0 & & \downarrow \Pi_{r+}^1 & & & & \downarrow \Pi_{r+}^n \\
0 \to \mathbb{R} & \xrightarrow{\subset} & \mathcal{P}_r^+\Lambda^0(\mathcal{T}) & \xrightarrow{d} & \mathcal{P}_r^+\Lambda^1(\mathcal{T}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{P}_r^+\Lambda^n(\mathcal{T}) \to 0
\end{array}$$

Discrete exact sequences, continued

There is another exact sequence ending at $\mathcal{P}_r\Lambda^n(\mathcal{T})$

(Demkowicz, Monk, Vardapetyan, Rachowicz):

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_{r+n}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+n-1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r\Lambda^n(\mathcal{T}) \to 0$$

In fact, there are 2^{n-1} of them in n dimensions!

For $n = 3$ the other two are:

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^+\Lambda^2(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}) \to 0$$

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+1}^+\Lambda^1(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^2(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}) \to 0$$

All relate to the de Rham complex through a commutative diagram.

Consider the stability (in the $H\Lambda^k$ norms) of the discretized PDE:

$$\sigma \in \Lambda_h^{k-1}, \quad u \in \Lambda_h^k :$$
$$\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 \qquad \forall \tau \in \Lambda_h^{k-1}$$
$$\langle d\sigma, v \rangle + \langle du, dv \rangle = \langle f, v \rangle \qquad \forall v \in \Lambda_h^k$$

where the Λ_h^k are finite element subspaces of $H\Lambda^k(\Omega)$.

Bilinear form: $B(\sigma, u; \tau, v) = \langle \sigma, \tau \rangle - \langle d\tau, u \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle$

Need to show: $\forall\ \sigma$, u, $\exists\ \tau$, v:

$$B(\sigma, u; \tau, v) \geq \gamma(\|\sigma\|^2_{H\Lambda^{k-1}} + \|u\|^2_{H\Lambda^k})$$
$$\|\tau\|_{H\Lambda^{k-1}} + \|v\|_{H\Lambda^k} \leq C(\|\sigma\|_{H\Lambda^{k-1}} + \|u\|_{H\Lambda^k})$$

The key properties are: *1. exact sequence of discrete spaces*:

$$0 \to \mathbb{R} \xrightarrow{\subset} \Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_h^n \to 0$$

2. inf-sup condition: $\forall v \in \Lambda_h^k$ with $dv = 0$, $\exists \tau \in \Lambda_h^{k-1}$ with $d\tau = v$ and $\|\tau\|_{H\Lambda^{k-1}} \le C\|v\|_{H\Lambda^k}$

Stability proof: Given $u \in \Lambda_h^k$, $\sigma \in \Lambda_h^{k-1}$, define $\rho \in \Lambda_h^{k-1}$ and $y \in \Lambda_h^{k+1}$ by

$$u = d\rho + d^*y, \quad d^*r = 0, \quad dy = 0.$$

(Possible due to exact sequence.) Using inf-sup condition, $\|\rho\|_{H\Lambda^{k-1}} \le C\|u\|$, and $\|u\| \le C(\|du\| + \|d\rho\|)$.

Choose $\tau = \sigma - t\rho$, $v = u + d\sigma$, t sufficiently small.

The key to proving the inf-sup condition is the commuting diagram.

$$\begin{array}{ccc} \Lambda^{k-1}(\Omega) & \xrightarrow{d} & \Lambda^k(\Omega) \\ \Pi_h^{k-1}\big\downarrow & & \Pi_h^k\big\downarrow \\ \Lambda_h^{k-1} & \xrightarrow{d} & \Lambda_h^k \end{array}$$

Sketch: given $v \in \Lambda_h^k$ with $dv = 0$, $\exists \tau \in H\Lambda^{k-1}$ with $d\tau = v$ and $\|\tau\|_{H\Lambda^{k-1}} \le C\|v\|_{H\Lambda^k}$.

But we need $\tau \in \Lambda_h^{k-1}$.

The idea is to use $\Pi_h^{k-1}\tau$, for which $d\Pi_h^{k-1}\tau = \Pi_h^k d\tau = \Pi_h^k v = v$.

There are technical problems with regularity (Π_h^k is not defined on all of $H\Lambda^k$), but these can be overcome.

The elasticity complex:

$$\mathbb{T} \hookrightarrow C^\infty(\Omega,\mathbb{R}^3) \xrightarrow{\epsilon} C^\infty(\Omega,\mathsf{Sym}) \xrightarrow{J} C^\infty(\Omega,\mathsf{Sym}) \xrightarrow{\mathrm{div}} C^\infty(\Omega,\mathbb{R}^3) \longrightarrow 0$$

↑ displacement ↑ strain ↑ stress ↑ load

$J = \mathrm{curl}_{\mathsf{c}}\,\mathrm{curl}_{\mathsf{r}}$, second order

$\mathbb{T}$ is the space of infinitesmal rigid motions

A complex from relativity:

$$\mathcal{P}_1 \hookrightarrow C^\infty(\Omega,\mathbb{R}) \xrightarrow{\nabla\nabla} C^\infty(\Omega,\mathsf{Sym}) \xrightarrow{\mathrm{curl}} C^\infty(\Omega,\mathbb{R}^{3\times 3}_{\mathsf{TF}}) \xrightarrow{\mathrm{div}} C^\infty(\Omega,\mathbb{R}^3) \to 0$$

For what PDEs is there an associated differential complex which it is useful to consider and, maybe, useful to discretize

?